Efficient Gossiping by Short Messages

Jean–Claude Bermond*
I3S, CNRS
Université de Nice, Sophia Antipolis
Bat ESSI, 930 Route de Colles, BP 145
06903 Sophia Antipolis Cedex, France

Luisa Gargano†
Dipartimento di Informatica ed Applicazioni
Università di Salerno
84081 Baronissi (SA), Italy

Adele A. Rescigno†
Dipartimento di Informatica ed Applicazioni
Università di Salerno
84081 Baronissi (SA), Italy

Ugo Vaccaro†
Dipartimento di Informatica ed Applicazioni
Università di Salerno
84081 Baronissi (SA), Italy

November 7, 1994
(19:42)

Abstract

Gossiping is the process of information diffusion in which each node of a network holds a packet that must be communicated to all other nodes in the network. We consider the problem of gossiping in communication networks under the restriction that communicating nodes can exchange up to a fixed number p of packets at each round. In the first part of the paper we study the extremal case $p = 1$ and we exactly determine the optimal number of communication rounds to perform gossiping for several classes of graphs, including Hamiltonian graphs and complete k-ary trees. For arbitrary graphs we give asymptotically matching upper and lower bounds. We also study the case of arbitrary p and we exactly determine the optimal number of communication rounds to perform gossiping under this hypothesis for complete graphs, hypercubes, cycles, and paths.

*The work of this author was partially supported by the French GDR/PRC Project PRS. He wants also to thank the Dipartimento di Informatica ed Applicazioni of the Università di Salerno, where part of his research was done, for inviting him.

†The work of these authors was partially supported by Progetto Finalizzato Sistemi Informatici e Calcolo Parallelo of C.N.R. under Grant No. 92.01622.PF69 and by the Italian Ministry of the University and Scientific Research, Project: Algoritmi, Modelli di Calcolo e Strutture Informative.

1 Introduction

Gossiping (also called total exchange or all–to–all communication) in distributed systems is the process of distribution of information known to each processor to every other processor of the system. This process of information dissemination is carried out by means of a sequence of message transmissions between adjacent nodes in the network.

The gossiping problem was originally introduced by the community of discrete mathematicians, to which it owes most of its terminology, as a combinatorial problem in graphs. Nonetheless, it was soon realized that, once cast in more realistic models of communication, gossiping is a fundamental primitive in distributed memory multiprocessor system. There are a number of situations in multiprocessor computation, such as global processor synchronization, where gossiping occurs. Moreover, the gossip problem is implicit in a large class of parallel computation problems, such as linear system solving, Discrete Fourier Transform, and sorting, where both input and output data are required to be distributed across the network [8]. Due to the interesting theoretical questions it poses and its numerous practical applications, gossiping has been widely studied under various communication models. Hedetniemi, Hedetniemi and Liestman [23] provide a survey of the area. Two more recent surveys paper collecting the latest results are [17, 27]. The reader can also profitably see the book [39]. The problem of coping with malfunctionings during the execution of gossiping protocols is addressed in [10, 12, 13, 6, 19, 20, 22].

The great majority of the previous work on gossiping has considered the case in which the packets initially held by each node can be freely combined and the resulting messages transmitted in a constant amount of time, that is, that the time required to transmit a message is independent from its length. While this assumption is reasonable for short messages, it is clearly unrealistic in case the size of the messages becomes large. Notice that most of the gossiping protocols proposed in the literature require the transmission, in the last rounds of the execution of the protocol, of messages of size $\Theta(n)$, where n is the number of nodes in the network. Therefore, it would be interesting to have gossiping protocols that require only the transmission of bounded length messages between processors. In this paper we consider the problem of gossiping in communication networks under the restriction that communicating nodes can exchange up to a fixed number p of packets at each round.

1.1 The Model

Consider a communication network modeled by a graph $G = (V, E)$ where the node set V represents the set of processors of the network and E represents the set of the communication lines between processors.

Initially each node holds a packet that must be transmitted to any other node in the network by a sequence of *calls* between adjacent processors. During each call, communicating nodes can exchange up to p packets, where p is an *a priori* fixed integer. We assume that each processor can participate in at most one call at time. Therefore, we can see the gossiping process as a sequence of *rounds*: During each round a disjoint set of edges (matching) is selected and the nodes that are end vertices of these edges make a call. This communication model is usually referred to as *telephone model* [23] or *Full–Duplex 1–Port* (F_1) [33]. We denote by $g_{F_1}(p, G)$ the minimum possible number of rounds to complete the gossiping process in the graph G subject to the above conditions. Another popular communication model is the *mail model* [23] or *Half–Duplex 1–Port* (H_1) [33], in which in each round any node can send a message to one of its neighbors or receive a message from it but not simultaneously. The problem of estimating $g_{H_1}(p,G)$ has been considered in [5]. Analogous problems in bus networks have been considered in [18, 24]. Optimal bounds on $g_{H_1}(1,G)$ when the edges of G are subject to random failures are given in [10]. Packet routing in interconnection networks in the F_1 model has been considered in [1].

1.2 Results

We first study the extremal case in which gossiping is to be performed under the restriction that communicating nodes can exchange *exactly* one packet at each round. We provide several lower bound on the gossip time $g_{F_1}(1,G)$ and we provide matching upper bounds for Hamiltonian graphs, complete trees, and complete bipartite graphs. For general graphs we provide asymptotically tight upper and lower bounds. Subsequently, we study the case of p fixed arbitrary constant and we compute exactly $g_{F_1}(p,G)$ for complete graphs, hypercubes, cycles and paths. Our result for hypercubes allow us to improve the corresponding result in the H_1 model given in [5].

2 Gossiping by exchanging one packet at time

In this section we study $g_{F_1}(1,G)$, that is the minimum possible number of rounds to complete gossip in a graph G under the condition that at each round communicating nodes can exchange *exactly one* packet.

In order to avoid overburdening the notation, in all this section we will simply write $g(G)$ to denote $g_{F_1}(1,G)$.

2.1 Lower bounds on $g(G)$

In this section we give some lower bounds on the time needed to complete the gossiping process.

Lemma 2.1 *For any graph $G=(V,E)$, with $|V|=n$, let $\mu(G)$ be the size of a maximum matching in G, then*

$$g(G) \geq \left\lceil \frac{n(n-1)}{2\mu(G)} \right\rceil . \tag{1}$$

Proof. For any node $v \in V$ the packet initially resident in v must reach each of the remaining $n-1$ nodes in the graph. Therefore, during the gossiping process, at least $n(n-1)$ packet transmissions must be executed over the edges of G. Since in each communication round at most $\mu(G)$ calls can be performed and each call allows the transmission of 2 packets (one in each direction) the bound follows. □

Lemma 2.2 *Let $X \subset V$ be a vertex cutset of the graph $G=(V,E)$ whose removal disconnects G into the connected components $V_1,\ldots,V_d$, then*

$$g(G) \geq \left\lceil \sum_{i=1}^{d} \frac{\max\{|V_i|, n-|V_i|\}}{|M_X|} \right\rceil , \tag{2}$$

where $|M_X|$ is the size of a maximum matching M_X in G such that any edge in it has an endpoint in X and the other in $V-X$.

Proof. Consider a component V_i, for some $1 \leq i \leq d$. Nodes in V_i can receive the packets of nodes in $V - V_i$ only by means of calls between a node in X and one in V_i; moreover, at least $n - |V_i|$ calls are needed between nodes in X and nodes in V_i to bring all packets in $V - V_i$ to nodes in V_i. Analogously, packets of nodes in V_i can reach nodes in $V - V_i$ only by means of calls between a node in X and one in V_i and at least $|V_i|$ such calls are needed. Therefore, for each $i = 1, \ldots, d$, at least $\max\{|V_i|, n - |V_i|\}$ calls must take place between nodes in X and nodes in V_i. We then get that at least $\sum_{i=1}^{d} \max\{|V_i|, n - |V_i|\}$ calls are needed between nodes in X and nodes in $V - X = \cup_{i=1}^{d} V_i$. Since at most $|M_X|$ such calls can take place during each round, we get the desired lower bound of

$$\left\lceil \frac{\sum_{i=1}^{d} \max\{|V_i|, n - |V_i|\}}{|M_X|} \right\rceil$$

on the time necessary to gossip in G. □

Remark 2.1 The bound in the above Lemma 2.2 can sometimes be improved by observing that after the last call has been done between a node in some V_i and a node in X, the last exchanged message has still to reach all the other nodes of V_i (or of $V - V_i$). Therefore, we can add to the lower bound (2) the minimum of the eccentricities of the subgraphs induced by the V_i's and the $V - V_i$'s.

Corollary 2.1 *Let $\alpha(G)$ be the independence number of G, then*

$$g(G) \geq \left\lceil \frac{\alpha(G)(n-1)}{n - \alpha(G)} \right\rceil. \tag{3}$$

Proof. Let Y denote an independent set of G. Applying Lemma 2.2 with cutset $X = V - Y$ and connected components $V_1, \ldots, V_{|Y|}$, each consisting of just one element of Y, we get

$$g(G) \geq \left\lceil \sum_{i=1}^{|Y|} \frac{n - |V_i|}{|M_X|} \right\rceil \geq \left\lceil \sum_{i=1}^{|Y|} \frac{n - |V_i|}{|X|} \right\rceil = \left\lceil \sum_{i=1}^{|Y|} \frac{n-1}{n - |Y|} \right\rceil = \left\lceil \frac{|Y|(n-1)}{n - |Y|} \right\rceil.$$

Choosing an independent set of maximum size $|Y| = \alpha(G)$ we get (3). □

Let T be a tree and v one of its nodes, we indicate the connected components into which the node set of T is splitted by the removal of v by $V_1(v), \ldots, V_{\deg(v)}(v)$, ordered so that $|V_1(v)| \geq \ldots \geq |V_{\deg(v)}(v)|$.

Corollary 2.2 *Let T be a tree on n nodes of maximum degree $\Delta = \max_{v \in V} \deg(v)$, then*

$$g(T) \geq \max_{v \,:\, \deg(v) = \Delta} L(v),$$

where

$$L(v) = \begin{cases} (\deg(v) - 1)n + 1 & \text{if } |V_1(v)| \leq n/2; \\ (\deg(v) - 2)n + 1 + 2|V_1(v)| & \text{if } |V_1(v)| > n/2. \end{cases}$$

Proof Given a node v, Lemma 2.2 gives

$$\begin{aligned} g(T) &\geq \sum_{i=1}^{\deg(v)} \max\{|V_i(v)|, n - |V_i(v)|\} \\ &= \sum_{i=2}^{\deg(v)} n - |V_i(v)| + \begin{cases} |V_1(v)| & \text{if } |V_1(v)| > n/2, \\ n - |V_1(v)| & \text{if } |V_1(v)| \leq n/2, \end{cases} \\ &= L(v). \end{aligned}$$

Direct computation shows that if $\deg(v) > \deg(w)$ then $L(v) > L(w)$ thus proving that the maximum is always attained at a node of maximum degree. □

2.2 Upper bounds

In this section we will determine exactly $g(G)$ for several classes of graphs, including Hamiltonian graphs and complete k-ary trees. Moreover, we will provide good upper bounds on $g(G)$ for general graphs G.

2.2.1 Hamiltonian Graphs

We first note that in any graph $G = (V, E)$ the size of a maximum matching $\mu(G)$ is at most $\lfloor |V|/2 \rfloor$. Therefore, from Lemma 2.1 we get that the gossiping time $g(G)$ of *any* graph with n nodes is always lower bounded by

$$g(G) \geq \begin{cases} n-1 & \text{if } n \text{ is even;} \\ n & \text{if } n \text{ is odd.} \end{cases} \tag{4}$$

We will show that this lower bound is attained by Hamiltonian graphs.

Let $C_n = (V, E)$ denote the cycle of length n; we assume the vertex set be $V = \{0, \ldots, n-1\}$ and the edge set be $E = \{(v, w) \ : \ 1 = |v - w| (\text{mod } n)\}$[1].

Lemma 2.3

$$g(C_n) \leq \begin{cases} n-1 & \textit{if } n \textit{ is even;} \\ n & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. We distinguish two cases according to the parity of the number n of nodes.
Case n even. We shall give a gossiping protocol on the ring C_n that requires $n-1$ rounds. First, split the edge set of C_n into two disjoint perfect matchings M_{even} and M_{odd}, where

$$\begin{aligned} M_{\text{even}} &= \{(v, w) \ : \ v \text{ is even and } w = v + 1\} \\ M_{\text{odd}} &= \{(v, w) \ : \ v \text{ is odd and } w = v + 1 (\text{mod } n)\} \end{aligned}$$

and for each integer t let

$$M_t = \begin{cases} M_{\text{even}} & \text{if } t \text{ is even,} \\ M_{\text{odd}} & \text{otherwise.} \end{cases} \tag{5}$$

The gossiping algorithm is shown in Figure 1.

It is mmediate to see that each node receives a new packet at each round (this can be formally proved by induction on t). Therefore, at the end of round $n-1$ of algorithm **Gossiping-even**(C_n) each node has received all the packets of the other $n-1$ nodes.

Case n odd. Define the following maximum matchings M_t in C_n for each $t = 0, \ldots, n-1$

$$M_t = \{(v, w) \ : \ v - t (\text{mod } n) \text{ is odd, } w = v + 1 (\text{mod } n), \text{ and } v \neq t \neq w\}. \tag{6}$$

We give in Figure 2 a gossiping protocol on the ring C_n that requires n rounds. It is easy to see that at each round $t = 1, \ldots, n$ each node different from $t-1$ receives a new packet. Therefore, at the end of round n of algorithm **Gossiping-odd**(C_n) each node has received all the packets of the other $n-1$ nodes. □

[1] Here and in the rest of the paper with $x = a$ (mod b) we denote the unique integer $x < b$ such that $x = qb + a$.

Gossiping-even(C_n)

Round $t = 1$**:** each node v sends its own packet to the node w such that $(v, w) \in M_1$;

Round $t = 2$**:** each node v sends its own packet to the node w such that $(v, w) \in M_2$;

Round t**,** $3 \leq t \leq n-1$**:** For each node v let w be the node such that $(v, w) \in M_t$,

node v sends a new packet to w, namely v sends the oldest packet it knows among those v has neither received from w nor sent to w in any previous round.

Figure 1: Gossiping Algorithm in C_n, n even.

Gossiping-odd(C_n)

Round t**,** $1 \leq t \leq n$**:** For each node $v \neq t-1$ let w be the neighbor of v such that $(v, w) \in M_{t-1}$,

node v sends to w the oldest packet it knows that v has neither received from w nor sent to w in a previous round (v own packet is considered to be older than any other packet).

Figure 2: Gossiping Algorithm in C_n, n odd.

Example 2.1 *For* $n = 6$ *we have* $M_{odd} = M_1 = M_3 = M_5 = \{(1,2),(3,4),(50)\}$ *and* $M_{even} = M_2 = M_4 = \{(0,1),(2,3),(4,5)\}$*. The sets* $I_6(v,t)$*, for* $0 \leq v \leq 5$ *and* $1 \leq t \leq 5$*, are given in the following Table.*

$t \backslash v$	0	1	2	3	4	5
1	$\{5,0\}$	$\{1,2\}$	$\{1,2\}$	$\{3,4\}$	$\{3,4\}$	$\{5,0\}$
2	$\{5,0,1\}$	$\{0,1,2\}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{3,4,5\}$	$\{4,5,0\}$
3	$\{4,5,0,1\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{2,3,4,5\}$	$\{2,3,4,5\}$	$\{4,5,0,1\}$
4	$\{4,5,0,1,2\}$	$\{5,0,1,2,3\}$	$\{0,1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{2,3,4,5,0\}$	$\{3,4,5,0,1\}$
5	$\{3,4,5,0,1,2\}$	$\{5,0,1,2,3,4\}$	$\{5,0,1,2,3,4\}$	$\{1,2,3,4,5,0\}$	$\{1,2,3,4,5,0\}$	$\{3,4,5,0,1,2\}$

For $n = 5$ *we have* $M_0 = \{(1,2),(3,4)\}$, $M_1 = \{(2,3),(4,0)\}$, $M_2 = \{(3,4),(0,1)\}$ $M_3 = \{(4,0),(1,2)\}$, *and* $M_4 = \{(0,1),(2,3)\}$*. The sets* $K_5(v,t)$*, for* $0 \leq v \leq 4$ *and* $1 \leq t \leq 5$*, are given in the following Table.*

$t \backslash v$	0	1	2	3	4
1	$\{0\}$	$\{1,2\}$	$\{1,2\}$	$\{3,4\}$	$\{3,4\}$
2	$\{4,0\}$	$\{1,2\}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{3,4,0\}$
3	$\{4,0,1\}$	$\{0,1,2\}$	$\{1,2,3\}$	$\{2,3,4,0\}$	$\{2,3,4,0\}$
4	$\{3,4,0,1\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{2,3,4,0\}$	$\{2,3,4,0,1\}$
5	$\{3,4,0,1,2\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{1,2,3,4,0\}$	$\{2,3,4,0,1\}$

Theorem 2.1 *For any Hamiltonian graph G on n vertices we have*

$$g(G) = \begin{cases} n-1 & \text{if } n \text{ is even;} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If G has n vertices the lower bound (4) tells us that $g(G) \geq \begin{cases} n-1 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases}$
If G is Hamiltonian, from Lemma 2.3 we get that gossiping along the edges of the Hamiltonian cycle gives a protocol with gossiping time matching the lower bound and the theorem holds. □

2.3 Trees

In this section we investigate the gossiping time in trees. We first give an upper bound on the gossiping time in *any* tree and afterwards we exactly compute the gossiping time of complete k–ary trees.

Given a tree $T = (V, E)$, let us denote by δ the minimum degree of an internal node and by λ the maximum number of leaves connected to a same node.

Theorem 2.2 *For any tree $T = (V, E)$ on n nodes*

$$g(T) \leq (n-\delta)\Delta + (\delta - 1)\lambda.$$

where $\Delta = \max_{v \in V} \deg(v)$.

Proof. Since $\Delta = \max_{v \in V} \deg(v)$ we can color the edges of T with Δ colors, say $0, \ldots, \Delta - 1$, so that no two edges sharing a vertex are assigned the same color. Denote by $c(u, v) = c(v, u)$ the color assigned to the edge (u, v). Moreover, give to each edge (u, f), where f is a leaf of T, a second color $c'(u, f) \in \{0, \ldots, \lambda - 1\}$ so that $c'(u, f) \neq c'(u, f')$. Consider the gossiping algorithm in the tree T given in Figure 3.

Gossiping-tree(T)

Phase 1

Round t, for $t = 1, \ldots, \Delta(n - \delta)$: For each node u, if there is an edge (u, v) such that $c(u, v) = t - 1 (\text{mod } \Delta)$ then u sends a new packet to v, namely u sends to v the oldest packet among those that u has neither sent to v nor received from v in a previous round, if such a packet exists, otherwise u sends nothing.

Phase 2

Round $\Delta(n - \delta) + \tau$, for $\tau = 1, \ldots, (\delta - 1)\lambda$: For each leaf f, if (u, f) is the edge on f and $c'(u, f) = \tau - 1 (\text{mod } \lambda)$ then u sends to f any packet f does not know.

Figure 3: Gossiping Algorithm in a tree T.

In order to prove the theorem, we shall now prove the correctness of algorithm **Gossiping-tree**(T). Let us say that the edge (u, v) of T is *saturated* from u to v at time t of **Gossiping-tree**(T) if no packet is sent from u to v at *any time* $t' \geq t$, that is, if by time $t-1$ node v has received the packet of each node w connected to v through u. We need the following property of **Gossiping-tree**(T).

Property 2.1 *In any round* $t \leq \Delta(n-\delta)$ *of gossiping algorithm* **Gossiping-tree**(T), *if the edge* (u, v) *has color* $c(u, v) = t-1 (\bmod \Delta)$ *and it is not saturated from* u *to* v *at time* t, *then* u *sends a new packet to* v *at round* t.

Proof. The proof is by induction on the time unit t. Let $t \leq \Delta$: At time unit t, for each edge (u, v) of color $c(u, v) = t-1 \in \{0, \ldots, \Delta-1\}$ node u and v exchange a call for the first time and have at least their own packet to send each other.
Let now $t > \Delta$ and suppose that the hypothesis holds for each $t' < t$.
Consider an edge (u, v) such that $c(u, v) = t-1 (\bmod \Delta)$. Suppose by contradiction that at time t the edge (u, v) is not saturated from u to v but u has no packets to send to v among those u has not received through v, that is, all packets known to u and not received through v have been already sent from u to v.

In particular, node u has already sent to v all the packets it has received from its other neighbors, call them $w_1, \ldots, w_k$. Notice that the last call from u to v has taken place at time $t-\Delta$. Let τ_i be the only integer such that both $t-d < \tau_i < t$ and $c(u, w_i) = \tau_i - 1 (\bmod \Delta)$ hold. If the edge (u, w_i) is not saturated at time τ_i, by inductive hypothesis we know that u has received a packet by w_i at time τ_i. We can have now two cases: The first case is that all the edges (u, w_i) are saturated at time $\tau_i < t$, this immediately implies that (u, v) is saturated at time t, contradicting our assumption that (u, v) not saturated from u to v at time t. The second case is that at least one edge (u, w_i) is not saturated at time τ_i; in such a situation we know by the inductive hypothesis that u has received a new packet by w_i at time τ_i that can be now forwarded to v, again getting a contradiction. □

We can now complete the proof of the theorem by showing that at the end of **Gossiping**–k–**tree**($T_{k,n}$) each node knows all the other $n-1$ packets. Above Property 2.1 shows that a new packet is sent from u to v at each round t such that $c(u, v) + 1 = t \ (\bmod \Delta)$, until the edge (u, v) is saturated and no more packets will be sent from u to v. Therefore, for any internal node u and for any Δ consecutive rounds, either u has already received all packets or it will receive a new packet from one of its neigbors during one of these Δ consecutive rounds. Moreover, it is obvious that u receives a new packet from each of its neighbors during the first $\Delta \geq \deg(v)$ initial rounds. Therefore, being $\deg(u) + (n-\delta-1) \geq n-1$, by time unit $\Delta(n-\delta)$ node u gets all the necessary $n-1$ packets.
Analogously, by round $\Delta(n-\delta)$, any leaf f knows $n-\Delta$ packets. During the $\lambda(\delta-1)$ rounds of Phase 2 any leaf f will receive a new packet during one out of λ consecutive rounds, thus getting the remaining $\delta-1$ packets that f needs to complete the gossip.

□

The following corollary is immediate.

Corollary 2.3 *For any connected graph* $G = (V, E)$ *with* n *vertices and maximum degree* Δ *we have*

$$g(G) \leq (n-1)\Delta.?????????????????? \tag{7}$$

From Corollary 2.2 and Theorem 2.2 we have that for any tree with n nodes and maximum degree Δ it holds $n\Delta - n + 1 \leq g(T) \leq n\Delta - \Delta$. Let us consider now the tree $S_{n,\Delta}$ of Figure 4. If $\Delta = n - 1$ then $S_{n,n-1}$ is the star on n nodes and from Corollary 2.2 and Theorem 2.2 we have $g(S_{n,n-1}) = (n-1)^2$. If Δ is constant with respect to $n > 2\Delta$ then from Corollary 2.2 and Theorem 2.2 we get $\Delta(n-1) - (\Delta - 1) \leq g(S_{n,\Delta}) \leq \Delta(n-1) - 1$. It is not difficult to obtain a specific gossiping algorithm attaining the lower bound. Therefore, we have that for any n and Δ there exists a graph $G_{n,\Delta}$ with n vertices and maximum degree Δ such that $g(G_{n,\Delta}) = \Omega((n-1)\Delta)$, thus showing that the bound (7) is asymptotically tight.

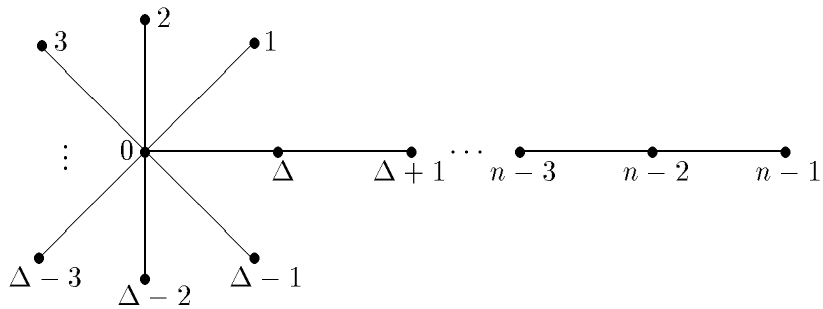

Figure 4: Tree $S_{n,\Delta}$

We shall now exactly compute the gossiping time of k-ary trees, that is, rooted trees in which each internal node has exactly k sons.

Corollary 2.4 *For any k-ary tree on n nodes $T_{k,n}$ it holds*

$$g(T_{k,n}) = (k+1)(n-1) - k.$$

Proof Let u be a node of $T_{k,n}$ whose sons are all leaves, applying Corollary 2.2 to $T_{k,n}$ we get

$$g(T_{k,n}) \geq \max_v L(v) \geq L(u) = (k-1)n + 1 + 2(n-k-1) = (k+1)(n-1) - k.$$

On the other hand from Theorem 2.2, since for any k–ary tree $\Delta = \delta = k+1$ and $\lambda = k$, we get $g(T_{k,n}) \leq (k+1)(n-1) - k$. □

The above Corollary 2.4 holds for any value of k, in particular it holds for $k = 1$, that is, in case $T_{k,n}$ is the path on n nodes P_n.

Corollary 2.5 *Let P_n be the path on n nodes, for each $n \geq 4$*

$$g(P_n) = 2n - 3.$$

2.4 Complete bipartite graphs

Let $K_{r,s} = (V(K_{r,s}), E(K_{r,s}))$ be the complete bipartite graph on the node set $V(K_{r,s}) = \{a_0, \ldots, a_{r-1}\} \cup \{b_0, \ldots, b_{s-1}\}$, with $\{a_1, \ldots, a_{r-1}\} \cap \{b_0, \ldots, b_{s-1}\} = \emptyset$, $r \geq s$, and edge set $E(K_{r,s}) = \{a_0, \ldots, a_{r-1}\} \times \{b_0, \ldots, b_{s-1}\}$. In the next theorem we determine the gossiping time of $K_{r,s}$.

Theorem 2.3 *For each r and s with $r \geq s \geq 1$*

$$g(K_{r,s}) = \lceil (r+s-1)r/s \rceil .$$

Proof. The lower bound $g(K_{r,s}) \geq \lceil (r+s-1)r/s \rceil$ is an immediate consequence of Corollary 2.1 since the complete bipartite graph has $\alpha(K_{r,s}) = r$.

In order to give a gossiping algorithm in $K_{r,s}$ requiring $\lceil (r+s-1)r/s \rceil$ communication rounds, we define the matchings

$$M_j = \{(b_i, a_{i+j \ (\mathrm{mod}\ r)}) \ : \ 0 \leq i \leq s-1\},$$

for $j = 0, \ldots, r-1$. The algorithm is shown in Figure ???.

According to the protocol, at the end of **Phase 1** of **Gossiping–bipartite**$(K_{r,s})$ each node a_i (resp. b_i) knows the message of each b_i (resp. a_i). Consider now **Phase 2**. It is immediate to see that during the first $s-1$ rounds of **Phase 2** each of the b_i's receives the packet of each b_j for $j \neq i$, thus completing its knowledge. Moreover, after the $\lceil r(r+s-1)/s \rceil - r = \lceil r(r-1)/s \rceil$ rounds of **Phase 2** each node a_i has been involved in a call at least $r-1$ times and has then received the packet of each of the a_j, for $j \neq i$, thus completing its knowledge. □

> **Gossiping–bipartite**$(K_{r,s})$
>
> **Phase 1**
>
> **round t, for $t = 1, \ldots, r$:** For each edge $(b_i, a_{i+t-1 \ (\mathrm{mod}\ r)}) \in M_{t-1}$
> nodes b_i and $a_{i+t-1 \ (\mathrm{mod}\ r)}$ exchange their own packets;
>
> **Phase 2**
>
> **round t, for $t = r+1, \ldots, \lceil r(r+s-1)/s \rceil$:** For each edge $(b_i, a_j) \in M_{(t-1-s)s \ (\mathrm{mod}\ r)}$
> node b_i sends to a_j any packet that a_j has not received in a previous round;
> if $t \leq r+s-1$ then b_i receives from a_j the packet of $b_{i+t-r (\mathrm{mod}\ s)}$.

Figure ????: Gossiping Algorithm in $K_{r,s}$.

2.5 Generalized Petersen Graphs

In Section 2.2.1 we have seen that Hamiltonian graphs have the minimum possible gossiping time among all graphs with n nodes. A natural question to ask is to see if there are non–Hamiltonian graphs on n vertices with gossiping time equal to n if n is odd and $n-1$ if n is even. A quick check shows that this is not the case for rectangular grids $G_{t,s}$ with both t and s odd [2]. In fact, we know that $\alpha(G_{t,s}) = \lceil \frac{s \cdot t}{2} \rceil$ and from Corollary 2.1 we get $g(G_{t,s}) \geq s \cdot t + 1$. Moreover, it is also easy to check that the gossiping time of the Petersen graph on 10 vertices is at least 10. Therefore, one could be tempted to conjecture that the gossiping time $g(G)$ of a graph G is equal to the minimum possible only if G is Hamiltonian. This conjecture, although nice sounding, would be wrong as the following classes of graphs, including the Generalized Petersen Graphs, shows.

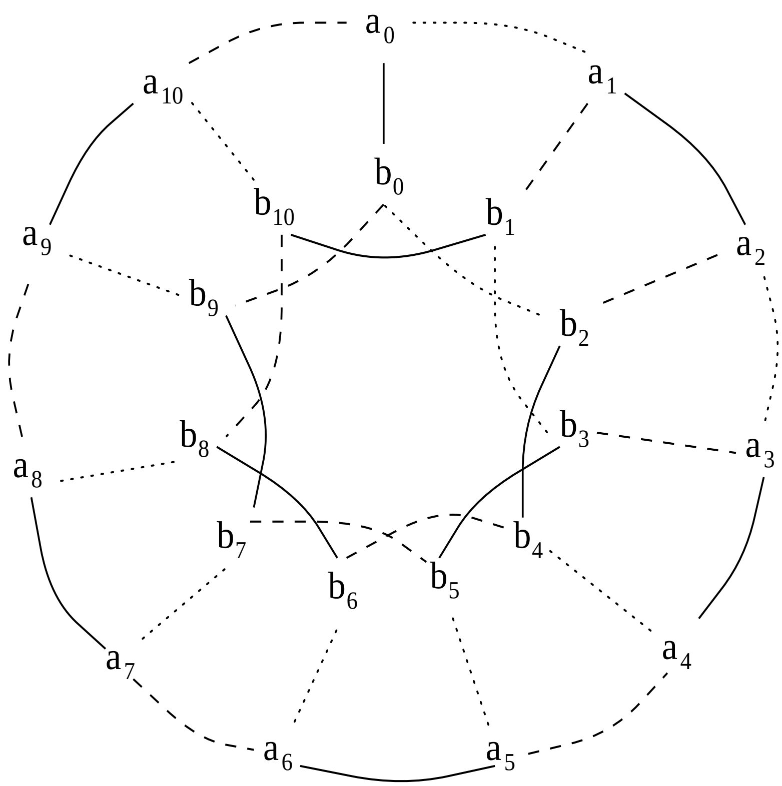

Figure ????: A 3-coloration of the GPG with $n = 11$ and $s = 2$.

Let $P_{n,\pi}$ be the graph consisting of two cycles of size n connected by a perfect matching in the following way: given a permutation π of $\{0, \ldots, n-1\}$ the graph $P_{n,\pi} = (V(P_{n,\pi}), E(P_{n,\pi}))$ has vertex set $V(P_{n,\pi}) = \{a_0, \ldots, a_{n-1}\} \cup \{b_0, \ldots, b_{n-1}\}$ and edge set

$$E(P_{n,\pi}) = \{(a_i, a_{i+1 (\text{mod } n)}) : 0 \leq i \leq n-1\} \cup \{(b_i, b_{i+1 (\text{mod } n)}) : 0 \leq i \leq n-1\} \cup \{(a_i, b_{\pi(i)}) : 0 \leq i \leq n-1\}.$$

The Petersen Graph has $n = 5$ and $\pi(i) = 3i \,(\text{mod } 5)$, for $i = 0, 1, 2, 3, 4$; Generalized Petersen Graphs (GPG) have n odd and $\pi(s \cdot i \,(\text{mod } n)) = i$, $i = 0, \ldots, n-1$, for a fixed integer s.

From Lemma 2.1 we know that $g(P_{n,\pi}) \geq 2n - 1$. We will show that for any n and π such that $P_{n,\pi}$ is 3–edge–colorable, we have the equality

$$g(P_{n,\pi}) = 2n - 1.$$

Notice that each cubic GPG, other than the Petersen graph itself, is 3–edge–colorable; this class include the family of non Hamiltonian GPGs with $n = 5 \,(\text{mod } 6)$ and $s = 2$ (see [2] and references therein quoted). Figure ????? shows a 3-coloration of the GPG with $n = 11$ and $s = 2$.

[2]It is well known that all rectangular grids $G_{t,s}$ are Hamiltonian but for values of t and s both odd.

Gossiping–3–color$(P_{n,\pi})$

Phase q $(1 \le q \le (n-1)/2)$ *[it consists of 3 communication rounds]*:

round t $(t = 1, 2, 3)$: make a call between the endpoints of each edge of color t.

Calls are made so that:

when an edge $(a_i, a_{i+1 (\mathrm{mod}\ n)})$ is used, then a_i receives the packet of $a_{i+p (\mathrm{mod}\ n)}$, and $a_{i+1 (\mathrm{mod}\ n)}$ receives the packet of $a_{i+1-p (\mathrm{mod}\ n)}$;

when an edge $(b_i, b_{i+1 (\mathrm{mod}\ n)})$ is used then b_i receives the packet of $b_{i+p (\mathrm{mod}\ n)}$, and $b_{i+1 (\mathrm{mod}\ n)}$ receives the packet of $b_{i+1-p (\mathrm{mod}\ n)}$;

when the edge $(a_i, b_{\pi(i)})$ is used then a_i receives the packet of some b_j, $0 \le j \le n-1$ and $b_{\pi(i)}$ receives the packet of some a_j, $0 \le j \le n-1$.

Phase $3(n-1)/2 + q$ $(1 \le q \le (n+1)/2)$ *[it consists of 1 communication round]*:

node a_i (resp. $b_{\pi(i)}$), for $i = 0, \ldots, n-1$, sends to $b_{\pi(i)}$ (resp. a_i) the packet of some a_j (resp. b_j) it has not already sent to it.

Figure ????: Gossiping Algorithm in $P_{n,\pi}$.

The gossiping algorithm is described in Figure ???; it assumes that the edges of the graph are colored with the three colors 1,2, and 3. It is easy to prove by induction on q that all the calls of **Phase** q, for $q \le (n-1)/2$, can actually be done. Therefore, after the first $(n-1)/2$ phases each node a_i has the packet of $a_{i \pm j (\mathrm{mod}\ n)}$ for $j = 0, \ldots, (n-1)/2$, that is, it knows the packet of each other node in its own cycle; moreover it knows the packet of $(n-1)/2$ nodes in the cycle on $\{b_0, \ldots, b_{n-1}\}$. Analogously, each b_i knows the packet of each other node in its own cycle and of $(n-1)/2$ nodes in $\{a_0, \ldots, a_{n-1}\}$.

Therefore, the calls between nodes in $\{a_0, \ldots, a_{n-1}\}$ and in $\{b_0, \ldots, b_{n-1}\}$ of the last $(n+1)/2$ communication rounds allow to complete the knowledge of each node in the graph.

3 Gossiping by exchanging more than one packet at time

In this section we shall study the minimum number of time units $g_{F_1}(p, G)$ necessary to perform gossiping in a graph G, under the restriction that at each time instant communicating nodes can exchange up to p packets, p fixed but arbitrary otherwise. Again, for ease of notation, we shall write $g(p, G)$ to denote $g_{F_1}(p, G)$.

3.1 Lower Bounds

First of all we shall present a simple lower bound on $g(p, G)$ based on elementary counting arguments. Nonetheless, we shall prove in the sequel that the obtained lower bound is tight for complete graphs with an even number of nodes and for hypercubes. In order to derive the lower bound, let us define $I(p, t)$ as the maximum number of packets a vertex can have possibly received after t communication rounds in *any* graph. Since at each round i, with $1 \leq i \leq t$, any vertex can receive at most $\min\{p, 2^{i-1}\}$ packets, it follows that

$$I(p,t) = 1 + \sum_{i=1}^{t} \min\{p, 2^{i-1}\}, \tag{8}$$

or, equivalently

$$\begin{aligned} I(p,t) &= 1 + \sum_{i=1}^{\lceil \log p \rceil} 2^{i-1} + p(t - \lceil \log p \rceil) \\ &= 2^{\lceil \log p \rceil} + p(t - \lceil \log p \rceil) \end{aligned} \tag{9}$$

for any $t \geq \lceil \log p \rceil$. Therefore, for any graph $G = (V, E)$, the gossiping time $g(p, G)$ is always lower bounded by the smallest integer $t^\star$ for which $I(p, t^\star) \geq |V|$. Since $t^\star$ is obviously greater or equal to $\lceil \log |V| \rceil \geq \lceil \log p \rceil$, we can use (9) and obtain

$$g(p,G) \geq \lceil \log p \rceil + \left\lceil \frac{1}{p}(N - 2^{\lceil \log p \rceil}) \right\rceil.$$

Moreover, notice that if the number of nodes in the graph is odd then at each round there is a node that does not receive any message. This implies that after any round t there exists a node who can have possibly received at most $I(p, t-1)$ packets. Therefore, we get

$$g(p,G) \geq \lceil \log p \rceil + \left\lceil \frac{1}{p}(N - 2^{\lceil \log p \rceil}) \right\rceil + 1.$$

The above arguments give the following lemma.

Lemma 3.1 *For any graph $G = (V, E)$, $|V| = N$, and integer p such that $2^{\lceil \log p \rceil} \leq N$ we have*

$$g(p,G) \geq \begin{cases} \lceil \log p \rceil + \left\lceil \frac{1}{p}(N - 2^{\lceil \log p \rceil}) \right\rceil & \text{if } N \text{ is even,} \\ \lceil \log p \rceil + \left\lceil \frac{1}{p}(N - 2^{\lceil \log p \rceil}) \right\rceil + 1 & \text{if } N \text{ is odd.} \end{cases}$$

Using similar arguments, we can also generalize the lower bound (1) that we extablished in Section 2.1 for $p = 1$ to general values of p.

Lemma 3.2 *Let $G = (V, E)$ be a graph with N vertices and let $\mu(G)$ be the size of a maximum matching in G. For any integer $p < N$ we have*

$$g(p,G) \geq \lceil \log p \rceil + \left\lceil \frac{1}{p} \left(\frac{N(N-1)}{2\mu(G)} - 2^{\lceil \log p \rceil} + 1 \right) \right\rceil.$$

Proof. The proof is similar to that of Lemma 2.1 but now one has to take into account that at each communication round t, $1 \leq t \leq g(p, G)$, at most $\min\{p, 2^{t-1}\}\mu(G)$ packets out of $N(N-1)$ can be exchanged in the graph. □

3.2 Rings and Paths

Let $g(\infty, G)$ denote the gossiping time of the graph G in absence of any restriction on the size of the messages. We show that $g(p, G) = g(\infty, G)$, for each $p \geq 2$, when G is either the ring C_n or the path P_n on n nodes.

Theorem 3.1 *For each $p \geq 2$ it holds* $g(p, C_n) = g(2, C_n) = \begin{cases} n/2 & \textit{if } n \textit{ is even,} \\ (n+3)/2 & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. The lower bound is immediate by noting that for any $p \geq 2$ one has $g(p, C_n) \geq g(\infty, C_n)$ and that [27]

$$g(\infty, C_n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n+3)/2 & \text{if } n \text{ is odd.} \end{cases}$$

We give now a gossiping algorithm C_n that uses $g(\infty, C_n)$ rounds and in which nodes exchange 2 packets at the time, thus showing that $g(2, C_n) = g(\infty, C_n)$. Consider the matchings M_t be as defined in (5) if n is even, and as defined in (6) if n is odd. Moreover define the sets

$$W_i = \{2i-1, 2i\}, \text{ for } i = 1, \ldots, \lceil n/2 \rceil - 1 \text{ and } W_0 = \begin{cases} \{n-1, 0\} & \text{if } n \text{ is even} \\ \{0\} & \text{if } n \text{ is odd.} \end{cases}$$

The gossiping algorithm is shown in Figure 6. □

Gossiping(p,C_n)

Round $t = 1$**:** each node v sends its packet to its neighbor in M_1, but 0 if n is odd;

[Now each node knows the packet of the other node in the same W_i]

Round t**,** $2 \leq t \leq t(n)$**:** Along each edge $(v, w) \in M_t$, v sends to w the oldest message consisting of the packets of the set W_i that v knows and that v has neither sent to w nor received from w in a previous round (if any).

Figure 6: Gossiping Algorithm in C_n, n even, and $p \geq 2$.

Let us consider now the path P_n on the n nodes $0, \ldots, n-1$.

Theorem 3.2 *For each $n \geq 2$ and $p \geq 2$ it holds* $g(p, P_n) = g(2, P_n) = 2\lceil \frac{n}{2} \rceil - 1$.

Proof. The lower bound follows from $g(p, P_n) \geq g(\infty, P_n)$ and by $g(\infty, P_n) = 2\lceil \frac{n}{2} \rceil - 1$ [27].

We give in Figure ??? a gossiping algorithm showing that $g(2, P_n) = g(P_n)$. In the algorithm we use the mathchings M_t and the sets V_i defined as

$$M_t = \begin{cases} \{(v, v+1) \ : \ v \text{ is even, } 0 \leq v \leq n-2\} & \text{if } t \text{ is odd,} \\ \{(v, v+1) \ : \ v \text{ is odd, } 1 \leq v \leq n-2\} & \text{if } t \text{ is even,} \end{cases} \quad \text{for } t = 1, \ldots, n$$

$$V_i = \begin{cases} \{2i, 2i+1\} & \text{if } 0 \leq i \leq \lfloor n/2 \rfloor - 1, \\ \{n-1\} & \text{if } n \text{ is odd and } i = \lfloor n/2 \rfloor. \end{cases} \tag{10}$$

Gossiping–path(p, P_n)

Round 1**:** Each node v sends its packet to its neighbor in M_1.

Round t**,** $2 \le t \le 2\lceil \frac{n}{2} \rceil - 1$**:** Along each edge $(v, v+1) \in M_t$

node v sends to $v+1$ a message containing the packets of the nodes in a new set V_i with $2i+1 \le v$, if any;

node $v+1$ sends to v a message containing the packets of the nodes in a new set V_i with $2i \ge v+1$, if any.

Figure 8: Gossiping Algorithm in P_n, $p \ge 2$.

3.3 Complete graphs

In this section we study the gossiping time of the complete graph K_n on n nodes. We shall denote by $\{0, 1, \ldots, n-1\}$ the vertex set of K_n. We recall that $g(\infty, K_n)$ is equal to $\lceil \log n \rceil$ if n is even, and $\lceil \log n \rceil + 1$ if n is odd.

Theorem 3.3 *For each even integer n and integer p such that $2^{\lceil \log p \rceil} \le n$ it holds*

$$g(p, K_n) = \lceil \log p \rceil + \left\lceil \frac{1}{p}(n - 2^{\lceil \log p \rceil}) \right\rceil .$$

Proof. The lower bound follows from Lemma 3.1. We give now a gossiping algorithm in K_n that uses the optimal number of rounds. For each node v, with v even and $0 \le v \le n-1$, define the sequence of nodes v_t as

$$v_t = \begin{cases} v + 2^t - 1 \pmod n & \text{if } 1 \le t \le \lceil \log p \rceil, \\ v + 2^{\lceil \log p \rceil} - 1 + (\tau - 1)p + 2\lceil \frac{p}{2} \rceil \pmod n & \text{if } t = \lceil \log p \rceil + \tau, \text{ with } \tau \ge 1 \text{ even}, \\ v + 2^{\lceil \log p \rceil} - 1 + \tau p \pmod n & \text{otherwise.} \end{cases} \quad (11)$$

Note that for each t $M_t = \{(v, v_t) \ : \ v \text{ even}, 0 \le v < n\}$ is a perfect matching between even and odd nodes.
Finally, for each integer $\tau \ge 1$, for each even node v, with $0 \le v \le n-1$, define

$$P_{\text{even}}(v, \tau) = \begin{cases} \{v + i (\bmod n) : 1 \le i \le p\} & \text{if } p \text{ and } \tau \text{ are odd}, \\ \{v + i (\bmod n) : 0 \le i \le p-1\} & \text{otherwise,} \end{cases} \quad (12)$$

and for each odd node v, with $0 \le v \le n-1$

$$P_{\text{odd}}(v, \tau) = \begin{cases} \{v - i (\bmod n) : 1 \le i \le p\} & \text{if } p \text{ and } \tau \text{ are odd}, \\ \{v - i (\bmod n) : 0 \le i \le p-1\} & \text{otherwise.} \end{cases} \quad (13)$$

Gossiping–even$(p,\, K_n)$
Phase 1
Round $t,\, 1 \le t \le \lceil \log p \rceil$: For each even node v
nodes v and v_t exchange all the packets they knows;

Phase 2
Round $t = \lceil \log p \rceil + \tau,\, 1 \le \tau \le \left\lceil \frac{1}{p}(n - 2^{\lceil \log p \rceil}) \right\rceil$: For each even node v
node v sends to v_t the packets of nodes in $P_{\text{even}}(v, \tau)$ and
node v_t sends to v the packets of nodes in $P_{\text{odd}}(v_t, \tau)$.

Figure ????

Consider the gossiping algorithm given in Figure ???? and let $I_n(v,t)$ denote the set of nodes whose packets are known by v by the end of round t. For each node v the size of $I_n(v,t)$ doubles at each round of **Phase 1** and increases of p in every round of **Phase 2**. Indeed, it is immediate to see that for each $t = 1, \ldots, \lceil \log p \rceil$

$$I_n(v,t) = \begin{cases} \{v + i(\text{mod} n) : 0 \le i \le 2^t - 1\} & \text{if } v \text{ is even,} \\ \\ \{v - i(\text{mod} n) : 0 \le i \le 2^t - 1\} & \text{if } v \text{ is odd,} \end{cases} \tag{14}$$

and for each $\tau = 1, \ldots, \left\lceil \frac{1}{p}(n - 2^{\lceil \log p \rceil}) \right\rceil$

$$I_n(v, \lceil \log p \rceil + \tau) = \begin{cases} \{v + i(\text{mod} n) : 0 \le i \le 2^{\lceil \log p \rceil} + \tau p - 1\} & \text{if } v \text{ is even,} \\ \\ \{v - i(\text{mod} n) : 0 \le i \le 2^{\lceil \log p \rceil} + \tau p - 1\} & \text{if } v \text{ is odd.} \end{cases} \tag{15}$$

Hence, at the end of round $\lceil \log p \rceil + \left\lceil \frac{1}{p}(n - 2^{\lceil \log p \rceil}) \right\rceil$ we have $I_n\left(v, \lceil \log p \rceil + \left\lceil \frac{1}{p}(n - 2^{\lceil \log p \rceil}) \right\rceil\right) = \{0, \ldots, n-1\} = V$ for each node v. □

Remark 3.1 A close look to the algorithm **Gossiping-even**(p, K_n) reveals that the calls are always made between even and odd nodes. Therefore, the same protocol works in the complete bipartite graphs $K_{r,r}$ from which we get that for any p and r

$$g(p, K_{r,r}) = g(p, K_{2r}) = \lceil \log p \rceil + \left\lceil \frac{1}{p}(2r - 2^{\lceil \log p \rceil}) \right\rceil .$$

We consider now the case of complete graphs with odd number of nodes.

Theorem 3.4 *For each odd integer N and integer p such that $2^{\lceil \log p \rceil} \le N + 1$ it holds*

$$\lceil \log p \rceil + \left\lceil \frac{N - 2^{\lceil \log p \rceil}}{p} \right\rceil + 1 \le g(p, K_N) \le \lceil \log p \rceil + \left\lceil \frac{N + 1 - 2^{\lceil \log p \rceil}}{p} \right\rceil + 2.$$

Gossiping–odd$(p,\, K_N)$
Phase 1

Round t, $1 \le t \le \lceil \log p \rceil$: For each even node v, with $v \neq N+1-2^t$, nodes v and v_t exchange all the packets they know;

Round $t = \lceil \log p \rceil + 1$: each node v with

$v \in \{3+4i \;:\; 0 \le i \le 2^{\lceil \log p \rceil -2} - 2\} \cup \{N-3-4i \;:\; 0 \le i \le 2^{\lceil \log p \rceil -2} - 1\}$ receives from $v+2$ a message containing the packets of all the nodes in $\{N - 2^{\lceil \log p \rceil -1} + 1, \ldots, N-1\}$.

Phase 2

Round $t = \lceil \log p \rceil + 1 + \tau$, $1 \le \tau \le \lceil (N+1-2^{\lceil \log p \rceil})/p \rceil$: For each even v with $v_{t-1} \neq N$ node v sends to v_{t-1} the packets of nodes in $P_{\text{even}}(v,\tau)$ and v_{t-1} sends to v the packets of nodes in $P_{\text{odd}}(v_{t-1},\tau)$.

Round $t = \lceil \log p \rceil + \lceil (N+1-2^{\lceil \log p \rceil})/p \rceil + 2$: Each node v such that $v_{t-1} = n-1$ for some $t = \lceil \log p \rceil + 1 + \tau$ with $1 \le \tau \le \lceil (N+1-2^{\lceil \log p \rceil})/p \rceil + 1$ receives from $v+1$ a message containing the packets of the nodes in $P_{\text{odd}}(N,\tau)$.

FIGURE ????

Proof. The lower bound follows from Lemma 3.1.

To prove the upper bound, we show that the algorithm **Gossiping–odd**(p, K_N) given in Figure ???? completes gossiping in K_N in $\lceil \log p \rceil + \left\lceil \frac{N+1-2^{\lceil \log p \rceil}}{p} \right\rceil + 2$ rounds. The algorithm **Gossiping–odd**(p, K_N) is described in terms of the algorithm **Gossiping–even**(p, K_N), where $n = N+1$.

Let V_t, $P_{\text{even}}(v,\tau)$, and $P_{\text{odd}}(v,\tau)$ be defined as in (11), (12), and (13), respectively. In order to show the correctness of **Gossiping–odd**(p, K_n), let us first consider **Phase 1**. At round t, for $1 \le t \le \lceil \log p \rceil$, node $N+1-2^t$ does not receive the information of the nodes in $I_n(N,t) - \{N\}$ It is easy to see that the set of nodes that *have not* the packet of *all* the nodes in $I_n(v,t)$ are the nodes in the set X_t defined by $X_1 = \emptyset$ and

$$X_t = X_{t-1} \cup \{v+2^t-1 \pmod{n} \;:\; v \in X_{t-1} \text{ even }\} \cup \{v-2^t+1 \pmod{n} \;:\; v \in X_{t-1} \text{ odd }\} \cup \{N+1-2^t\},$$

for $2 \le t \le \lceil \log p \rceil$, that gives

$$X_t = \{3+4i \;:\; 0 \le i \le 2^{t-2}-2\} \cup \{N-3-4i \;:\; 0 \le i \le 2^{t-2}-1\}, \qquad \text{for } t = 2, \ldots, \lceil \log p \rceil.$$

Moreover, each node in X_t has at least the packets of all nodes in $I(v,t) - I(N,t-1)$. Therefore, at the end of round $\lceil \log p \rceil$ each node in $X_{\lceil \log p \rceil}$ misses at most the packets of the nodes in $I(N, \lceil \log p \rceil - 1) = \{N - 2^{\lceil \log p \rceil -1} + 1, N - 2^{\lceil \log p \rceil -1} + 2, \ldots, N-1\}$ and the calls of Round $\lceil \log p \rceil + 1$ between each node $v \in X_{\lceil \log p \rceil}$ and $v+2 \notin X_{\lceil \log p \rceil}$ assure that each node knows the packets of all nodes in $I(v, \lceil \log p \rceil)$. Consider now **Phase 2**. It is immediate that at round t each node receives p new packets, but for the even node v such that $v_{t-1} = n-1$. Hence after the calls of round $\lceil \log p \rceil + \left\lceil \frac{1}{p}(n - 2^{\lceil \log p \rceil}) \right\rceil + 2$ each node knows the packet of each of the other $N-1$ nodes. □

3.4 Hypercube

In the next theorem we shall determine $g(p, G)$ for any p when the graph G is the n-dimensional hypercube H_n.

Theorem 3.5 *For each integer $p < 2^n$.*

$$g(p, H_n) = \lceil \log p \rceil + \left\lceil \frac{1}{p}(2^n - 2^{\lceil \log p \rceil}) \right\rceil .$$

Proof. The lower bound follows from Lemma 3.1. We prove now the matching upper bound. Let p be fixed. Denote by t_n the minimum integer such that $I(p, t_n) \geq 2^n$, where $I(p, t_n)$ is given in (8). We shall show that there exists a gossiping protocol that requires t_n rounds. Notice that $t_n = \lceil \log p \rceil + \left\lceil \frac{1}{p}(2^n - 2^{\lceil \log p \rceil}) \right\rceil$.

The proof is by induction on n. The assertion is trivially true for $n = 1$; suppose now that there exists a gossiping protocol in H_n that takes t_n rounds to be completed and that satisfies the additional property that after any round $t \leq t_n - 1$ each vertex knows exactly $I(p, t)$ packets. We shall exhibit a gossiping protocol in H_{n+1} that takes t_{n+1} rounds to be completed and that also satisfies the aforesaid additional property.

Case 1: $I(p, t_n) = 2^n$. This implies that in the last round of the gossiping protocol in H_n — the t_n-th — each vertex must receive exactly $\min\{p, 2^{n-1}\}$ packets. Consider now the following protocol in the $n + 1$–dimensional hypercube H_{n+1}: Split H_{n+1} into two hypercubes of dimension n according to the value of its $n + 1$-th dimension; during the first t_n rounds gossip separately in each n-dimensional subcube according to the protocol whose existence is guaranteed by the induction hypothesis. After t_n rounds each vertex has received all the information of the subcube it belongs to, i.e., according to the hypothesis of this Case each vertex has received exactly $I(p, t_n) = 2^n$ packets. Now, exchange in the successive rounds packets along dimension $n + 1$ in H_{n+1} by sending either all the 2^n packets in one round, if $p > 2^n$, or p packets per round except may be in the last one where one sends $2^n - p\lfloor 2^n/p \rfloor$ (if non zero) packets. It is clear that this protocol requires t_{n+1} rounds to be completed. Moreover, for each t, with $0 \leq t \leq \lfloor 2^n/p \rfloor$, after round $t_n + t \leq t_{n+1} - 1$ each node in H_{n+1} knows exactly $I(p, t_n) + p\,t = I(p, t_n + t)$ packets. Hence the protocol for H_{n+1} satisfies all inductive hypothesis.

Case 2: $I(p, t_n) > 2^n$. This implies that $p < 2^{n-1}$, otherwise it is easy to check that one would have $t_n = n$ and $I(p, t_n) = 1 + \sum_i 2^{i-1} = 2^n$. Consider the protocol in H_n whose existence is implied by the induction hypothesis. By inductive hypothesis at round $t_n - 1$ each vertex has received $I(p, t_n - 1)$ packets and in the last round receives α packets, with $\alpha < p$, otherwise, we would be again in Case 1.

Let $\mathcal{M} = \cup_{i=1}^{2^{n-1}} (x_i, y_i)$ be the perfect matching used in the last round, i.e., the round t_n, of the protocol on H_n and let A_i (resp. B_i) be the set of new packets that x_i (resp. y_i) receives in this last round. Note that $A_i \cap B_i = \emptyset$ and $|A_i| = |B_i| = \alpha$. For what follows, let C_i and D_i be two sets of packets such that $|C_i| = |D_i| = p - \alpha$ and $C_i \cap A_i = \emptyset$, $D_i \cap A_i = \emptyset$, $C_i \cap B_i = \emptyset$, $D_i \cap B_i = \emptyset$, and $C_i \cap D_i = \emptyset$. Such sets exist since $|A_i| + |B_i| + |C_i| + |D_i| = 2p < 2^n$. Consider now the following gossiping protocol in H_{n+1}. Split H_{n+1} according to the value of the $n + 1$-th dimension in two subcubes H_n and H'_n of dimension n; during the first $t_n - 1$ rounds gossip in H_n and H'_n separately. At the end of this phase each vertex knows $2^n - \alpha$

packets. Now, for each node x in H_n denote by x' its neighbour in H'_n. Next round exchange p packets along dimension $n+1$ in such a way x_i (resp. y_i, x'_i, y'_i) sends to x'_i (resp. y'_i, x_i, y_i) p packets including C_i (resp. D_i, C'_i, D'_i) and not D_i (resp. C_i, D'_i, C'_i).
In the next round exchange p packets along the matching $\mathcal{M}$ in such a way x_i (resp. y_i) sends to y_i (resp. x_i) all packets in $B_i \cup C'_i$ (resp. $A_i \cup D'_i$) and x'_i (resp. y'_i) sends to y'_i (resp. x'_i) all packets in $B'_i \cup C_i$ (resp. $A'_i \cup D_i$).
After the above t_n+1 rounds we are sure that each vertex x_i (resp. x'_i) knows all the packets of the subcube it belongs to and so we can finish the protocol by sending packets along dimension $n+1$ in such a way p new packets are received during each round (except possibly the last final round). Therefore, for each t, with $1 \leq t \leq 1 + \lfloor 2^n/p \rfloor$, each node in H_{n+1} after round $t_n + t - 1 \leq t_{n+1} - 1$ knows exactly $I(p, t_n - 1) + p\,t = I(p, t_n + t - 1)$ packets. Hence this protocol in H_{n+1} satisfies all the induction hypothesis. □

Remark 3.2 It is worth pointing out that the obvious inequality $g_{H_1}(p,G) \leq 2g_{F_1}(p,G)$ and above theorem allow us to improve the upper bound on $g_{H_1}(p, H_n)$ given by Theorem 4 of [5] for all values of p not power of two. Indeed, the authors of [5] have $g_{H_1}(p, H_n) \leq 2n + 2^{n+1}/p - 2/p$ while from Theorem 3.5 we get $g_{H_1}(p, H_n) \leq 2g_{F_1}(p, H_n) = 2\lceil \log p \rceil + 2\left\lceil \frac{1}{p}(2^n - 2^{\lceil \log p \rceil})\right\rceil$.

4 Conclusions and open problems

We have considered the problem of gossiping in communication networks under the restriction that communicating nodes can exchange up to a fixed number p of packets at each round. In the extremal case $p = 1$ we have exactly determined the optimal number of communication rounds to perform gossiping for several classes of graphs, including Hamiltonian graphs, paths, complete k-ary trees, complete bipartite graphs, 3–colorable generalized Petersen graphs. For arbitrary graphs we give asymptotically matching upper and lower bounds.

In the case of arbitrary p we have determined the optimal number of communication rounds to perform gossiping under this hypothesis for complete graphs, hypercubes, cycles, and paths.

Several open problems remain in the area. We list the most important of them here.

- It would be interesting to determine the computational complexity of computing $g_{F_1}(1, G)$ for general graphs, we suspect that it is NP–hard. We can ask the same question for $g_{F_1}(p, G)$ (we know that computing $g_{F_1}(\infty, G)$ is NP–hard, see [33]).

- We have left open the problem of determining the gossiping time $g_{F_1}(1, G_{t,s})$ for non hamiltonian rectangular grids $G_{t,s}$ with both t and s odd. We know that $\alpha(G_{t,s}) = \lceil \frac{s \cdot t}{2} \rceil$ and, therefore, from Corollary 2.1 we have that $g_{F_1}(1, G_{t,s}) \geq st + 1$. Does equality holds? It can be shown that $g_{F_1}(1, G_{3,3}) = 10$.

- It would be interesting to determine the exact value of $g_{F_1}(p, G)$, $p \geq 2$, for other classes of graphs like grids, complete k–ary trees, complete bipartite graphs.

References

[1] N. Alon, F.R.K. Chung, and R.L. Graham "Routing Permutations on Graphs via Matchings", *Proc. 25th ACM Symposium on the Theory of Computing*, San Diego, CA (1993), 583–591.

[2] B. Alspach, "The Classification of Hamiltonian Generalized Petersen Graphs", *J. Combin. Theory Series B*, 34 (1983), 293 - 312.

[3] B. Backer and R. Shostah, "Gossips and Telephones", *Discr. Math.*, 2 (1972), 191–193.

[4] A. Bagchi, E.F. Schmeichel, and S.L. Hakimi, "Sequential Information Dissemination by Packets", *NETWORKS*, to appear.

[5] A. Bagchi, E.F. Schmeichel, and S.L. Hakimi, "Parallel Information Dissemination by Packets", *SIAM J. on Computing*, to appear.

[6] K.A. Berman and M. Hawrycz, "Telephone Problems with Failures", *SIAM J. on Alg. Discr. Meth.*, 7 (1986), 13–17.

[7] J.–C. Bermond and P. Fraigniaud, "Broadcasting and Gossiping in de Bruijn Networks", *LIP Technical Report*, 91–04 (1991).

[8] D. P. Bertsekas, and J. N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*, Prentice–Hall, Englewood Cliffs, NJ, 1989.

[9] R.T. Bumby, "A Problem with Telephone", *SIAM J. on Alg. Discr. Meth.*, 2 (1981), 13–19.

[10] B.S. Chlebus, K. Diks, A. Pelc, "Optimal Gossiping with Short Unreliable Messages", *Disc. Appl. Math*, to appear.

[11] N. Cot, "Extensions of the Telephone Problem", *Proc. 7th SE Conf. on Comb. Graph Theory and Comp.*, Utilitas Mathematica, Winnipeg, Manitoba, Canada (1976), 239–256.

[12] K. Diks, A. Pelc, "Almost Safe Gossiping in Bounded Degree Networks", *SIAM J. Disc. Math.* 5 (1992), 338–344.

[13] K. Diks, A. Pelc, "Linear Time Gossiping with Random Faults", Université du Québec á Hull, Technical Report RR 93/02–3 (1993).

[14] S. Even and B. Monien, "On the Number of Rounds Necessary to Disseminate Information", *Proc. 1st ACM Symp. on Parallel Algorithms and Architectures*, Santa Fe, NM (1989), 318–327.

[15] R.C. Entringer and P. J. Slater, "Gossips and Telegraphs", *J. Franklin Institute*, 307 (1979), 353–360.

[16] A. M. Farley and A. Proskurowski, "Gossiping in Grid Graphs", *J. Comb. Inform. & System Sci.*, 5 (1980), 161–172.

[17] P. Fraignaud, E. Lazard, "Methods and Problems of Communication in Usual Networks", *Discr. Appl. Math.*, to appear.

[18] S. Fujita, "Gossiping in Mesh–Bus Computers by Packets with Bounded Length", *IPS Japan SIGAL*, 36-6 (1993), 41–48.

[19] L. Gargano, "Tighter Time Bounds on Fault–Tolerant Broadcasting and Gossiping", *NETWORKS*, 22 (1992), 469–486.

[20] L. Gargano and A. Rescigno, "'Communication Complexity of Fault–Tolerant Information Diffusion", in: *Proceedings of Fifth IEEE Symposium on Parallel and Distributed Processing*, (1993) 564 – 572.

[21] A. Hajnal, E.C. Milner, and E. Szemeredi, "A Cure for Telephone Diseases", *Canad. Math. Bull.*, 15 (1976), 447–450.

[22] R.W. Haddad, S. Roy, and A. A. Schäffer, "On Gossiping with Faulty Telephone Lines", *SIAM J. on Alg. Discr. Meth.*, 8 (1987), 439–445.

[23] S. M. Hedetniemi, S. T. Hedetniemi, and A. Liestman, "A Survey of Gossiping and Broadcasting in Communication Networks", *NETWORKS*, 18 (1988), 129–134.

[24] A. Hily and D. Sotteau, "Communications in Bus Networks", in: *Parallel and Distributed Computing*, M. Cosnard, A. Ferreira, and J. Peters (Eds.), Lectures Notes in Computer Science, 805, Springer – Verlag, (1994),197–206.

[25] J. Hromkovič, C,-D. Jeschke, B. Monien, "Note on Optimal Gossiping in some Weak–Connected Graphs", *Theoretical Computer Science*, to appear.

[26] J. Hromkovič, R. Klasing, W. Unger, H. Wagener, and D. Pardubská, "The Complexity of Systolic Dissemination of Information in Interconnection Networks", in:*Parallel and Distributed Computing*, M. Cosnard, A. Ferreira, and J. Peters (Eds.), Lectures Notes in Computer Science, 805, Springer – Verlag, (1994), 235–249.

[27] J. Hromkovič, R. Klasing, B. Monien, and R. Peine, "Dissemination of Information in Interconnection Networks (Broadcasting and Gossiping)", to appear in: F. Hsu, D.-Z. Du (Eds.) *Combinatorial Network Theory*, Science Press & AMS.

[28] S.L. Johnsson, C.–T. Ho, "Optimum Broadcasting and Personalized Communication in the Hypercube", *IEEE Trans. Comp.* 38(9) (1989), 1249–1268.

[29] D.J. Kleitman, J.B. Shearer, "Further Gossip Problems", *Discr. Math.*, 30 (1980), 151–156.

[30] W. Knödel, "New Gossips and Telephons", *Discr. Math.*, 13 (1975), 95.

[31] D. W. Krumme, "Representations of Gossip Schemes", in: *Parallel and Distributed Computing*, M. Cosnard, A. Ferreira, and J. Peters (Eds.), Lectures Notes in Computer Science, 805, Springer – Verlag, (1994), 251–265.

[32] D. W. Krumme, "Fast Gossiping for the Hypercube", *SIAM J. on Computing*, 21 (1992), 365–380.

[33] D. W. Krumme, K.N. Venkataraman, and G. Cybenko, "Gossiping in Minimal Time", *SIAM J. on Computing*, 21 (1992), 111–139.

[34] R. Labhan, S. T. Hedetniemi, and R. Laskar, "Periodic Gossiping on Trees", *Discr. Appl. Math.*, to appear.

[35] R. Labhan and I. Warnke, "Quick Gossipinng by Telegraphs", *Discr. Math.* 126 (1994), 421–424.

[36] A. Liestman and D. Richards, "Network Communication in Edge–Coloured Graphs: Gossiping", *IEEE Trans. Par. Distr. Computing*, 4 (1993), 438–445.

[37] A. Liestman and D. Richards, "Perpetual Gossiping", Technical report, Simon Fraser University, Vancouver, Canada, (1993).

[38] D. Richards and A. Liestman, "Generalization of Broadcasting and Gossiping", *NETWORKS* 18 (1988), 125–138.

[39] J. de Rumeur, *Communication dans les Reseaux de Processeur*, MASSON, Paris (1994).

[40] A. Seress, "Gossiping Old Ladies", *Discr. Math.*, 46 (1983), 75–81.

[41] V.S. Sunderam, P. Winkler, "Fast Information Sharing in Complete Networks", *Discr. Appl. Math.* 42 (1993), 75–86

[42] D.B. West, "A Class of Solutions to the Gossip Problem, part.1", *Discr. Math.*, 46 (1983), 307–326.

[43] O. Wolfson and A. Segall, "The Communication Complexity of Atomic Commitment and of Gossiping", *SIAM J. on Computing*, 20 (1991), 423–450.