Girth in Digraphs
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ABSTRACT

For an integer $k > 2$, the best function $m(n, k)$ is determined such that every strong digraph of order $n$ with at least $m(n, k)$ arcs contains a circuit of length $k$ or less.

INTRODUCTION

Let $D$ be a digraph (without loops or multiple arcs) with vertex set $V(D)$ and arc set $E(D)$ where $|V(D)| = n$. The girth $g(D)$ of a digraph $D$ which has at least one circuit (directed cycle) is the length of the smallest circuit in $D$. Definitions not given here can be found in [3].

A problem which has been studied is to find the minimum number $f(r, g)$ of vertices an $r$-regular digraph ($d^+(x) = d^-(x) = r$ for all $x$) with girth $g$ may possess. In particular it has been conjectured that:

**Conjecture 1.** $f(r, g) = r(g - 1) + 1$ [1].

If $h(r, g)$ is the minimum number of vertices in a digraph of girth $g$ in which every vertex has out degree at least $r$ it has been conjectured that:

**Conjecture 2.** $h(r, g) = r(g - 1) + 1$ [4].

These conjectures are respectively equivalent to the following:

**Conjecture 1'.** If $d^+(x) = d^-(x) = r$ for every vertex $x$ and $n \leq kr$ then $g(D) \leq k$ [1].

**Conjecture 2'.** If $d^+(x) \geq r$ for every vertex $x$ and $n \leq kr$ then $g(D) \leq k$ [4].

(Note that Conjecture 2 implies Conjecture 1.)

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Only particular cases of these conjectures are solved: essentially the cases \( r = 2, 3 \), any \( k \), for the Conjecture 1, plus some couples of values \((r, k)\) (see [2, 4]); the case \( r = 2 \) any \( k \) for Conjecture 2 (see [5]).

Thomassen [6] asked the problem of finding the best function \( m(n, k) \) such that a strong digraph of order \( n \), with at least \( m(n, k) \) arcs satisfies \( g(D) \leq k \). We solve this problem by showing

**Theorem.** Let \( D \) be a strong digraph of order \( n \), and let \( k \geq 2 \). Then

\[
|E(D)| \leq \frac{n^2 + (3 - 2k)n + k^2 - k}{2},
\]

implies that \( g(D) \leq k \).

**Remark.** We will also show that the theorem is the best possible.

**Proof.** (1) Since \( D \) is strong \( g(D) \) exists and \( 2 \leq g(D) \leq n \). Thus the theorem is true for \( n = k \). The theorem is also true for \( k = 2 \), because

\[
|E(D)| \geq \frac{n^2 - n + 2}{2} = \frac{n(n-1)}{2} + 1.
\]

(2) Thus we will suppose \( k \geq 3 \) in what follows and prove the theorem by induction on \( |V(D)| = n \). The theorem is true for \( n = k \); we suppose that it is true for every \( n' \) with \( k \leq n' \leq n - 1 \) and let \( D \) be a strong digraph with \( n \) vertices, where then \( n \geq k + 1 \).

Exactly we will prove that if \( g(D) \geq k + 1 \) then

\[
|E(D)| \leq \frac{n^2 + (3 - 2k)n + k^2 - k}{2} - 1 = \varphi(n, k).
\]

Furthermore since \( k + 1 \geq 3 \), \( D \) is antisymmetric: let

\[
\tilde{E}(D) = \{(x, y), x, y \in V(D), (x, y) \notin E(D) \text{ and } (y, x) \notin E(D)\}
\]

then

\[
|\tilde{E}(D)| = \frac{n(n-1)}{2} - |E(D)|.
\]

Thus it suffices to prove that if \( g(D) \geq k + 1 \), then

\[
|\tilde{E}(D)| \leq (k - 2)n - \frac{(k + 1)(k - 2)}{2}.
\]

(3) If \( g(D) = n \), then \( D \) is a circuit of length \( n \) and \( |E(D)| = n \). The condition \( n \leq \varphi(n, k) \) reduces to \((n - k + 2)(n - k - 1) \geq 0\), which is true for every \( n \geq k + 1 \).

Thus we will suppose that \( k + 1 \leq g(D) \leq n - 1 \).

(4) Let \( D' \) be a strong proper subdigraph of \( D \) of maximum order. A subdigraph is called proper if it has strictly less than \( n \) vertices. Strong
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proper subdigraphs of $D$ exist: for instance the subdigraph generated by
the vertices of a smallest circuit of $D$.

Let $p = |V(D')|$ then $k + 1 \leq p \leq n - 1$. Since $g(D') \geq g(D) \equiv k + 1$ we
have by induction hypothesis that

$$|\bar{E}(D')| \geq (k-2)p - \frac{(k+1)(k-2)}{2}.$$ 

To prove the theorem it suffices thus to prove that

$$|\bar{E}(D)| - |\bar{E}(D')| \geq (k-2)(n-p).$$

(5) Suppose that $|V(D')| = p = n - 1$. Let $V(D) = V(D') \cup \{a\}a \notin D'$. As
$D$ is strong and $g(D) \geq k + 1$, there exists a circuit of length at least $k + 1$
containing $a$. Let $C_i = (a, y_1, y_2, \ldots, y_t, a)$ be such a circuit of minimum
length $l + 1$ with $l \geq k$. As $C_i$ is of minimum length for $2 \leq i \leq l - 1$
$\{a, y_i\} \in \bar{E}(D) - \bar{E}(D')$ and thus

$$|\bar{E}(D)| - |\bar{E}(D')| \geq l - 2 \geq k - 2.$$ 

(6) Thus we can assume that $|V(D')| = p \leq n - 2$. Let $P$ be a directed
path having only its end vertices in common with $D'$ and containing a
vertex of $D - D'$ (such a path exists as $D$ is strong and antisymmetric).
Every other vertex of $D - D'$ belongs to $P$ otherwise $D' \cup P$ would be a
larger strong proper subdigraph of $D$.

Thus let us write $P = (a', x_1, \ldots, x_i, \ldots, x_{n-p}, b')$, where $a' \in D'$, $b' \in D'$,
$x_i \in D - D'$ (with eventually $a' = b'$).

(7) Suppose, now that $p \leq n - 3$. (i) As $D'$ is of maximum cardinality
$\{x_i, u\} \in \bar{E}(D)$ for $2 \leq i \leq n - p - 1$ and $u \in V(D')$. We have thus exhibited
$p(n-p-2)$ pairs of vertices of $\bar{E}(D) - \bar{E}(D')$. (ii) Furthermore let $C = (x_1, \ldots, x_{n-p}, b' = x_{n-p+1}, \ldots, x_{l-1}, a' = x_{l}, x_l)$ be a circuit of length $l$
containing $P$; thus $l \geq k + 1$. For $2 \leq i \leq k$ there is no arc $(x_i, x_i)$ in $D$,
otherwise $(x_1, \ldots, x_l, x_1)$ would be a circuit of length at most $k$, and
$k < g(D)$. For $3 \leq i \leq l$, there is no arc $(x_i, x_i)$, otherwise $D'$ would not be
of maximum order. Thus for $3 \leq i \leq k$, $(x_i, x_i) \in \bar{E}(D) - \bar{E}(D')$. Similarly
$(x_{n-p}, x_l) \in \bar{E}(D) - \bar{E}(D')$ for $n - p - k + 1 \leq i \leq n - p - 2$, where the indices
are to be taken modulo $l$. Thus we have exhibited $2(k-2)$ pairs of vertices of $\bar{E}(D) - \bar{E}(D')$ which are distinct, except possibly we have
counted $(x_1, x_{n-p})$ twice, and which are distinct from the $p(n-p-2)$ pairs
exhibited in (i). In summary

$$|\bar{E}(D)| - |\bar{E}(D')| \geq p(n-p-2) + 2(k-2) - 1 = (k-2)(n-p)$$
$$+ (p-k+2)(n-p-2) - 1$$
$$\geq (k-2)(n-p) + 3(n-p-2) - 1 > (k-2)(n-p),$$

and thus the theorem is proved for $p \leq n - 3$ (with a strict inequality).
(8) The only remaining case is $p = n - 2$. As we have seen in (6) $D$ consists of $D'$ plus an arc $(a, b)$ with at least one arc from $D'$ to $a$ and one arc from $b$ to $D'$. Let us consider now a circuit of minimum length containing the arc $(a, b): C = (a, b, y_1, y_2, \ldots, y_{l-1}, a)$ of length $l \geq k + 1$.

The arc $(a, y_i) \not\in E(D)$ otherwise $D'$ will not be of minimum order and the arc $(y_i, a) \not\in E(D)$ for $1 \leq i \leq l-3$ otherwise the circuit $C$ will not be of minimum length. Thus for $1 \leq i \leq l-3$ \{$(a, y_i) \in \bar{E}(D) - \bar{E}(D')$\}. Similarly for $2 \leq i \leq l-2$ \{$(b, y_i) \in \bar{E}(D) - \bar{E}(D')$\}. Thus

$$|\bar{E}(D)| - |\bar{E}(D')| \geq 2(l-3) = 2(k-2) + 2(l-k-1).$$

Thus the theorem is also true for $p = n - 2$. In order to characterize the extremal graphs we will show that in the case $p = n - 2$ we have in fact $|\bar{E}(D)| - |\bar{E}(D')| > 2(k-2)$. That is the case if $l > k + 1$. If $l = k + 1$, then since $n - 1 \geq k + 1$ there exists a vertex $z$ not on the circuit $C$. Neither the arcs $(a, z)$ nor $(z, b)$ belongs to $E(D)$ otherwise $D'$ will not be of maximum order. Furthermore at least one of the arcs $(z, a)$ and $(b, z)$ does not belong to $E(D)$ otherwise we will have a circuit of length 3 contradicting $g(D) \geq k + 1 \geq 4$. So we have one more pair of vertices in $\bar{E}(D) - \bar{E}(D')$ and thus the inequality is strict.

Remark. The theorem is the best possible in the sense that there exist strong digraphs of order $n$, which have $\varphi(n, k)$ arcs and girth $k + 1$. For example let $D$ be the digraph consisting of a Hamiltonian circuit $(x_1, x_2, \ldots, x_n, x_1)$ plus the arcs $(x_i, x_j)$, where $k - 1 \leq i < j - 1 \leq n - 1$ except the arc $(x_{n-1}, x_n)$. For $n = k + 1$, this digraph reduces to a circuit of length $k + 1$ and is the only strong digraph of order $k + 1$, having $\varphi(k + 1, k) = k + 1$ arcs and girth $k + 1$. However for $n > k + 1$, the digraph $D$ above is not the unique digraph of order $n$ with $\varphi(n, k)$ arcs and girth $k + 1$. Such digraphs can be constructed recursively from the circuit of length $k + 1$. Indeed the proof of the theorem, in particular the fact that the equality $|\bar{E}(D)| - |\bar{E}(D')| = (k-2)(n-p)$ occurs only for $n = p + 1$ shows that a strong digraph of order $n$, with $\varphi(n, k)$ arcs and girth $k + 1$ is obtained from a strong digraph $D'$ of order $n - 1$, with $\varphi(n - 1, k)$ arcs and girth $k + 1$ by adding a vertex of degree $n - k + 1$ in such a way that no circuit of length less than or equal to $k$ is created.

Note added in proof: Other partial results on Conjectures 1 and 2 have been obtained by Y. O. Hamidoune, A note on the girth of digraphs, to appear.

References


