Girth in Digraphs

J. C. Bermond
A. Germa
M. C. Heydemann
D. Sotteau
UNIVERSITE DE PARIS SUD[1]

ABSTRACT

For an integer $k>2$, the best function $m(n, k)$ is determined such that every strong digraph of order n with at least $m(n, k)$ arcs contains a circuit of length k or less.

INTRODUCTION

Let D be a digraph (without loops or multiple arcs) with vertex set $V(D)$ and arc set $E(D)$ where $|V(D)|=n$. The girth $g(D)$ of a digraph D which has at least one circuit (directed cycle) is the length of the smallest circuit in D. Definitions not given here can be found in [3].

A problem which has been studied is to find the minimum number $f(r, g)$ of vertices an r-regular digraph ($d^+(x)=d^-(x)=r$ for all x) with girth g may possess. In particular it has been conjectured that:

Conjecture 1. $f(r, g)=r(g-1)+1$ [1].
If $h(r, g)$ is the minimum number of vertices in a digraph of girth g in which every vertex has out degree at least r it has been conjectured that:

Conjecture 2. $h(r, g)=r(g-1)+1$ [4].

These conjectures are respectively equivalent to the following:

Conjecture 1′. If $d^+(x)=d^-(x)=r$ for every vertex x and $n\le kr$ then $g(D)\le k$ [1].

Conjecture 2′. If $d^+(x)\ge r$ for every vertex x and $n\le kr$ then $g(D)\le k$ [4].

(Note that Conjecture 2 implies Conjecture 1.)

[1] 91405—ORSAY, France, Informatique bât, 490.

Journal of Graph Theory, Vol. 4 (1980) 337–341
 0364–9024/80/0004–0337$01.00

Only particular cases of these conjectures are solved: essentially the cases $r=2$, 3, any k, for the Conjecture 1, plus some couples of values (r, k) (see [2, 4]); the case $r=2$ any k for Conjecture 2 (see [5]).

Thomassen [6] asked the problem of finding the best function $m(n, k)$ such that a strong digraph of order n, with at least $m(n, k)$ arcs satisfies $g(D) \leq k$. We solve this problem by showing

Theorem. Let D be a strong digraph of order n, and let $k \geq 2$. Then

$$|E(D)| \geq \frac{n^2+(3-2k)n+k^2-k}{2},$$

implies that $g(D) \leq k$.

REMARK. We will also show that the theorem is the best possible.

Proof. (1) Since D is strong $g(D)$ exists and $2 \leq g(D) \leq n$. Thus the *theorem is true for* $n=k$. The theorem is also *true for* $k=2$, because

$$|E(D)| \geq \frac{n^2-n+2}{2} = \frac{n(n-1)}{2}+1.$$

(2) Thus we will suppose $k \geq 3$ in what follows and prove the theorem *by induction* on $|V(D)|=n$. The theorem is true for $n=k$; we suppose that it is true for every n' with $k \leq n' \leq n-1$ and let D be a strong digraph with n vertices, where then $n \geq k+1$.

Exactly we will prove that if $g(D) \geq k+1$ then

$$|E(D)| \leq \frac{n^2+(3-2k)n+k^2-k}{2} - 1 = \varphi(n, k).$$

Furthermore since $k+1 \geq 3$, D is antisymmetric: let

$$\tilde{E}(D) = \{\{x, y\}, x, y \in V(D), (x, y) \notin E(D) \quad \text{and} \quad (y, x) \notin E(D)\}$$

then

$$|\tilde{E}(D)| = \frac{n(n-1)}{2} - |E(D)|.$$

Thus it suffices to *prove that if* $g(D) \geq k+1$, then

$$|\tilde{E}(D)| \geq (k-2)n - \frac{(k+1)(k-2)}{2}.$$

(3) If $g(D)=n$, then D is a circuit of length n and $|E(D)|=n$. The condition $n \leq \varphi(n, k)$ reduces to $(n-k+2)(n-k-1) \geq 0$, which is true for every $n \geq k+1$.

Thus we will suppose that $k+1 \leq g(D) \leq n-1$.

(4) *Let D' be a strong proper subdigraph of D of maximum order.* A subdigraph is called proper if it has strictly less than n vertices. Strong

proper subdigraphs of D exist: for instance the subdigraph generated by the vertices of a smallest circuit of D.

Let $p=|V(D')|$ then $k+1 \le p \le n-1$. Since $g(D') \ge g(D) \ge k+1$ we have by induction hypothesis that

$$|\tilde{E}(D')| \ge (k-2)p - \frac{(k+1)(k-2)}{2}.$$

To prove the theorem it suffices thus to prove that

$$|\tilde{E}(D)| - |\tilde{E}(D')| \ge (k-2)(n-p).$$

(5) *Suppose that* $|V(D')| = p = n-1$. Let $V(D) = V(D') \cup \{a\} a \notin D'$. As D is strong and $g(D) \ge k+1$, there exists a circuit of length at least $k+1$ containing a. Let $C_l = (a, y_1, y_2, \ldots, y_l, a)$ be such a circuit of minimum length $l+1$ with $l \ge k$. As C_l is of minimum length for $2 \le i \le l-1$ $\{a, y_i\} \in \tilde{E}(D) - \tilde{E}(D')$ and thus

$$|\tilde{E}(D)| - |\tilde{E}(D')| \ge l-2 \ge k-2.$$

(6) *Thus we can assume that* $|V(D')| = p \le n-2$. Let P be a directed path having only its end vertices in common with D' and containing a vertex of $D-D'$ (such a path exists as D is strong and antisymmetric). *Every other vertex of* $D-D'$ *belongs to* P otherwise $D' \cup P$ would be a larger strong proper subdigraph of D.

Thus let us write $P=(a', x_1, \ldots, x_i, \ldots, x_{n-p}, b')$, where $a' \in D'$, $b' \in D'$, $x_i \in D-D'$ (with eventually $a'=b'$).

(7) *Suppose, now that* $p \le n-3$. (i) As D' is of maximum cardinality $\{x_i, u\} \in \tilde{E}(D)$ for $2 \le i \le n-p-1$ and $u \in V(D')$. We have thus exhibited $p(n-p-2)$ pairs of vertices of $\tilde{E}(D)-\tilde{E}(D')$. (ii) Furthermore let $C = (x_1, \ldots, x_{n-p}, b' = x_{n-p+1}, \ldots, x_{l-1}, a' = x_l, x_1)$ be a circuit of length l containing P; thus $l \ge k+1$. For $2 \le i \le k$ there is no arc (x_i, x_1) in D, otherwise $(x_1, \ldots, x_i, x_1)$ would be a circuit of length at most k, and $k < g(D)$. For $3 \le i \le l$, there is no arc (x_1, x_i), otherwise D' would not be of maximum order. Thus for $3 \le i \le k$, $\{x_i, x_1\} \in \tilde{E}(D) - \tilde{E}(D')$. Similarly $\{x_{n-p}, x_i\} \in \tilde{E}(D) - \tilde{E}(D')$ for $n-p-k+1 \le i \le n-p-2$, where the indices are to be taken modulo l. Thus we have exhibited $2(k-2)$ pairs of vertices of $\tilde{E}(D)-\tilde{E}(D')$ which are distinct, except possibly we have counted $\{x_1, x_{n-p}\}$ twice, and which are distinct from the $p(n-p-2)$ pairs exhibited in (i). In summary

$$\begin{aligned} |\tilde{E}(D)| - |\tilde{E}(D')| &\ge p(n-p-2) + 2(k-2) - 1 = (k-2)(n-p) \\ &\quad + (p-k+2)(n-p-2) - 1 \\ &\ge (k-2)(n-p) + 3(n-p-2) - 1 > (k-2)(n-p), \end{aligned}$$

and thus the theorem is proved for $p \le n-3$ (with a strict inequality).

(8) *The only remaining case is* $p=n-2$. As we have seen in (6) D consists of D' plus an arc (a, b) with at least an arc from D' to a and one arc from b to D'. Let us consider now a circuit of minimum length containing the arc $(a, b): C=(a, b, y_1, y_2, \ldots, y_{l-2}a)$ of length $l \geq k+1$.

The arc $(a, y_i) \notin E(D)$ otherwise D' will not be of minimum order and the arc $(y_i, a) \notin E(D)$ for $1 \leq i \leq l-3$ otherwise the circuit C will not be of minimum length. Thus for $1 \leq i \leq l-3$ $\{a, y_i\} \in \tilde{E}(D)-\tilde{E}(D')$. Similarly for $2 \leq i \leq l-2$ $\{b, y_i\} \in \tilde{E}(D)-\tilde{E}(D')$. Thus

$$|\tilde{E}(D)|-|\tilde{E}(D')| \geq 2(l-3)=2(k-2)+2(l-k-1).$$

Thus the theorem is also true for $p=n-2$. In order to characterize the extremal graphs we will show that in the case $p=n-2$ we have in fact $|\tilde{E}(D)|-|(\tilde{E}(D')|>2(k-2)$. That is the case if $l>k+1$. If $l=k+1$, then since $n-1 \geq k+1$ there exists a vertex z not on the circuit C. Neither the arcs (a, z) nor (z, b) belongs to $E(D)$ otherwise D' will not be of maximum order. Furthermore at least one of the arcs (z, a) and (b, z) does not belong to $E(D)$ otherwise we will have a circuit of length 3 contradicting $g(D) \geq k+1 \geq 4$. So we have one more pair of vertices in $\tilde{E}(D)-\tilde{E}(D')$ and thus the inequality is strict. ∎

REMARK. The theorem is the best possible in the sense that there exist strong digraphs of order n, which have $\varphi(n, k)$ arcs and girth $k+1$. For example let D be the digraph consisting of a Hamiltonian circuit $(x_1, x_2, \ldots, x_n, x_1)$ plus the arcs (x_i, x_j), where $k-1 \leq i<j-1 \leq n-1$ except the arc (x_{k-1}, x_n). For $n=k+1$, this digraph reduces to a circuit of length $k+1$ and is the only strong digraph of order $k+1$, having $\varphi(k+1, k)=k+1$ arcs and girth $k+1$. However for $n>k+1$, the digraph D above is not the unique digraph of order n with $\varphi(n, k)$ arcs and girth $k+1$. Such digraphs can be constructed recursively from the circuit of length $k+1$. Indeed the proof of the theorem, in particular the fact that the equality $|\tilde{E}(D)|-|\tilde{E}(D')|=(k-2)(n-p)$ occurs only for $n=p+1$ shows that a strong digraph of order n, with $\varphi(n, k)$ arcs and girth $k+1$ is obtained from a strong digraph D' of order $n-1$, with $\varphi(n-1, k)$ arcs and girth $k+1$ by adding a vertex of degree $n-k+1$ in such a way that no circuit of length less than or equal to k is created.

Note added in proof: Other partial results on Conjectures 1 and 2 have been obtained by Y. O. Hamidoune, A note on the girth of digraphs, to appear.

References

[1] M. Behzad, G. Chartrand, and C. E. Wall, On minimal regular digraphs with given girth. *Fund. Math.* 69 (1970) 227–231.

[2] M. Behzad, Minimal 2-regular digraphs with given girth. *J. Math. Soc. Japan* 25 (1973) 1–6.

[3] C. Berge, *Graphs and hypergraphs.* North Holland, Amsterdam (1973).

[4] J. C. Bermond, 1-graphes réguliers minimaux de girth donné. *Cahiers du C.E.R.O.*, Bruxelles (1975) 123–135.

[5] L. Caccetta and R. Häggkvist., On minimal digraphs with given girth. *Congressus Numerantium XXI.* Edited by F. Hoffman, D. McCarty, R. C. Mullin, and R. G. Stanton. Utilitas Mathematica, Boca Raton (1978) 181–187.

[6] C. Thomassen, private communication.