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HAMILTON CYCLE DECOMPOSITION OF THE BUTTERFLY NETWORK*

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ABSTRACT

In this paper, we prove that the wrapped Butterfly graph $\mathcal{WBF}(d,n)$ of degree d and dimension n is decomposable into Hamilton cycles. This answers a conjecture of Barth and Raspaud who solved the case $d = 2$.

Keywords: Butterfly graph, graph theory, Hamilton decomposition, Hamilton cycle, Hamilton circuit, perfect matching.

1. Introduction and notations

The construction of one, and if possible many edge-disjoint Hamilton cycles in a network can provide advantage for algorithms that make use of a ring structure. As example, the existence of many edge-disjoint Hamilton cycles allows the message traffic to be evenly distributed across the network. Furthermore, a partition of the edges into Hamilton cycles can be used in various distributed algorithms (termination, garbage collector, ...). So, many authors have considered the problem of finding how many edge-disjoint Hamilton cycles can be found in a given network. The most significant results have been obtained for the class of Cayley graphs on abelian groups, and for (underlying) line digraphs. Here we solve this problem for the *Butterfly networks*. These networks have been proposed as suitable topologies for parallel computers, due to their interesting structure (see [11,13]) because they are, when properly defined, both Cayley digraphs (on a non-abelian group) and iterated line digraphs.

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1.1. Definitions

First, we have to warn the reader that under the name *Butterfly* and with the same notation, different networks are described in the literature. Indeed, while some authors consider the *Butterfly networks* to be multistage networks used to route permutations, others consider them to be point-to-point networks. In what follows, we will study the point-to-point version, and use Leighton's terminology [11], namely, *wrapped Butterfly*. Also, when we use the terms edge-disjoint or arc-disjoint, it obviously means *pairwise* edge-disjoint or arc-disjoint.

In this article, we will use the following definitions and notation, where $\mathbb{Z}_q$ denotes the set of integers modulo q - For definitions not given here see [13].

Definition 1 *The* **wrapped Butterfly digraph** *of degree d and dimension n, denoted $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$, has as vertices the ordered pairs (x,l) where x is an element of $\mathbb{Z}_d^n$, that is, a word $x_{n-1}x_{n-2}\cdots x_1x_0$ where the letters belong to $\mathbb{Z}_d$, and $l \in \mathbb{Z}_n$ (l is called the level). For any l, a vertex $(x_{n-1}x_{n-2}\cdots x_l \cdots x_1x_0, l)$ is joined by an arc to the d vertices $(x_{n-1}\cdots x_{l+1}\ x_l+\alpha\ x_{l-1}\cdots x_0,\ l+1)$ where α is any element of $\mathbb{Z}_d$. Each one of these arcs is said to have the* **slope** *α.*

$\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is a d-regular digraph with nd^n vertices; its diameter is $2n-1$. This network is sometimes considered as undirected, but its structure being indeed directed, we will always consider the digraph.

For convenience, we repeat the level 0 when drawing the wrapped Butterfly digraph. Hence, the reader has to remember that the two occurrences of level 0 have to be identified. Figure (**1**) displays $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,2)$ with the arcs directed from left to right. Note that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is often represented (for example in [11,13]) in an opposite way to our drawing as the authors denote the nodes $(x_0x_1\cdots x_{n-1}, l)$.

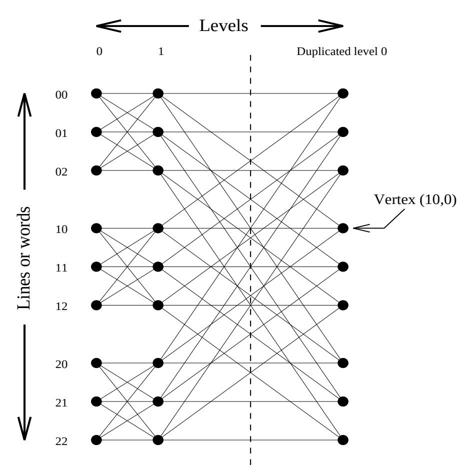

Figure 1: *The digraph $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,2)$, the arcs being directed from left to right.*

Now, we define two other useful digraphs.

- $\mathcal{K}_d^+$ denotes the complete symmetric digraph with a loop on each vertex,
- $\vec{\mathcal{K}}_{d,d}$ denotes the complete bipartite digraph where each set of the bipartition has size d and with all the arcs directed from one part of the bipartition, called *left part*, to the other, called *right part*.

Note that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,1)$ is nothing else than $\mathcal{K}_d^+$.

In digraphs, the concept of dipaths and circuits (directed cycles) is well-known. Here, we need to use more general concepts valid for digraphs of paths and cycles (which are also called oriented elementary paths and oriented elementary cycles).

Definition 2 *A* **path** *of a digraph is a sequence $\mu = (v_0, e_0, v_1, e_1, \cdots, v_k, e_k, v_{k+1})$ where the v_i's are vertices and the e_i's are arcs such that the end vertices of e_i are v_i and v_{i+1} and where the sequence μ does not meet twice the same vertex except maybe v_0 and v_{k+1}.*

Definition 3 *A path such that $v_{k+1} = v_0$ in the sequence μ is called a* **cycle***.*

Note that the arc e_i can be either directed from v_i to v_{i+1} or from v_{i+1} to v_i. If all the arcs of the path (*resp. cycle*) are directed from v_i to v_{i+1} we have a dipath (*resp. circuit also called dicycle*).

Definition 4 *A vertex v_i of a cycle is said to be* **of type** $+$ (resp. **of type** $-$) *for the cycle, if v_i is the terminal vertex of e_{i-1}* (resp. e_i) *and the initial vertex of e_i* (resp. e_{i-1}).

Note that the type is not necessarily defined for all the vertices of a cycle. In a circuit, all vertices are of type $+$.

Definition 5 *A vertex v is said* **to be crossed by a cycle***, or* **a cycle crosses the vertex** *v, if v is of type $+$ or of type $-$ for the cycle. When a vertex v is crossed by a cycle, we will define its* **sign function** *ϵ by $\epsilon(v) = +1$* (resp. $\epsilon(v) = -1$) *if v is of type $+$* (resp. of type $-$).

Remark 1 We can also define the predecessor $p(v)$ and the successor $s(v)$ of the vertex v in the order induced by the cycle. Then, the vertex v is of type $+$ (or has sign $\epsilon(v) = +1$) if $(p(v), v)$ and $(v, s(v))$ are both arcs of the digraph, and is of type $-$ (or has sign $\epsilon(v) = -1$) if both $(s(v), v)$ and $(v, p(v))$ are arcs of the digraph.

Definition 6 *A* **Hamilton cycle** (resp. **circuit**) *of a digraph is a cycle* (resp. circuit) *which contains every vertex exactly once.*

Definition 7 *We say that a digraph is* **decomposable into Hamilton cycles** (resp. **circuits**) *if its arcs can be partitioned into Hamilton cycles* (resp. circuits).

Definition 8 *A Hamilton cycle of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is said to be l*-**crossing** *if the cycle crosses all the vertices of level l and furthermore $\sum_{v=(x,l),\, x\in\mathbb{Z}_d^n} \epsilon(v) \equiv 0 \pmod{d}$.*

Figure (**3**) shows examples of 1-crossing Hamilton cycles in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,2)$ and $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,3)$. Note that a Hamilton circuit is l-crossing for all l.

1.2. Results

Various results have been obtained on the existence of Hamilton cycles in classical networks (see for example the surveys [2,9]). For example, it is well-known that any Cayley graph on an abelian group is Hamiltonian. Furthermore, it has been conjectured by Alspach [1] that:

Conjecture 1 (Alspach) *Every connected Cayley graph on an abelian group has a Hamilton decomposition.*

This conjecture has been verified for all connected 4-regular graphs on abelian groups in [8]. This includes in particular the toroidal meshes (grids). It is also known that $\mathcal{H}(2d)$, the hypercube of dimension $2d$, is decomposable into d Hamilton cycles (see [2,3]).

Concerning line digraphs, it has been shown in [10] that d-regular line digraphs always admit $\lfloor \frac{d}{2} \rfloor$ Hamilton circuits. In the case of de Bruijn and Kautz digraphs which are the simplest line digraphs, partial results have been obtained successively in [12] and [5] respectively, and near optimal results have been obtained for undirected de Bruijn and Kautz graphs [4].

The wrapped Butterfly digraph is actually a Cayley graph (on a non-abelian group) and a line digraph. So, the decomposition into Hamilton cycles (*resp. circuits*) of this digraph has received some attention. It is well-known that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ has one Hamilton circuit (see [11, page 465] for a proof in the case $d=2$ or [15]). In [6], Barth and Raspaud proved that the underlying multigraph associated with $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(2,n)$ contains two arc-disjoint Hamilton cycles answering a conjecture of Rowley and Sotteau (private communication). In our terminology, their result can be stated as:

Theorem 1 (Barth, Raspaud) *$\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(2, n)$ is decomposable into 2 Hamilton cycles.*

They conjectured that this result can be generalized for any degree:

Conjecture 2 (Barth, Raspaud) *For $n \geq 2$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d, n)$ is decomposable into d Hamilton cycles.*

In this paper, we prove the conjecture (**2**). To do so, we use some techniques introduced in [7] where we studied the decomposition of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d, n)$ into Hamilton circuits. In fact, we prove that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d, n)$ is decomposable into d l-crossing Hamilton cycles. Indeed, the l-crossing property, combined with the recursive structure of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d, n)$, enables us to prove that the number of l-crossing arc-disjoint Hamilton cycles that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d, n)$ contains can only increase when n increases. Then, we prove mainly that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d, 2)$ contains d arc-disjoint l-crossing Hamilton cycles, by constructing two arc-disjoint l-crossing Hamilton cycles using only arcs of slopes 0 and 1 and $d-2$ arc-disjoint Hamilton circuits using arcs of other slopes. The results are summarized in the following theorem:

Theorem 2 *For $n \geq 2$,*

- *for $d \notin \{3, 4, 6\}$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d, n)$ is decomposable into $d-2$ Hamilton circuits and 2 Hamilton cycles,*
- *for $d \in \{4, 6\}$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d, n)$ is decomposable into d Hamilton circuits,*
- *$\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3, n)$ is decomposable into 1 Hamilton circuit and 2 Hamilton cycles.*

2. The general construction

We give below some additional definitions and properties enabling us to establish lemma (**2**) which is a strengthened version of the inductive lemma of [7]. This lemma is then applied in section 3 to construct inductively the decomposition.

2.1. Cyclic-potent families of permutations

In this paper, M will always denote a permutation of $\mathbb{Z}_d$ which associates the element a with the element $M(a)$. To such a permutation, one can associate a perfect matching (denoted also M) of $\vec{\mathcal{K}}_{d,d}$ containing all the arcs $(a, M(a))$.

Let $x \in \mathbb{Z}_d^n$, M_x will denote a permutation; the label x will be useful in the proof of lemma (**2**), as we will associate M_x with a perfect matching of $\vec{\mathcal{K}}_{d,d}(x)$ where $\vec{\mathcal{K}}_{d,d}(x)$ denote the bipartite subgraph of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d, n+1)$ with left part the vertices $(\star x, n)$ and right part the vertices $(\star x, 0)$. M_x contains the arcs joining (ax, n) to $(M_x(a)x, 0)$. In [7], M_x is said to be a permutation realizable in $\vec{\mathcal{K}}_{d,d}(x)$.

Definition 9 *Let S be a set of slopes (that is a subset of $\mathbb{Z}_d$). Then, a* **permutation** M *of* $\mathbb{Z}_d$ **uses the slopes in S** *if, for any $a \in \mathbb{Z}_d$, $M(a) \in \{a+s,\, s \in S\}$. A* **family of d^n permutations** $\mathcal{M} = \{M_x,\, x \in \mathbb{Z}_d^n\}$ *of* $\mathbb{Z}_d$ **uses the slopes in S** *if, for any permutation M_x of the family, M_x uses the slopes in S.*

Definition 10 *A set of p permutations M_j, with $1 \leq j \leq p$, is said to be* **compatible** *if, $\forall a, M_j(a) \neq M_{j'}(a)$ for $j \neq j'$.*

In other words the perfect matchings associated with the M_j are arc-disjoint.

Definition 11 *For $1 \leq j \leq p$, let $\mathcal{M}_j = \{M_{x,j} \mid x \in \mathbb{Z}_d^n\}$ be p families, each consisting of d^n permutations. The* **families** $\mathcal{M}_j$ *are said to be* **compatible** *if, for each x in $\mathbb{Z}_d^n$, the p permutations $M_{x,j}$ are compatible, i.e. $\forall a$, $M_{x,j}(a) \neq M_{x,j'}(a)$ for $j \neq j'$.*

The **composition** $M \cdot M'$ **of two permutations** M and M' is the permutation which associates the element a with the element $M(M'(a))$.

Definition 12 *A permutation M is* **cyclic** *if, for some x, all the elements $M^i(x)$ are distinct for $0 \leq i < d^n$.*

Remark 2 Note that if M is cyclic, then for every x, the elements $M^i(x)$ are all distinct. In fact, to verify that M is cyclic, it suffices to verify that for a given x, $M^i(x) \neq x$, for $1 \leq i < d^n$. Indeed, if there exists j and k, with $j > k$, such that $M^j(x) = M^k(x)$, then $M^{j-k}(x) = x$.

For example, the permutation M which associates a with the element $a + \delta$ is clearly cyclic if and only if δ is prime with d, as $M^i(a) = a + \delta i$.

Definition 13 *A family $\mathcal{M} = \{M_x,\, x \in \mathbb{Z}_d^n\}$ of d^n permutations of $\mathbb{Z}_d$ satisfies the* **cyclic-potent property** *if, for any order of composition of the M_x and any set of sign $\{\epsilon_x \mid x \in \mathbb{Z}_d^n\ \epsilon_x \in \{-1,1\}\}$ such that $\sum_x \epsilon_x \equiv 0 \pmod d$, the permutation $\Pi_x M_x^{\epsilon_x}$ is cyclic.*

Definition 14 *A* **family of d^n permutations** $\mathcal{M} = \{M_x,\, x \in \mathbb{Z}_d^n\}$ *is* **of type** (i,j) *if for $x \neq 0$, $M_x(a) = a + i$; and for $x = 0$, $M_0(a) = a + j$.*

Lemma 1 *A family of permutations of type (i,j), $\mathcal{M} = \{M_x,\, x \in \mathbb{Z}_d^n\}$ is cyclic-potent if and only if $j - i$ is relatively prime to d.*

Proof. As the permutations of the family commute, the permutation $\Pi_x M_x^{\epsilon_x}$ of definition (**13**) can be simply expressed as $a \rightarrow a + \delta$. So, this permutation will be cyclic if and only if δ is prime with d. Here $\delta = (\sum_{x \neq 0} \epsilon_x)i + \epsilon_0 j$. As $\sum_x \epsilon_x = 0$, we have $\delta = (\sum_x \epsilon_x)i + \epsilon_0(j-i) = \epsilon_0(j-i)$. So, δ is clearly prime with d if and only if $j-i$ is prime with d. □

We will represent a set of p families of permutations of type (i,j): $\{(i_0, j_0), (i_1, j_1), ..., (i_{p-1}, j_{p-1})\}$ by the array:

Families						
i_0	i_1	i_2	i_3	...	i_{p-2}	i_{p-1}
j_0	j_1	j_2	j_3	...	j_{p-2}	j_{p-1}

In section 3, we will need some very simple cyclic-potent families of permutations that we give as examples.

Families 1 There exist d compatible cyclic-potent families of permutations:

Families 1								
0	1	2	3	4	5	...	$d-2$	$d-1$
1	2	3	4	5	6	...	$d-1$	0

These families are cyclic-potent as, applying lemma (**1**), $1-0 = 2-1 = \cdots = d-1-(d-2) = 0-(d-1) = 1$ which is prime with d. These families use all the slopes.

Families 2 There exist 2 compatible cyclic-potent families using slopes $\{0,1\}$:

Families 2	
0	1
1	0

According to lemma (**1**) they are two compatible cyclic-potent families and they use the slopes $\{0,1\}$.

Families 3 When $d \neq 3$, there exist $d-2$ compatible cyclic-potent families of permutations using the slopes $\{2, \ldots, d-1\}$. One possible solution is given below:

- when d is odd and $d \neq 3$, the following families can be used:

Families 3 (d odd and $d \neq 3$)							
2	3	4	5	...	$d-3$	$d-2$	$d-1$
4	5	6	7	...	$d-1$	2	3

- when d is even, we use the following families:

Families 3 (d even)						
2	3	4	5	...	$d-2$	$d-1$
3	2	5	4	...	$d-1$	$d-2$

These families are cyclic-potent as, applying lemma (**1**), we get:

- for d odd, $4-2=5-3=\cdots=d-1-(d-3)=2$ and $2-(d-2)=3-(d-1)=4$, as 2 and 4 are prime with d;
- for d even, $3-2=5-4=\cdots=(d-1)-(d-2)=1$ and $2-3=4-5=\cdots=(d-2)-(d-1)=-1$, which are prime with d.

In both cases, the slopes used are in $\{2,\ldots,d-1\}$.

2.2. Inductive construction

Lemma 2 *If $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ admits p arc-disjoint l-crossing Hamilton cycles and if there exist p compatible cyclic-potent families each of d^n permutations, then $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$ admits p arc-disjoint l-crossing Hamilton cycles.*

Proof. Let H be an l-crossing Hamilton cycle of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$. As all the levels are equivalent, we can suppose without loss of generality and for simplicity in the notations that $l=0$. Let $\mathcal{M}=\{M_x,\ x\in\mathbb{Z}_d^n\}$ be a cyclic-potent family of d^n permutations. The vertices of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$ can be labeled (ax,l) with $a\in\mathbb{Z}_d$, $x\in\mathbb{Z}_d^n$ and $l\in\mathbb{Z}_{n+1}$. Now, we associate H and $\mathcal{M}$ with a partial digraph H' in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$ as follows (for an example of such a construction see figure (**3**)):

- for $0\leq l\leq n-1$ and for each a, if the arc $(x,l)(x',l+1)$ belongs to H, we put in H' the arc $(ax,l)(ax',l+1)$ where the indices are taken modulo $n+1$, which means that to the arc $(x,n-1)(x',0)$ of H is associated the arc $(ax,n-1)(ax',n)$ in H';
- between levels n and 0 of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$ we put the arcs joining (ax,n) to $(M_x(a)x,0)$.

With such a definition, each vertex of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$ is incident to two arcs of H'. Hence, we can define for each vertex a predecessor and a successor on H' that enables us to prove that we can order H' in a cycle.

For $1\leq l\leq n-1$, let (x',l') (*resp.* (x'',l'')) be the predecessor (*resp. successor*) of (x,l) in H, then the predecessor (*resp. successor*) of (ax,l) in H' will be (ax',l') (*resp.* (ax'',l'')).

For $l=0$ and n, as H is a 0-crossing Hamilton cycle, vertices $(x,0)$ are either of type $+$ or $-$ on H.

- When $(x,0)$ is of type $+$, its predecessor (*resp. successor*) in the cycle H is $(x',n-1)$ (*resp.* $(x'',1)$). Then, in H' the predecessor (*resp. successor*) of (ax,n) will be $(ax',n-1)$ (*resp.* $(M_x(a)x,0)$); the predecessor (*resp. successor*) of $(ax,0)$ will be $(M_x^{-1}(a)x,n)$ (*resp.* $(ax'',1)$).

- When $(x,0)$ is of type $-$, its predecessor (*resp. successor*) in H is $(x',1)$ (*resp.* $(x'',n-1)$). Then, in H' the predecessor (*resp. successor*) of $(ax,0)$ will be $(ax',1)$ (*resp.* $(M_x^{-1}(a)x,n)$); the predecessor (*resp. successor*) of (ax,n) will be $(M_x(a)x,0)$ (*resp.* $(ax'',n-1)$) in H'.

Therefore, when $(x,0)$ is of type $+$ (*resp.* $-$), (ax,n) and $(ax,0)$ are vertices of type $+$ (*resp.* $-$) in H'. Hence, all the vertices of levels 0 and n are crossed by H'; furthermore, the sum of the signs of the vertices of H' of levels 0 or n will be d times the sum of the signs of the vertices of H of level 0, that is, by hypothesis, 0. Hence, H' is 0-crossing (and also n-crossing).

Now, we have to prove that H' is effectively a Hamilton cycle. For this it suffices to prove that if we start at some vertex $(ax,0)$ and follow H', we meet successively all the vertices of level 0 and n before coming back to $(ax,0)$. Indeed, suppose that (y,l) was on the portion of cycle H between $(x_1,0)$ and $(x_2,0)$. Then, (ay,l) will be on the portion of H' between (ax_1,α) and (ax_2,β), where $\alpha=0$ (*resp.* $\alpha=n$) if $(x_1,0)$ is of type $+$ (*resp.* $-$), and $\beta=0$ (*resp.* $\beta=n$) if $(x_2,0)$ is of type $-$ (*resp.* $+$). These cases are described on figure (**2**).

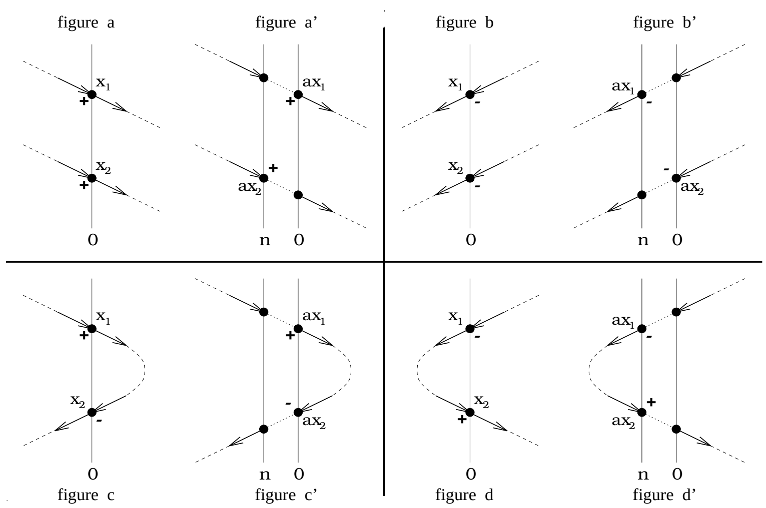

Figure 2: *This figure shows the four possible cases when we perform the inductive construction of* $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$ *from* $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$. *In figure a and a'* (resp. *b* and *b'*) *the vertices* x_1 *and* x_2 *are of type* $+$ (resp. $-$). *Figure c and c'* (resp. *d* and *d'*) *displays the case where the vertex* x_1 *is of type* $+$ (resp. $-$) *and the vertex* x_2 *is of type* $-$ (resp. $+$).

Now, let $(x_0,0),(x_1,0),\ldots,(x_{d^n}=x_0,0)$ be the sequence of vertices of H at level 0 in the order we meet them on H. Starting from $(a_0x_0,0)$ we will meet successively $(a_1x_1,0),(a_2x_2,0),\ldots,(a_{d^n}x_{d^n}=a_{d^n}x_0,0)$ on H'. Following such a path, we can meet either x_i of type $+$ by going from level n to level 0, in which case we will apply the permutation M_{x_i} to some a, or x_i of type $-$ by going from level 0 to n, in which case we will apply $M_{x_i}^{-1}$ to a. So $a_{d^n} = \Pi M_{x_i}^{\epsilon_{x_i}}(a)$ where the product is taken in an order depending on x_0. As all the x_i differ, we can meet again $(a_0x_0, 0)$ only at some $a_{qd^n}x_0$, but M being cyclic-potent, the values $a_{d^n}, a_{2d^n}, \ldots, a_{qd^n}, \ldots, a_{(d)d^n}$ are all distinct. So, we meet again $(a_0x_0, 0)$ only after having encountered the d^{n+1} vertices of level 0.

Now, note that we can perform this construction with p arc-disjoint 0-crossing cycles and p compatible cyclic-potent families. From construction, the p 0-crossing cycles that we will obtain, will be arc-disjoint. □

Remark 3 When the 0-crossing Hamilton cycles used in the lemma above are circuits of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$, all the vertices are of type $+$, and the construction leads to circuits of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$, giving another proof of the inductive lemma of [7].

3. Decomposition of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$

We will use a decomposition of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ into two partial digraphs.

Definition 15 *The Butterfly digraph $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is the sum of* **two partial digraphs** *$\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,n)$ and $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\ldots,d-1}(d,n)$ defined as follows:*

- *$\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,n)$ contains the arcs which slopes belong to $\{0,1\}$,*
- *$\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\ldots,d-1}(d,n)$ contains the arcs which slopes belong to $\{2,\ldots,d-1\}$.*

3.1. Decomposition of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\ldots,d-1}(d,n)$

The proof is by induction on n. We start the induction for $n = 1$.

Lemma 3 *When $d \notin \{4,6\}$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\ldots,d-1}(d,1)$ is decomposable into $d-2$ Hamilton circuits.*

Proof. As $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,1) = \mathcal{K}_d^+$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\ldots,d-1}(d,1)$ is obtained from $\mathcal{K}_d^+$ by removing the loops and the arcs of slope 1. Following Tillson [14], we know that $\mathcal{K}_d^+$ without the loops contains $d-1$ arc-disjoint Hamilton circuits when $d \neq 4, 6$. So, using Tillson's decomposition, we can label the vertices of $\mathcal{K}_d^+$ such that one of the circuits uses all the arcs of slope 1. By removing it, we get $d-2$ arc-disjoint Hamilton circuits in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\ldots,d-1}(d,1)$. □

Proposition 1 *For $d \notin \{3,4,6\}$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\ldots,d-1}(d,n)$ is decomposable into $d-2$ Hamilton circuits.*

Proof. As $d \notin \{4,6\}$, the proposition is proved for $n = 1$ by lemma (**3**). Then, as $d \neq 3$, the $d-2$ compatible cyclic-potent families (**3**) use the slopes

$\{2, \ldots, d-1\}$ and satisfy the hypothesis of lemma (**2**). Hence, we can apply that lemma inductively, in order to construct $d-2$ arc-disjoint Hamilton circuits (see remark (**3**)) in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\ldots,d-1}(d,n)$. □

3.2. Decomposition of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,n)$

Lemma 4 *$\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,2)$ is decomposable into 2 l-crossing Hamilton cycles.*

Proof. For this proof, the vertices of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,2)$ will be denoted by the ordered pairs (xy,l) with $x \in \mathbb{Z}_d$, $y \in \mathbb{Z}_d$ and $l \in \mathbb{Z}_2$. We will show that we can build two arc-disjoint 1-crossing Hamilton cycles in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,2)$ by using two sets of arcs of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,2)$ defined by the next two rules:

1. $\boxed{\textit{Arcs of } H_0\textit{:}}$

$$\begin{cases} \text{if } \ x \neq y, & (x(y-1),0) \overset{+1}{\to} (xy,1) \overset{+0}{\to} (xy,0), & (1) \\ \text{if } \ x = y, & (xx,0) \overset{+0}{\to} (xx,1) \overset{+1}{\to} ((x+1)x,0). & (2) \end{cases}$$

2. $\boxed{\textit{Arcs of } H_1\textit{:}}$

$$\begin{cases} \text{if } \ x \neq y, & (xy,0) \overset{+0}{\to} (xy,1) \overset{+1}{\to} ((x+1)y,0), & (1) \\ \text{if } \ x = y, & (x(x-1),0) \overset{+1}{\to} (xx,1) \overset{+0}{\to} (xx,0). & (2) \end{cases}$$

It is easy to verify that H_0 and H_1 are arc-disjoint. With the arcs (1) of H_0, we can define for each $x \in \mathbb{Z}_d$ a dipath P_x as follows:

$$P_x \begin{cases} & (xx,0) & \to & (x(x+1),1) & \to & (x(x+1),0) & \to \\ \to & (x(x+2),1) & \to & \cdots & \to & (x(x+d-2),1) & \to \\ \to & (x(x+d-2),0) & \to & (x(x+d-1),1) & \to & (x(x+d-1),0) & \end{cases}$$

The d dipaths P_x, $x \in \mathbb{Z}_d$, are clearly vertex-disjoint. Only the vertices noted $(xx,1)$ are not in these d dipaths. The arcs (2) of H_0 allows us to join the end vertices of the d dipaths through the missing vertices $(xx,1)$ as follows:

$$\begin{array}{lllll} P_x & \leftarrow & ((x+d-1)(x+d-1),1) & \leftarrow & P_{x+d-1} & \leftarrow \\ & \leftarrow & ((x+d-2)(x+d-2),1) & \leftarrow & \cdots & \leftarrow \\ & \leftarrow & ((x+1)(x+1),1) & \leftarrow & P_{x+1} & \leftarrow \\ & \leftarrow & (xx,1) & \leftarrow & P_x & \end{array}$$

One can easily check that we have defined a Hamilton cycle. The d dipaths are joined through their extremal vertices in a cyclic way, using only arcs (2) of H_0.

By construction, all the vertices at level 1 are crossed. In order to compute the sign of the vertices at level 1, we can choose to walk along the cycle in the direction $(xx,0) \to (x(x+1),1)$. Therefore, all the vertices $(xy,1)$ with $x \neq y$ are of type $+$ and have $+1$ as sign, while the vertices $(xx,1)$ are of type $-$ and have -1 as sign. So, the sum of the signs is $(d^2-d)-(d) \equiv 0 \pmod d$.

To prove that the second set of rules builds a second 1-crossing Hamilton cycle, it suffices to notice that we can rewrite this rule up to a permutation of the letters x and y as being:

- $\boxed{\textit{Arcs of } H_1 \textit{ (with permutation of } x \textit{ and } y\textit{):}}$

$$\begin{cases} \text{if } \; y \neq x, & (y(x+1),0) \overset{+1}{\leftarrow} (yx,1) \overset{+0}{\leftarrow} (yx,0), \\ \text{if } \; y = x, & (yy,0) \overset{+0}{\leftarrow} (yy,1) \overset{+1}{\leftarrow} ((y-1)y,0). \end{cases}$$

Construction 2 is then clearly similar to construction 1; to be convinced, just exchange x and y, and replace 1 by -1 in the proof for construction (1).

Hence, H_0 and H_1 are two arc-disjoint 1-crossing Hamilton cycles. As the levels are equivalent, the result holds also for level 0. □

Figure (**3**) gives a decomposition of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(3,2)$ into two 1-crossing Hamilton cycles.

Proposition 2 *For $n \geq 2$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,n)$ is decomposable into 2 l-crossing Hamilton cycles.*

Proof. The proposition is proved for $n = 2$ by the lemma (**4**). Then, we use lemma (**2**) with the two compatible cyclic-potent families (**2**) which use the slopes $\{0,1\}$ to construct inductively two arcs-disjoint l-crossing Hamilton cycles in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,n)$. □

Figure (**3**) gives the recursive construction of two 1-crossing arc-disjoint Hamilton cycles in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(3,3)$ from two 1-crossing arc-disjoint cycles in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,2)$.

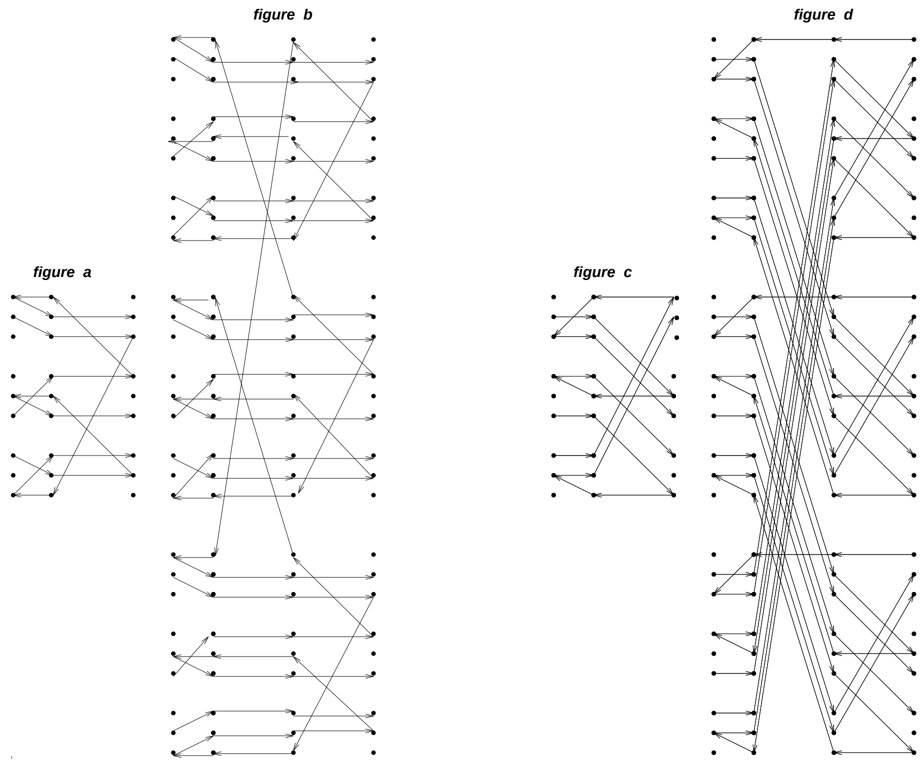

Figure 3: *Figures a and c show the two* 1*-crossing arc-disjoint Hamilton cycles of* $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(3,2)$*. We display on figures b and d, two* 1*-crossing arc-disjoint Hamilton cycles in* $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(3,3)$ *obtained by applying lemma (**2**) with the families (**2**).*

3.3. Global decomposition

We are now ready to prove the main result:

Theorem 2 *For* $n \geq 2$*,*

- *for* $d \notin \{3,4,6\}$*,* $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ *is decomposable into* $d-2$ *Hamilton circuits and* 2 *Hamilton cycles,*
- *for* $d \in \{4,6\}$*,* $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ *is decomposable into* d *Hamilton circuits,*
- $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,n)$ *is decomposable into* 1 *Hamilton circuit and* 2 *Hamilton cycles.*

Proof. According to propositions (**1**) and (**2**) we have, when $d \notin \{3,4,6\}$, $d-2$ arc-disjoint circuits in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\ldots,d-1}(d,n)$ and 2 arc-disjoint cycles in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,n)$. So, the result holds in these cases. For $d \in \{4,6\}$ and $n=2$, an exhaustive computer search shows that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is decomposable into Hamilton circuits, and so, for $n \geq 2$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(4,n)$ and $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(6,n)$ are decomposable into Hamilton circuits. For $d = 3$, we can construct two 1-crossing arc-disjoint Hamilton cycles and one arc-disjoint Hamilton circuit in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,2)$ (see figure (**4**)). Then, we can apply lemma (**2**) with families (**1**) and the result holds for $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,n)$ with $n \geq 2$. □

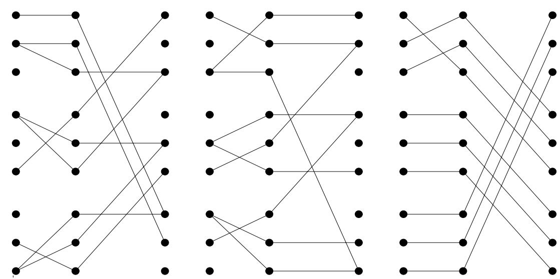

Figure 4: *The decomposition of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,2)$ into two 1-crossing arc-disjoint Hamilton cycles and one arc-disjoint Hamilton circuit.*

The preceding result implies the conjecture of Barth and Raspaud:

Theorem 3 *For any d and $n \geq 2$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is decomposable into d Hamilton cycles.*

Remark 4 We could also have derived theorem (**3**) by proving that, if $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is decomposable into l-crossing Hamilton cycles, then $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$ is also decomposable into l-crossing Hamilton cycles. This can be done by applying lemma (**2**) with the families (**1**). But to start the induction we needed to split the Butterfly digraph into two partial digraphs in order to prove that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,2)$ is decomposable into l-crossing Hamilton cycles for $n = 2$ and $d \neq 3$.

4. Conclusion

In this paper we have proved that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is always decomposable into Hamilton cycles. In the paper [7], we considered the problem of decomposing $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ into Hamilton circuits and conjectured that such a decomposition into d Hamilton circuits exists for $n \geq 2$, except for ($d = 2$ and ($n = 2$ or $n = 3$)) and ($d = 3$ and $n = 2$). The difficulty in that case was to start the induction; indeed in [7] we were able to reduce the problem to the case $n = 2$ and d prime and to solve it in many cases. Consequently, we proposed as an open problem the following conjecture:

Conjecture 3 ([7]) *For any prime number $p > 3$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(p,2)$ is decomposable into Hamilton circuits.*

Note added in proof

Recently, Helen Verrall[a] has informed us that she has been able to prove conjecture (**3**), thus closing completely the problem of Hamilton decomposition of the Butterfly network.

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