Hierarchical Ring Network design

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Abstract

A *Hierarchical Ring Network* is obtained from a ring network by appending at most one subsidiary ring to each *node* of the ring and, recursively, to each node of each subsidiary ring. The depth d is the number of levels of the recursive appending of subsidiary rings. There are different definitions according to which rings are appended to nodes created at the preceding level (called HRN) or to any node (called here HBN for Hierarchical Bubble Network). The case of HRN was considered by Aiello, Bhatt, Chung, Rosenberg and Sitaraman who give bounds (not tight) on the diameter of such HRN as a function of the depth and the number of nodes. Here we determine the exact order of the diameter both for HRN and HBN. In fact we consider the optimization problem of maximizing the number of nodes of an HBN (or HRN) of given depth d and diameter D. We reduce the problem to a system of equation with a complex objective function. Solving this system enables us to determine precisely the structure of optimal HBN and to show that the maximum number of nodes is of order $D^d/d!$.

Keywords

Combinatorial Optimization, Network Design, Ring Networks, Diameter.

1 Introduction

Hierarchical networks are important structure in telecommunications; indeed some of the current telecommunication networks are built by connecting rings in a hierarchical fashion. Two parameters are of importance : the number of levels of the hierarchy (going from one level to another implies change in the transmission mode and need for interface routers) and the number of hops needed to establish a connection (whose minimization corresponds to minimizing the transmission time). In [1], the authors define a *Hierarchical Ring Network* as follows : "A *Hierarchical Ring Network* (*HRN*, for short) is obtained from a ring network by appending at most one subsidiary ring to each *node* of the ring and, recursively, to each node of each subsidiary ring. (Each node of an *HRN* thus belongs to either one or two rings) ...". The depth d of the *HRN* is the number of levels of the recursive appending of subsidiary rings. A *HRN* of depth 1 is nothing else than a cycle. A *HRN* of depth 2 consists of cycles interconnected via a cycle. More formally the sets of nodes V_i of level i, $1 \leqslant i \leqslant d$ consists of the disjoint union of chains $P_{i,1},\ldots,P_{i,j},\ldots,P_{i,k_i}$ $1 \leqslant j \leqslant k_i$, such that each $P_{i,j}$ is associated to a node v_j of V_{i-1} (that is of level $i-1$) and such that the subgraph generated by $P_{i,j} \cup v_j$ is a cycle (with $v_j \neq v_i$ for $j \neq i$). In [1], the authors show the relation of the model and other structures (Multi-Rings, Chordal-Rings and Express-Rings). Those structures have been proposed as interconnection networks for shared memory architectures [6, 5, 7] and are widely used by telecommunication operators. One can find a survey on these structures in [2], their diameter in [3, 8, 5], their broadcast time and information dissemination in [7]. Authors of [1] give the following bounds on the diameter $D_d^{HRN}(N)$ of a N-node depth-d *HRN* :

$$\frac{1}{2}(N \times d!)^{1/d} + o(N^{1/d}) \leqslant D_d^{HRN}(N) \leqslant (N \times d!)^{1/d} + o(N^{1/d})$$

From this result we get the following bounds on the number $N_d^{HRN}(D)$ of nodes of a diameter-D depth-d *HRN* : $\frac{D^d}{d!} \leqslant N_d^{HRN}(D) \leqslant 2^d \frac{D^d}{d!}$. Recently A. Rosenberg informed us that they improved their results and showed :

$$\left(\frac{1}{2}\right)^{2/d}(N \times d!)^{1/d} + o(N^{1/d}) \leqslant D_d^{HRN}(N) \leqslant (N \times d!)^{1/d} + o(N^{1/d})$$

but the bounds are still not tight. Motivated by [1] we prove here the following results which gives a tight estimate of $N_d^{HRN}(D)$ and so of $D_d^{HRN}(N)$:

Theorem 1 *For a given* $d \geqslant 2, N_d^{HRN}(D) = \frac{D^d}{d!} + \mathcal{O}(D^{d-1})$

Corollary 2 *For a given* $d \geqslant 2, D_d^{HRN}(N) = (N \times d!)^{1/d} + o(N^{1/d})$

In order to prove Theorem 1 we introduce a different kind of hierarchical ring network that we call *Hierarchical Bubble Network* (*HBN*, for short). The difference between *HBN* and *HRN* is that the cycle appended at level d can be appended to any node of lower level (and not only to a node of level $d-1$). We propose here an equivalent definition which emphasizes the recursive structure :

Definition 3 Recursive definition of *HBN*

- a depth-1 *HBN* is a ring.
- for $d \geqslant 2$, a depth-d *HBN* consists of a ring called *principal ring* and of a finite set of depth-$(d-1)$ *HBN* connected to nodes of the principal ring by a node of their own principal ring.

Remark 4 We can remark that for depth-1 and depth-2, there is no difference between a *HRN* and a *HBN* (see Figure 1 for examples). We can not

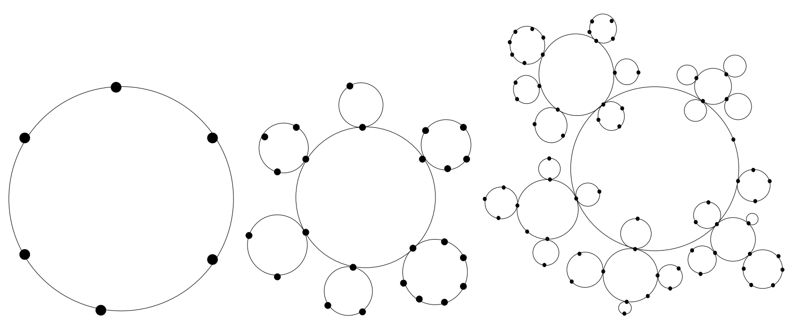

Figure 1: Examples of depth-1 *HRN*/*HBN*, depth-2 *HRN*/*HBN* and depth-3 *HBN*.

use such a recursive definition for the *HRN* because the structure appended on each node of the *principal* ring of the *HRN* is not a *HRN*. That is the reason why we use the *HBN* structure.

In this paper we determine the maximum number of nodes $N_d^{HBN}(D)$ of a diameter-D depth-d *HBN* (Theorem 22). The paper is organized as follows : we first reduce the problem to the solution of a system of equations with a complex objective function. Solving this system enables us to determine the structure of an optimal *HBN* and to prove Theorem 22. Then we prove that $N_d^{HRN}(D) = N_d^{HBN}(D) + O(D^{d-1}) = \frac{D^d}{d!} + O(D^{d-1})$ implying Theorem 1.

2 Characterization of the Optimal *HBN*

Definition 5 The *weight* of a *HBN* B, denoted by $w(B)$, is defined as the minimum eccentricity of the nodes of the principal ring of B.

Definition 6 We denote by $n(d, D, w)$ the maximum number of nodes of a depth-d diameter-D *HBN* of weight w.

Note that the maximum number of nodes of a depth-d diameter-D *HBN*:

$$N_d^{HBN}(D) = \max_{0 \leqslant w \leqslant D} n(d, D, w) \tag{1}$$

2.1 Another Formulation of the Problem

Proposition 7 *Determining $N_d^{HBN}(D)$ is equivalent to the following problem :*

$$\begin{aligned} Maximize \quad & \sum_{0 \leqslant i \leqslant k-1} n(d-1, D, w(B_i)) & (2)\\ where \quad & k \in I\!N, k \leqslant 2D+1 & (3)\\ \forall 0 \leqslant i \leqslant k-1 \quad , \quad & B_i \textit{ is a depth-}(d-1) \textit{ diameter } D \text{ HBN} & (4)\\ \forall 0 \leqslant i, j \leqslant k-1 \quad , \quad & w(B_i) + w(B_j) + d(i,j) \leqslant D & (5) \end{aligned}$$

with $d(i, j) = \min\{i - j \mod k, j - i \mod k\}$.

Proof. Consider an optimal depth-d diameter-D *HBN*. It satisfies (3,4), indeed it consists of a principal ring of some size k. At each node i is attached a depth-$(d-1)$, diameter-D *HBN* B_i with some weight $w(B_i)$. The number of nodes of B_i is at most $n(d-1, D, w(B_i))$. To prove (5) consider x in B_i (resp. $y \in B_j$) at distances $w(B_i)$ (resp. $w(B_j)$) from i (resp. j). Such vertices exist as i(resp. j) has eccentricity at least $w(B_i)$ (resp. $w(B_j)$) (which is the minimum of the eccentricities). Then $d(x, y) = w(B_i) + w(B_j) + d(i, j)$ where $d(i, j)$ is the distance between i and j in the principal ring. As $d(x, y) \leqslant D$ that implies (5) (see Figure 2). So $N_d^{HBN}(D)$ is less than or equal to the maximum solution of the problem.

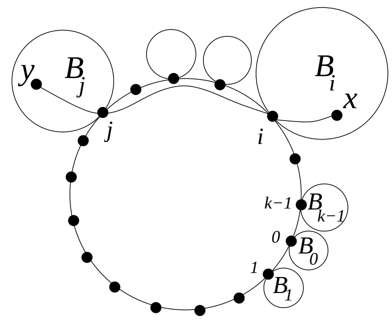

Figure 2: $\forall 0 \leqslant i, j \leqslant k-1, w(B_i) + w(B_j) + d(i, j) \leqslant D$

Conversely, consider a solution for which the maximum is attained and let k and $\{B_i\}_{0 \leqslant i \leqslant k-1}$ be an optimal solution. Consider the *HBN* obtained by taking a principal ring of size k and attaching to the node i a depth-$(d-1)$ diameter-D *HBN* B_i by its node with minimum eccentricity $w(B_i)$. We obtained a depth-d *HBN* whose diameter is at most D. Indeed the distance between any pair of nodes x and y is at most D if they belong to the same B_i (the diameter of B_i being at most D) and otherwise if $x \in B_i$ and $y \in B_j$, $d(x, y) \leqslant d(x, i) + d(i, j) + d(y, j)$ but $d(x, i) \leqslant ecc_{B_i}(i) = w(B_i)$ and $d(j, y) \leqslant ecc_{B_j}(j) = w(B_j)$.
So $d(x, y) \leqslant w(B_i) + w(B_j) + d(i, j) \leqslant D$. □

Proposition 8 *Initial values of* $n(d, D, w)$.

$$\forall D \geqslant 0, \forall 0 \leqslant w \leqslant D, n(1, D, w) = 2w + 1 \quad (6)$$
$$\forall d \geqslant 1, \forall D \geqslant 0, n(d, D, 0) = 1 \quad (7)$$

Proof.
(6) The number of vertices of a ring with a node of eccentricity w is $2w + 1$.
(7) There can be only one node in a graph of minimal eccentricity 0. □

2.2 Determination of $n(d, D, w)$ for $w \leqslant \lfloor D/2 \rfloor$

Proposition 9 $\forall d \geqslant 2, \forall 1 \leqslant w \leqslant \lfloor D/2 \rfloor$,

$$n(d, D, w) = n(d-1, D, w) + 2 \sum_{p=0}^{w-1} n(d-1, D, p) \tag{8}$$

$$= \sum_{p=0}^{\min(d,w)} 2^p \binom{d}{p} \binom{w}{p} = \sum_{p=0}^{d} \binom{d}{p} \binom{d+w-p}{p} \tag{9}$$

Proof.
(8) Let x be a node of the principal ring whose eccentricity is w, the weight of the depth-$(d-1)$ *HBN* based in x is at most w, by definition of the eccentricity. The weight of the depth-$(d-1)$ *HBN* at distance i is at most $w-i$; so we deduce that $n(d, D, w) \leqslant n(d-1, D, w) + 2\sum_{p=0}^{w-1} n(d-1, D, p)$. Conversely consider the *HBN* obtained by taking a ring of size $2w+1$ and appending a depth-$(d-1)$ diameter D *HBN* B_i to node i with weight $(w-i)$ for $i = 0, \ldots, w$ and weight $i - (w+1)$ for $i = w+1, \ldots, 2w$. Such a construction satisfies the constraint (5) and so we obtain Equation (8).
(9) Equation (8) can also be written as follows :
$\forall d \geqslant 2, \forall 1 \leqslant w \leqslant \lfloor D/2 \rfloor$,

$$n(d, D, w) - n(d, D, w-1) = n(d-1, D, w) + n(d-1, D, w-1) \tag{10}$$

Then (9) can be computed using generating series. It represents the number of integer points of a d-dimensional sphere of radius w; some other results on this function can be found in [4]. $\square$

Remark 10 *For all $d \geqslant 2$, the function $i \to n(d, D, i)$ is strictly increasing when i increases from 0 to $\lfloor D/2 \rfloor$.*

2.3 First Case : $\forall 0 \leqslant i \leqslant k-1, w(B_i) \leqslant \lfloor D/2 \rfloor$.

In this part, we consider that the weight of each depth-$(d-1)$ *HBN* is less or equal to $\lfloor D/2 \rfloor$, then using the Proposition 9, we will be able to determine the maximum number of nodes :

Corollary 11 *Determining* $N_d^{HBN}(D)$ *is equivalent to the following optimization problem :*

$$Maximize \qquad val(S) = \sum_{0 \leqslant i \leqslant k-1} n(d-1, D, w(i))$$

$$where\ n(d-1, D, w) \quad = \quad \sum_{p=0}^{d-1} \binom{d-1}{p} \binom{d-1+w-p}{d}$$

and where S *is the following system :*

$$\forall 0 \leqslant i, j \leqslant k-1 \quad , \quad w(i) + w(j) + d(i,j) \leqslant D \tag{11}$$

$$k \in I\!N, 1 \leqslant k \leqslant 2D+1 \tag{12}$$

$$\forall 0 \leqslant i \leqslant k-1 \quad , \quad 0 \leqslant w(i) \leqslant \lfloor D/2 \rfloor \tag{13}$$

Proof. $n(d-1, D, i)$ used in the Proposition 7 is given by the Proposition 9 as $w(i) \leqslant \lfloor D/2 \rfloor$. Then all the parameters of the system to optimize are known. □

A solution satisfying the inequalities (11, 12,13) will be said to be feasible. Among the feasible solutions, one which attains the maximum of $val(S)$ will be said optimal.

Lemma 12 *Consider an optimal solution of* S*, then for each* i*, there exists a* j *such that* $w(i) + w(j) + d(i,j) = D$*.*

Proof. Suppose that for all i and j, $w(i) + w(j) + d(i,j) \leqslant D - 1$. If $w(i) < \lfloor D/2 \rfloor$, then $w(i)$ could be increased without violating any constraint, and $val(S)$ will not be maximum. If $w(i) = \lfloor D/2 \rfloor$, then we have for all j_1 and j_2, $w(j_1)+w(j_2)+d(j_1,j_2) \leqslant 2D-2-2w(i) < D$ as $d(j_1,j_2) \leqslant d(i,j_1)+d(i,j_2)$ so one $w(j)$ for $j \neq i$ could be increased without violating constraints. □

Lemma 13 *Given an optimal solution* (w,k) *of the numerical system, the difference between two consecutive weights is at most 1, i.e.:* $\forall 0 \leqslant i \leqslant k-1, |w(i+1) - w(i)| \leqslant 1$.

Proof. Suppose that there exists some i such that $w(i) \leqslant w(i+\epsilon) - 2$, with $\epsilon = 1$ or -1, the inequality (11) applied with $i+\epsilon$ gives $\forall j, w(i+\epsilon) + w(j) + d(i+\epsilon, j) \leqslant D$, so we deduce $w(i) + w(j) + d(i+\epsilon, j) \leqslant D - 2$, then $w(i) + w(j) + d(i,j) \leqslant D - 1$, contradicting Lemma 12 (see Figure 3 for an example with $\epsilon = 1$). □

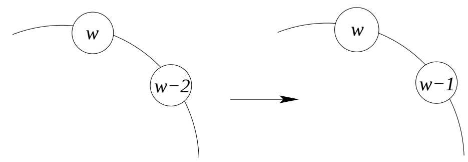

Figure 3: Example with $\epsilon = 1$.

Notation 14 *Let* $M = \max_{0 \leqslant i \leqslant k-1} \{w(i)\}$, $m = \min_{0 \leqslant i \leqslant k-1} \{w(i)\}$ *and* $h = \lfloor \frac{k}{2} \rfloor$.

Lemma 15 $M+m+h = D$ *and if* $w(i) = M$ *then* $w(i+h) = w(i-h) = m$.

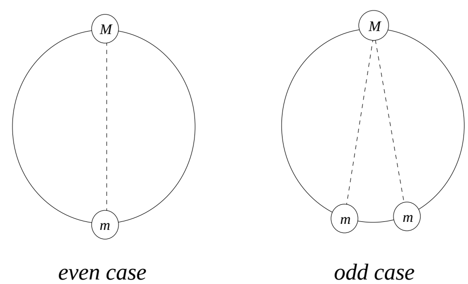

Figure 4: M is antipodal to m.

Proof. Let i_0 be such that $w(i_0) = M$. Consider one antipodal of i_0, i_0+h. By definition of m, $w(i_0+h) = m+\alpha$ with $\alpha \geqslant 0$. $w(i_0)+w(i_0+h)+d(i_0, i_0+h) = M+m+\alpha+h$. So by (11), $M+m+h \leqslant D-\alpha$. Now let i_1, be such that $w(i_1) = m$. $w(i_1)+w(j)+d(i_1, j) \leqslant m+M+h \leqslant D-\alpha$. So if $\alpha > 0$, we have a contradiction with Lemma 12 for $i = i_1$. So $\alpha = 0$ and $M+m+h = D$ and $w(i_0+h) = m$. If h is odd, i_0 has an other antipodal i_0-h and we can prove in the same way that $w(i_0-h) = m$. See Figure 4. □

Lemma 16 *Let* $d \geqslant 2$ *and* $\alpha \leqslant w \leqslant w' \leqslant \lfloor D/2 \rfloor - \alpha$, *then*

$$n(d, D, w-\alpha) + n(d, D, w'+\alpha) > n(d, D, w) + n(d, D, w')$$

Proof. It is sufficient to prove it for $\alpha = 1$, then repeated application will give the Lemma. By Equation (10) (in the proof of the Proposition

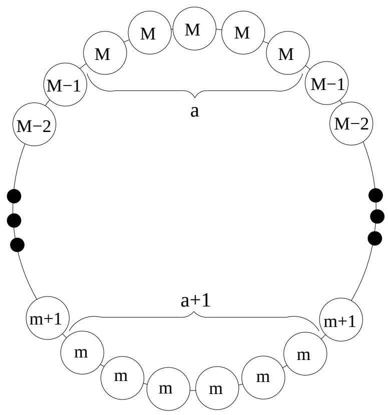

Figure 5: Structure of an optimal solution in the first case if $m < M$.

9), $n(d, D, w-1) + n(d, D, w'+1) = n(d, D, w) + n(d, D, w') + \epsilon$ where $\epsilon = n(d-1, D, w'+1) + n(d-1, D, w') - n(d-1, D, w) - n(d-1, D, w-1)$ and $\epsilon \geqslant 0$ by Remark 10. □

Proposition 17 (Structure of an optimal solution in the first case) *In an optimal solution of S, we have :*

- *k is odd, $k = 2h+1$*
- *$M + m + h = D$, (recall that $M = \max_i\{w(i)\}$ and $m = \min_i\{w(i)\}$)*
- *either $M = m$ and there are $k = 2h+1$ values $w(i) = M$*
- *either $M > m$ and we have (up to a cyclic shift of the indices) :*
  - *for $0 \leqslant i \leqslant a-1, w(i) = M$*
  - *for $a \leqslant i \leqslant h-1, w(i) = M - 1 - (i-a)$*
  - *for $h \leqslant i \leqslant h+a, w(i) = m$*
  - *for $h+a+1 \leqslant i \leqslant 2h, w(i) = M - 1 - 2h + i$*

  where a satisfies $2M + a - 1 = D$ or $a = h + 1 - M + m$.

Proof. First let us show that a solution of the kind described above, denoted by S', is feasible. That is the case if $M = m$ as $w(i)+w(j)+d(i,j) \leqslant$

$2M+h=D$. If $M \neq m$ we will show by induction on i that if $i < j \leqslant i+h$, $w(i)+w(j)+d(i,j) \leqslant 2M+a-1 = D$. That is true for $i=0$ as for $0 < j \leqslant k, w(0)+w(j)+d(0,j) \leqslant D$. Now we suppose by induction that it is true till i. For $i+1 < j \leqslant i+h$, $w(i+1)+w(j)+d(i+1,j) \leqslant w(i)+1+w(j)+d(i,j)-1 \leqslant D$ by induction hypothesis and $d(i+1,i+1+h) \leqslant D$. Furthermore $w(i+1)+w(i+1+h)+d(i+1,i+1+h)$ is $\leqslant M+m+h = D$ if $0 \leqslant i+1 \leqslant h$ or $h \leqslant i+1 \leqslant h+a$ or $\leqslant 2M+a-1$ otherwise.

Now we will show that $val(S')$ is optimum.
Suppose that we have an optimal solution S_{opt} with value $val(S_{opt})$. If $m = M$ then it contains $2h+1$ weights M with $2M+h=D$. So now let us suppose $m < M$. Consider one i_0 such that $w(i_0) = M$ so by Lemma 15, $w(i_0+h) = w(i_0-h) = m$. By Lemma 13 there should exist between i_0 and i_0+h a sequence $I : i_0 < i_1 < i_2 < i_\alpha < i_0+h$ with $w(i_\alpha) = M-\alpha$ ($w(i_1) = M-1, w(i_2) = M-2, \ldots, w(i_{M-m-1}) = m+1$). Now note that for $0 \leqslant i \leqslant i+h$, if $w(i) = M-\alpha$ then $w(i-h) \leqslant m+\alpha$ as $w(i)+w(i-h)+d(i,i-h) = M-\alpha+w(i-h)+h \leqslant M+m+h = D-w(i-h)$. If we replace $w(i-h)$ by $m+\alpha$, we increase the value of S_{opt} by remark 10 and so obtain a solution S_1 (not necessarily feasible) with $val(S_1) \geqslant val(S_{opt})$. Now for $0 < i < i+h, i \notin I$ ($i \neq i_0, i \neq i_1, \ldots, i_{M-m+1}$), we replace $w(i)$ by M and $w(i-h)$ by m we obtain a solution S_2 (not necessarily feasible). If $w(i) = M-\alpha$, $w(i-h) = m+\alpha$.
If $M-\alpha \geqslant m+\alpha$, Lemma 16 applied with $w = m+\alpha$ and $w' = M-\alpha$ shows that the solution S_2 where $w(i) = M$ and $w(i-h) = m$ is better. If $M-\alpha \leqslant m+\alpha$, Lemma 16 applied with $w = M-\alpha$ and $w' = m+\alpha$ shows that S_2 is better so $val(S_2) \geqslant val(S_1) \geqslant val(S_{opt})$.
Furthermore if there exists $i, i_0 < i < i_0+h, i \notin I$ for which $w(i) = M-\alpha$ with $\alpha > 0$, then $val(S_2) > val(S_{opt})$. If k is even, the sequence S_2 consists of a indices with weight M, a indices with weight m and exactly two with weight $M-\alpha$, $1 \leqslant \alpha \leqslant M-m-1$ where $2M+a-1 = D$. In that case we can add one weight m obtaining a solution S_3 with the same weight sequence as that S' of the theorem. In summary as S' is feasible, an optimal solution contains an odd number of indices so k is odd, $k = 2h+1$, and there are exactly a weight M, $a+1$ weights m and exactly two weights $M-\alpha, 1 \leqslant \alpha M-m-1$. Now in a feasible solution, the distance between two indices with weight M is at most $a-1$ as $2M+a-1 = D$ so the a indices of weight M are consecutive so for $i_0 \leqslant i \leqslant i_0+a-1, w(i) = M$. Furthermore as $w(i_0+h) = m$ and between i_0+a and i_0+h-1 there are exactly $M-m-1 = h-a$ indices $w(a-1+\alpha) = M-\alpha$ for $1 \leqslant \alpha \leqslant M-m-1$ and similarly as $w(i_0+a-1-h) = w(h+a) = m$

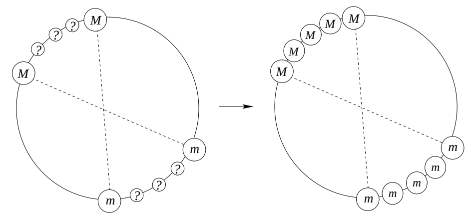

Figure 6: Transformation from S_{opt} to S_3.

we have for $h+a+1 \leqslant i \leqslant 2h, w(i) = M-1-2h+i$.

So an optimal solution can be obtained with this structure. □

Now to find the optimal solution we only have to compare a small number of structures. Given D, we have to find the better couple (M,m) with $0 \leqslant m \leqslant M \leqslant \lfloor D/2 \rfloor$ where $M+m+h=D$. The value is

- if $m = M$: $val(S) = (2h+1)n(d-1,D,M)$ with $2M+h=D$.
- if $m < M$: $val(S) = a\ n(d-1,D,M) + 2\sum_{\alpha=1}^{M-m-1} n(d-1,D,M-\alpha) + (a+1)n(d-1,D,m)$ with $2M+a-1=D$.

2.3.1 Case $d=2$

Proposition 18 *If $d=2$, an optimal solution of the first case is obtained with $M = m = \lfloor \frac{D+1}{4} \rfloor$. It is unique except for $D \equiv 2$ or $3 \mod 4$ where the solution with $M = m+1 = \lfloor \frac{D+3}{4} \rfloor$ is also optimal.*

Proof. First suppose there exists an optimal solution S_{opt} with $m \leqslant M-2$. If we consider the solution obtained with the parameters $M-1$ instead of M and $m+1$ as minimum (which is also feasible) that corresponds to replace $(a-2)$ indices M by $(a-2)$ indices $M-1$ and $(a-1)$ indices m by $(a-1)$ indices $m+1$ but as $n(d-1,D,w) = 2w+1$ for $d=2$ the value is increased by 2 so the solution was not optimal.
If $M=m$, $val(S) = (2h+1)(2M+1)$ with $2M+h=D$ so $val(S) = -8M^2 + (2M+1)(2D-1)+2$. A study of this function shows that the optimum is attained for $M = \lfloor \frac{D+1}{4} \rfloor$. Indeed in real number, the maximum is attained for $M = \frac{D}{4} - \frac{1}{8}$; then it is sufficient to compare for $D = 4q+1+\epsilon$ the value for $M=q$ and $M=q+1$. If $\epsilon = 0$ or 1 : the best is $M = q = \lfloor \frac{D+1}{4} \rfloor$. For $\epsilon = 2$ or 3 : the best is $M = q+1 = \lfloor \frac{D+1}{4} \rfloor$.

For $M = m+1$, $val(S') = a(2M+1)+(a+1)(2M-1)$ with $2M+a-1 = D$ so $val(S') = -8M^2 + 2M(2D+3) - 1$. By comparing the values we always have $val(S') \leqslant val(S)$ and equality is attained for $D = 4q+2$ for which the value for $M = q+1$, $m = q$, $val(S') = 8q^2 + 14q + 5$ that is the same as $val(S)$ for $M = m = q$ and for $D = 4q+3$ for which the value for $M = q+1$, $m = q$, $val(S') = 8q^2 + 18q + 9$ is the same as $val(S)$ for $M = m = q+1$. □

2.3.2 Case $d \geqslant 3$

Proposition 19 *In an optimal solution of the first case, if $d \geqslant 3$, then $a \leqslant 2$ except for $d = 3$ and $D = 4, 6, 7, 9$.*

Proof. We will use the following equality : $\forall d \geqslant 2, D, w \leqslant \lfloor D/2 \rfloor$,

$$\begin{aligned} n(d,D,w) - n(d,D,w-1) &= \sum_{p=0}^{d} \binom{d}{p} \left[\binom{d+w-p}{d} - \binom{d+w-1-p}{d} \right] \\ &= \sum_{p=0}^{d} \binom{d}{p} \binom{d+w-1-p}{d-1} \end{aligned}$$

Consider an optimal solution with a indices of weight M where $2M+a-1 = D$. Remember that if $m < M$ there are $(a+1)$ indices of weight m and if $m = M$, $(a-1)$ indices will be considered of weight m.

Claim 1: In an optimal solution, $m \leqslant \frac{(a+1)(d-1)}{2}$.
Either $m \leqslant d-1$ and it is true as $a \geqslant 1$, otherwise consider the transformation T_1 which consists in replacing $(a-1)$ indices of weight m by $(a+1)$ indices of weight $m-1$. The solution obtained is feasible and should have value less or equal to that of the optimal solution. So

$$\begin{aligned} T_1(a,d,D,m) = (a+1)n(d-1,D,m-1) - (a-1)n(d-1,D,m) &\leqslant 0 \\ \sum_{p=0}^{d-1} \binom{d-1}{p} \left[(a+1)\binom{d-1+m-1-p}{d-1} - (a-1)\binom{d-1+m-p}{d-1} \right] &\leqslant 0 \\ \text{If } m \geqslant d-1,\ \sum_{p=0}^{d-1} \binom{d-1}{p} \frac{(d-1+m-1-p)!}{(d-1)!(m-p)!} P(p,a,d,m) &\leqslant 0 \end{aligned}$$

where $P(p,a,d,m) = (a+1)(m-p) - (a-1)(d-1+m-p)$. $P(p,a,d,m)$ is decreasing in p, so we should have $P(d-1,a,d,m) \leqslant 0$, that is $(a+1)(m-d+1) - (a-1)m \leqslant 0$ or $m \leqslant \frac{(a+1)(d-1)}{2}$.

Claim 2: In an optimal solution, $m \geqslant (a-2)(d-1) - (a-1)$.
We suppose $a \geqslant 3$, otherwise the formula gives $m \geqslant -1$. Consider the transformation T_2 which consists in replacing $(a-2)$ indices of weight M by $(a-2)$ indices of weight $M+1$ and $(a-1)$ indices of weight m by $(a-1)$ indices of weight $m-1$. The solution obtained is feasible and should have a value less or equal to that of the optimal solution. Let $T_2(a,d,D,M,m) =$

$$\begin{aligned} & (a-2)[n(d-1,D,M+1) - n(d-1,D,M)] \\ - \quad & (a-1)[n(d-1,D,m) - n(d-1,D,m-1)] \end{aligned}$$

then $T_2(a,d,D,m,m) \leqslant T_2(a,d,D,M,m) \leqslant 0$

$$T_2(a,d,D,m,m) = \sum_{p=0}^{d-1} \binom{d-1}{p} \frac{(d-2+m-p)!}{(d-2)!(m-p+1)!} Q(p,a,d,m) \quad \leqslant \quad 0$$

where $Q(p,a,d,m) = (a-2)(d-1+m-p) - (a-1)(m-p+1)$ $Q(p,a,d,m)$ is increasing in p, so we should have $Q(0,a,d,m) \leqslant 0$, that is $m \geqslant (a-2)(d-1) - (a-1)$.

Claim 3: If $d \geqslant 4$ then $a \leqslant 5 + \frac{8}{d-3}$.
Combining claim 1 with claim 2, we obtain : $(a-2)(d-1) - (a-1) \leqslant m \leqslant (a+1)\frac{d-1}{2}$ that is $(d-3)a \leqslant 5d-7$ or if $d \neq 3, a \leqslant 5 + \frac{8}{d-3}$.

Claim 4: If $d \geqslant 12$, then $a \leqslant 2$.
Consider the transformation T_3 which consists in replacing $(a-2)$ indices of weight M by $(a-2)$ indices of weight $M+1$ and $(a-1)$ indices of weight m by 2 indices of weight $m-1$ and $(a-1)$ indices of weight $m-2$. We will show that this transformation increase the value *val* if $d \geqslant 12$ and $a \geqslant 3$, so giving a contradiction.

So we want to prove that
$T_3(a,d,D,M,m) = (a-2)\,[n(d-1,D,M+1) - n(d-1,D,M)] - (a-1)\,[n(d-1,D,m) - n(d-1,D,m-2)] + 2n(d-1,D,m-1) > 0$. As $M \geqslant m$, it is sufficient to prove that $T_3(a,d,D,m,m) > 0$.

$T_3(a,d,D,m,m) = \sum_{p=0}^{d-1} \binom{d-1}{p} U_p(a,d,m)$ where $U_p(a,d,m) = (a-2)\binom{d-1+m-p}{d-2} - (a-1)\left[\binom{d-1+m-1-p}{d-2} + \binom{d-1+m-2-p}{d-2}\right] + 2\binom{d-1+m-1-p}{d-1}$.
If $p \geqslant m+1$, all binomial parts with a negative sign are null, then we only define $U_p(a,d,m)$ for $p \leqslant m$.

$$U_p(a,d,m) = \frac{(d-3+m-p)!}{(d-1)!(m-p+1)!} T_p(a,d,m)$$

where $T_p(a,d,m) = (a-2)(d-1)(d-1+m-p)(d-2+m-p) - (a-1)(d-1)(m-p+1)\{(d-2+m-p)+(m-p)\} + 2(d-2+m-p)(m-p+1)(m-p)$.
So it is sufficient to prove that $T_p(a,d,m) > 0$.
Let $pol_a(p) := p \to T_p(a,d,m)$, we will consider the cases $d \geqslant 12, 3 \leqslant a \leqslant 5 + \frac{8}{d-3}$ and prove that for $0 \leqslant p \leqslant m, pol_a(p) > 0$.

$a = 3$, $pol_3(p) = -2p^3 + (6m-(d-1))p^2 + (-6m^2 + 2m(d-1) + d + 1)p + (d-1)(d-1+m)(d-2+m) + 2(d-2+m)m(m+1) - 2(d-1)(m+1)(d-2+2m)$

$pol_3'(p) = -6p + 2(-d+1+6m)p + 2dm - 6m + 1 + d - 2m$, the roots of $pol'(p)$ are
$\alpha = \frac{1}{6} - \frac{1}{6}d + m - \frac{1}{6}\sqrt{d^2+4d+7}$ and $\beta = \frac{1}{6} - \frac{1}{6}d + m + \frac{1}{6}\sqrt{d^2+4d+7}$
then $pol_3(p)$ is decreasing from 0 to α and increasing from α to m (as $\beta \geqslant m$).

$pol_3(\alpha)$ is independent of m and strictly positive if $d \geqslant 5$. $\alpha \geqslant m$, then for $d \geqslant 5, p \leqslant m$, $T_p(3,d,m) > 0$.

$a = 4,5$, note that $pol_a(m) = (d-1)(d-2)\{(a-2)(d-2)-1\} > 0$ for $a \geqslant 3, d \geqslant 4$.

$a = 4$: $pol_4(p) = -2p^3 + (6m - 2(d-1))p^2 + (-6m^2 + 4(d-1)m - d^2 + 5d - 2)p + 2m^3 - 2(d-1)m + (d - 5d + 2)m + (d-1)(d-2)(2d-5)$
$pol_4'(p) = -6p + 2(6m - 2(d-1))p + (-6m^2 + 4(d-1)m - d^2 + 5d - 2)$
$\Delta pol_4'(p) = -8(d^2 - 11d + 4)$ which is strictly negative for $d \geqslant 11$.
$\lim_{p\to\infty} pol_4'(p) = -\infty$ then for all p, $pol_4'(p) \leqslant 0$, then for $d \geqslant 11$, $pol_4(p)$ is decreasing. $pol_4(m) > 0$ for $d \geqslant 4$.
Then for $0 \leqslant p \leqslant m, d \geqslant 11, T_p(4,d,m) > 0$.

$a = 5$: $pol_5(p) = -2p^3 + 3(2m - (d-1))p + (-6m^2 + 6(d-1)m - 2d^2 + 9d - 5)p + 2m^3 - 3(d-1)m^2 + (2d^2 - 9d + 5)m + (d-1)(d-2)(3d-7)$

$pol_5'(p) = -6p^2 + 6(2m-(d-1))p - 6m^2 + 6(d-1)m - 2d^2 + 9d - 5$
$\Delta pol_5{\prime}(p) = -12d + 144d - 84$ which is strictly negative for $d \geqslant 12$.
Then for $d \geqslant 12$, $pol^5(p)$ is decreasing. As $pol_5(m) > 0$ for $d \geqslant 4$, for $0 \leqslant p \leqslant m, d \geqslant 12, T_p(5,d,m) > 0$.

Then for $a \geqslant 3, d \geqslant 12$, $T_3(a,d,D,M,m) > 0$ then if $d \geqslant 12$, in an optimal solution, $a \leqslant 2$.

Claim 5: If $4 \leqslant d \leqslant 11$ then $a \leqslant 2$.
Now we can compute all other cases : for $4 \leqslant d \leqslant 11, 3 \leqslant a \leqslant 5 + \frac{8}{d-3}$ (implied by claim 2), $1 \leqslant m \leqslant (a+1)(d-1)/2$ (implied by claim 1), $T_3(a,d,D,m,m) > 0$ or $T_2(a,d,D,m,m) > 0$.

Claim 6: In an optimal solution, if $d = 3$ then $a \leqslant 2$ unless for D in $\{4,6,7,9\}$. There exist optimal solutions for $d = 3$, $D = 4,7$ such that $a \leqslant 2$.
If $m = 1$, $T_2(a,3,D,M,1) = 4((a-2)M - 1)$, which is positive for $a \geqslant 3, M \geqslant 2$ and for $a \geqslant 4$, $M \geqslant 1$ then the only case to study is $a = 3$, $M = m = 1$, i.e. $D = 4$. In this case, there are two optimal solutions: one with $h = 2, M = m = 1$ and one with $a = 1$, $M = 2$, $m = 0$.

We denote by T_2^2 the transformation which consists in applying twice the transformation T_2.

If $m \geqslant 2$ and $a \geqslant 4$, applying the transformation T_2^2, a similar computation to that of claim 2 gives $m \geqslant 2a - \frac{13}{2}$. Combining this result with claim 1, that gives $a \leqslant 7$. Then we can compute all other cases, for $d = 3, 4 \leqslant a \leqslant 7, 2 \leqslant m \leqslant a+1$ (implied by claim 2), we obtain :
If $M \geqslant m$, $T_2^2(a,d,M,m) \geqslant T_2^2(a,d,m,m) > 0$ or $T_1(a,d,D,m) > 0$ except for $(a,m) \in \{(3,2),(4,2),(4,3),(5,4)\}$.
If $M \geqslant m+1$, $T_2(a,d,M,m) \geqslant T_2(a,d,m+1,m) > 0$ or $T_1(a,d,D,m) > 0$ except for $a = 5$ and $m = 4$.
If $M \geqslant m+2$, $T_2(a,d,M,m) \geqslant T_2(a,d,m+2,m) > 0$ or $T_1(a,d,D,m) > 0$.
Then we just have to study the optimal solutions in then following cases :

- $a = 3, m = M = 2$, that implies $D = 6$.

  The optimal solution for $d = 3, D = 6$ is with $a = 3$ (5 nodes of weight 2.)

- $a = 4, m = M = 2$, that implies $D = 7$.

  There are two optimal solutions for $d = 3, D = 7$: $a = 4$ (7 nodes of weight 2) or $a = 2$ (2 nodes of weight 3, 2 nodes of weight 2 and 3 nodes of weight 1).

- $a = 4, m = M = 3$, that implies $D = 9$.

  The optimal solution for $d = 3, D = 9$ is with $a = 4$ (7 nodes of weight 3).

- $a = 5, m = M = 4$, that implies $D = 12$.

  The optimal solution for $d = 3, D = 12$ is with $a = 1$;

- $a = 3, m = 2, M = 3$, that implies $D = 8$.

  The optimal solution for $d = 3, D = 8$ is with $a = 1$.

Then we conclude by : for all $d \geqslant 3$, there exists an optimal solution such that $a \leqslant 2$ except for $d = 3, D = 6, 9$. □

2.4 Second Case : $\exists 0 \leqslant i \leqslant k-1, w(B_i) > \lfloor D/2 \rfloor$.

Suppose that there exists a solution S_α for d with some $w(B_i) = \lfloor D/2 \rfloor + \alpha, \alpha \geqslant 1$. Then condition 11 $w(B_i)+w(B_j)+d(i,j) \leqslant D$ implies that the two B_j at distance p of B_i have weight at most $\lceil D/2 \rceil - \alpha - p$ so $val(S_\alpha) \leqslant n(d-1, D, \lfloor D/2 \rfloor + \alpha) + 2\sum_{w=0}^{\lceil D/2 \rceil - \alpha - 1} n(d-1, D, w)$. We will show by induction on d that this value is less than the value of a system S' of the previous case with $M = \lfloor D/2 \rfloor, m = 0$ and $a = 1$ or $a = 2$, according D is even or odd, implying that a solution of the second case can not be optimal.

2.4.1 Case $D = 2q$

It is true for $d = 2$. Indeed $val(S_\alpha) = 2(q+\alpha) + 1 + 2\sum_{w=0}^{q-\alpha-1}(2w+1)$ and for $M = q$ and $a = 1$, $val(S') = n(1, D, q) + 2\sum_{w=0}^{q-1} n(1, D, w) = 2q + 1 + 2\sum_{w=0}^{q-1}(2w+1)$. So $val(S') - val(S_\alpha) = -2\alpha + 2\sum_{w=q-\alpha}^{q-1}(2w+1)$. As $\alpha < q$ ($\alpha = q$ will imply a unique B_i and so a depth $d-1$) and $q > 1$, $val(S') - val(S_\alpha) \geqslant -2\alpha + 2(2q-1) > 0$. Now suppose it is true up to $d-1$, then by induction hypothesis, $n(d-1, D, q+\alpha) \leqslant n(d-1, D, q)$. So $val(S_\alpha) \leqslant n(d-1, D, q) + 2\sum_{w=0}^{q-\alpha-1} n(d-1, D, w)$. As $val(S') = n(d-1, D, q) + 2\sum_{w=0}^{q-1} n(d-1, D, w)$, $val(S') - val(S_\alpha) \geqslant 2n(d-1, D, q-1) > 0$.

2.4.2 Case $D = 2q + 1$

It is true for $d = 2$. Indeed $val(S_\alpha) = 2(q + \alpha) + 1 + 2\sum_{w=0}^{q-\alpha}(2w + 1)$ and for $M = q, a = 2, val(S') = 2(2q + 1) + 2\sum_{w=0}^{q-1}(2w + 1)$. So $val(S') - val(S_\alpha) \geqslant 2q-2\alpha+1$. As $\alpha \leqslant q$, $2q-2\alpha+1 > 0$ and $val(S')-val(S_\alpha) > 0$. Now suppose it is true till $d - 1$, then by induction hypothesis, $val(S_\alpha) \leqslant n(d - 1, D, q) + 2\sum_{w=0}^{q-\alpha-1} n(d - 1, D, w)$. $val(S') = 2n(d - 1, D, q) + 2\sum_{w=0}^{q-1} n(d - 1, D, w)$, $val(S') - val(S_\alpha) \geqslant n(d - 1, D, q) > 0$.

2.5 Results

Now we can state the final results:

Theorem 20

- *For $d = 2$, the optimal solutions are obtained with $M = m = \lfloor \frac{D+1}{4} \rfloor$ plus for $D \equiv 2$ or $3 \mod 4, M = m + 1 = \lfloor \frac{D+3}{4} \rfloor$.*
- *For D even, $d \geqslant 3$, the unique optimal solution satisfies $a = 1, M = D/2, m = 0$ except for $d = 3, D = 4$ where there is another optimal solution with 5 nodes of weight $M = m = 1$ and $d = 3, D = 6$ where the optimal solution consists in 5 nodes of weight $M = m = 2$.*
- *For D odd, $d \geqslant 3$, the unique optimal solution satisfies $a = 2, M = \frac{D-1}{2}$ except for $d = 3, D = 7$ where there is another solution with 7 nodes of weight 2 and $d = 3, D = 9$ where the optimal solution consists in 9 nodes of weight 3.*

Proof. As we have seen in section 2.4 the optimal solution is always obtained in the first case (all $w(B_i) \leqslant \lfloor D/2 \rfloor$). So the theorem follows for $d = 2$ from Proposition 18 and for $d \geqslant 3$ from Proposition 19 by noting that if D is even (resp. D odd), $a = D + 1 - 2M$ being odd (resp. even), $a \leqslant 2$ implies $a = 1$ for D even and $a = 2$ for D odd. As we are in the first case, by Proposition 19, if D is even, we have, except for $d = 3, D \in \{4, 6\}$ that $a \leqslant 2$ but $a = D + 1 - 2M$ being odd, $a = 1$, $M = D/2$ and if m was positive, we could add 2 nodes of weight $m - 1$ obtaining a better solution. $\square$

Thus we have characterized the optimal solution for D even and almost characterized for D odd where we propose the following conjecture :

Conjecture 21 *The optimal value of m in the case D odd, $d \geqslant 3$ is $m = M = \frac{D-1}{2}$ for $D \leqslant 2\left\lfloor \frac{3(d-1)}{4} \right\rfloor + 1$ and $m = \left\lfloor \frac{3(d-1)}{4} \right\rfloor$ for $D \geqslant 2\left\lfloor \frac{3(d-1)}{4} \right\rfloor + 1$ (except for $d = 3, D \in \{7, 9\}$).*

Now using Proposition 9 we are able to give the exact value of $N_d^{HBN}(D)$. Table 1 gives these values for $d \leqslant 9$ and $D \leqslant 12$.

Theorem 22

- *For $d = 2$, let $q = \left\lfloor \frac{D+1}{4} \right\rfloor$ then*

$$N_2^{HBN}(D) = -8q^2 + (2q+1)(2D-1) + 2$$

- *For $d \geqslant 3$ and D even, $D \geqslant 2d$*

$$N_d^{HBN}(D) = n(d, D, D/2) = \sum_{p=0}^{d} 2^p \binom{d}{p} \binom{D/2}{p}$$

- *For $d \geqslant 3$ and $D = 2q+1$ odd, $D \geqslant 2d+1$*

$$\begin{aligned} N_d^{HBN}(D) &= n(d, D, q) + n(d-1, D, q) \\ &+ \max_{O \leqslant m \leqslant 3d/2} \{n(d-1, D, m) - n(d, D, m) + n(d-1, D, m)\} \end{aligned}$$

Proof. If $d = 2$, recall that an optimal solution consists in $2D+1-4\left\lfloor \frac{D+1}{4} \right\rfloor$ nodes of weight $M = \left\lfloor \frac{D+1}{4} \right\rfloor$ and its values is $-8M^2 + (2M+1)(2D-1) + 2$. If $d \geqslant 3$ and D is even, $D \geqslant 2d$, the optimal value is $n(d-1, D, D/2) + 2\sum_{p=0}^{D/2-1} n(d-1, D, p) = n(d, D, D/2) = \sum_{p=0}^{d} 2^p \binom{d}{p} \binom{D/2}{p}$. For $d \geqslant 3$ and $D = 2q+1$ odd, $D \geqslant 2d+1$, the optimal value is $2\sum_{p=m}^{q} n(d-1, D, p) + n(d-1, D, m) = n(d, D, q) + n(d-1, D, q) + n(d-1, D, m) - 2\sum_{p=0}^{m-1} n(d-1, D, p) = n(d, D, q) + n(d-1, D, q) + n(d-1, D, m) - n(d, D, m) + n(d-1, D, m)$. But by claim 1 of Theorem 19, $m \leqslant \frac{3d}{2}$ and so we have to consider the maximum on a finite number values of m. □

Corollary 23 *For given $d \geqslant 2$, $N_d^{HBN}(D) = \frac{D^d}{d!} + O(D^{d-1})$.*

Proof. For $d = 2$ it follows from the exact value given above. For $d \geqslant 3$ and $D \geqslant 2d$, D even $\sum_{p=0}^{d} 2^p \binom{d}{p}\binom{D/2}{p} = O(D^{d-1})$ and $n(d, D, D/2) = 2^d \binom{d}{d} \frac{D^d}{2^d d!} + O(D^{d-1}) = \frac{D^d}{d!} + O(D^{d-1})$. □

The approximations can be done only for D large enough compared to d. Using Stirling formula one can show that if $D \geqslant Ad^2$ the constant before the D^{d-1} is of order 1/A.

The reasoning is similar for D odd as $n(d, D, m)$ is bounded by a value independent of D as $m \leqslant \frac{3d}{2}$.

3 Relation with *HRN*

Now we come back to the first model :

Definition 24 We call depth-0 *ear* a node and for $d \geqslant 1$ a depth-d *ear* of *base* x a set of depth-$(d-1)$ ears and a node x interconnected by a ring called *principal ring*.

Definition 25 Then a *HRN* can be defined as follows : A depth-d *HRN* is a set of depth-$(d-1)$ ears were the bases are interconnected by a ring.

We denote by $e(d, D, w)$ the maximum number of nodes that can be contained in a depth-d diameter-D ear whose base has an eccentricity w.

Proposition 26

$$\begin{aligned}
\forall d \geqslant 0, e(d, D, 0) &= 1 \\
\forall 0 \leqslant w \leqslant D, e(0, D, w) &= 1 \\
\forall d \geqslant 1, \forall 1 \leqslant w \leqslant \lfloor D/2 \rfloor, e(d, D, w) &= 1 + 2\sum_{i=0}^{w-1} e(d-1, D, i) \\
&= e(d, D, w-1) + 2e(d-1, D, w-1)
\end{aligned}$$

Proof. Same as the Proposition 9. □

Proposition 27 *The maximum number of nodes of a depth-d diameter-D* HRN *is*

$$N_d^{HRN}(D) = \max_{0 \leqslant w \leqslant D} \{e(d, D, w) + e(d-1, D, D-w)\} - 1$$

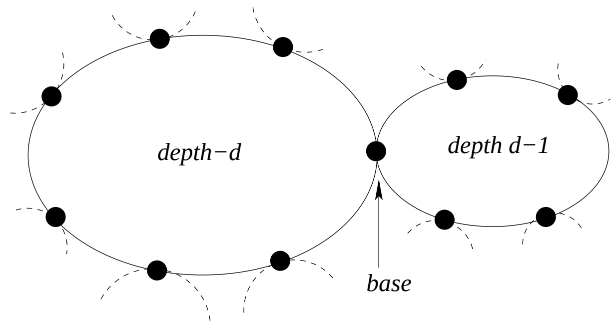

Figure 7: A *HRN* is the union of two ears

Proof. We can decompose every *HRN* in two ears : given a node x of the principal ring, we can extract an ear of depth d and another of depth $d-1$, then the number of nodes is the sum of the number of the two ears minus one as x is counted twice. In the other hand, any *HRN* can be construct with two ears of this kind (see Figure 7). □

Remark 28 $\forall D \geqslant 0, N_1^{HRN}(D) = N_1^{HBN}(D) = 2D+1$, in this case, the two constructions are the same.

Corollary 29 $e(d, D, \lfloor D/2 \rfloor) = n(d, D, \lfloor D/2 \rfloor) + O(D^{d-1})$

Proof. We prove it by induction on d. It's true for $d = 1$.
Suppose that it's true for $d-1$, then $n(d, D, w) - e(d, D, w) =$

$$n(d-1, D, w) + n(d-1, D, w-1) + n(d, D, w-1) - e(d, D, w-1) - 2e(d-1, D, w-1) =$$

$$n(d-1, D, w) - e(d-1, D, w) + n(d-1, D, w-1) - e(d-1, D, w-1) + n(d, D, w-1) - e(d, D, w-1) + e(d-1, D, w) - e(d-1, D, w-1)$$

So $n(d, D, w) - e(d, D, w) =$

$$n(d, D, w-1) - e(d, D, w-1) + e(d-1, D, w) - e(d-1, D, w-1) + O(D^{d-2})$$

Applying this w times, we obtain

$$n(d, D, w) - e(d, D, w) = e(d-1, D, w) + wO(D^{d-2})$$

therefore $n(d, D, \lfloor D/2 \rfloor) - e(d, D, \lfloor D/2 \rfloor) = e(d-1, D, \lfloor D/2 \rfloor) + O(D^{d-1})$
but $e(d-1, D, \lfloor D/2 \rfloor) \leqslant n(d-1, D, \lfloor D/2 \rfloor)$
and so $e(d-1, D, \lfloor D/2 \rfloor) = O(D^{d-1})$. □

Theorem 1 *For a given $d \geqslant 2$, $N_d^{HRN}(D) = \frac{D^d}{d!} + O(D^{d-1})$.*

Proof. By Proposition 27 and Equation (1),

$$\begin{aligned} N_d^{HRN}(D) &\leqslant \max_{0 \leqslant w \leqslant D} \{n(d, D, w) + n(d-1, D, D-w)\} - 1 \\ &\leqslant N_d^{HBN}(D) + N_{d-1}^{HBN}(D) - 1 \end{aligned}$$

and by Theorem 22, $N_d^{HBN}(D) + N_{d-1}^{HBN}(D) - 1 = \frac{D^d}{d!} + O(D^{d-1})$. On the other side $e(d, D, \lfloor D/2 \rfloor) \leqslant N_d^{HRN}(D)$, and by Corollary 29, $e(d, D, \lfloor D/2 \rfloor) = n(d, D, \lfloor D/2 \rfloor) + O(D^{d-1}) = \frac{D^d}{d!} + O(D^{d-1})$. Then $N_d^{HRN}(D) = \frac{D^d}{d!} + O(D^{d-1})$.
□

3.1 Optimal construction of HRN

Theorem 22 gives a tight estimation of the number of nodes of an optimal HRN and Proposition 17 gives a construction of an optimal HBN; but we don't have the optimal construction of a HRN. Using the Proposition 27, to compute $N_d^{HRN}(D)$ (and build the associate construction), we only have to compute (and build the associate construction) $e(d-1, D, w)$ and $e(d, D, w)$ for $w \leqslant D$ and take a pair which gives the maximum. Proposition 26 gives the value of $e(d, D, w)$ if $w \leqslant \lfloor \frac{D}{2} \rfloor$, then we have to find a way to build the ears of weight greater than $\lfloor \frac{D}{2} \rfloor$ to provide the optimal construction. We describe here a way to build these optimal constructions by using both dynamic and linear programming. We want to construct the optimal depth-d ear of diameter D and weight w by assuming that we have the optimal depth-$(d-1)$ ears of diameter D and weight w for $w \leqslant D$. For $d = 1$ an ear is a ring, then the maximum number of nodes for $w \leqslant D$ is $2w + 1$. Given a size k of the principal ring, we denote by w_i for $1 \leqslant i \leqslant k-1$ the weight of the i-th depth-$(d-1)$ ear considering that the base is the 0-th node. We consider two types of constraints between the base of the depth-d ear and the depth-$(d-1)$ ears connected on the principal ring :

- eccentricity constraint:

The distance between the base of the ear and the base of a depth-$(d-1)$ ear plus the weight of this ear has to be less than w :

$$\forall 1 \leqslant i \leqslant k-1, w_i + d(0,i) \leqslant w$$

- diameter constraint:

  similarly as for HBN, we have a diameter constraint between all pair of the depth-$(d-1)$ ears connected to the principal ring :

$$\forall 1 \leqslant i,j \leqslant k-1, w_i + w_j + d(i,j) \leqslant D$$

Then, for k fixed, we have to solve the optimization problem given by these two constraints with the objective function $1+\sum_{1\leqslant i\leqslant k-1} e(d-1,D,w_i)$ which has to be maximized. This can be solved by the following integer program :

$$\begin{array}{rl}
Maximize & 1 + \displaystyle\sum_{1\leqslant i\leqslant k-1} e_i \\
\forall 1 \leqslant i \leqslant k-1, & \displaystyle\sum_{0\leqslant v\leqslant D} X_{i,v} = 1 \\
\forall 1 \leqslant i \leqslant k-1, & e_i = \displaystyle\sum_{0\leqslant v\leqslant D} e(d-1,D,v) \times X_{i,v} \\
\forall 1 \leqslant i \leqslant k-1, & w_i = \displaystyle\sum_{0\leqslant v\leqslant D} v \times X_{i,v} \\
\forall 1 \leqslant i,j \leqslant k-1, & w_i + w_j + d(i,j) \leqslant D \\
\forall 1 \leqslant i \leqslant k-1, & w_i + d(0,i) \leqslant w \\
\forall 1 \leqslant i \leqslant k-1, \forall 0 \leqslant v \leqslant D, & X_{i,v} \in \{0,1\}
\end{array}$$

Then we have to solve this problem for each value of k less or equal than w to get the optimal value of $e(d,D,w)$ and the associate construction. After this, we only have to find w such that $N_d^{HRN}(D) = max_{0\leqslant w\leqslant D}\{e(d,D,w) + e(d-1,D,D-w)\} - 1$. Tables 2 and 3 give the values of $e(d,D,w)$ for $d \in \{6,7\}$ and $w \leqslant D$. Table 4 gives the values of w for which $N_d^{HRN}(D) = e(d,D,w) + e(d-1,D,D-w) - 1$ and Table 5(resp. 1) gives the values of $N_d^{HRN}(D)$(resp. $N_d^{HBN}(D)$) for $d \leqslant 9$, $D \leqslant 12$.

4 Conclusion

In this paper we have improved a result from Aiello, Bhatt, Chung, Rosenberg and Sitaraman by giving tight bounds on the diameter of a Hierarchical Ring Network according to the number of nodes. To do that we have defined another kind of Hierarchical Ring Network for which we give the optimal design according to depth and diameter constraints. This network due to its recursive structure might be more interesting for applications.

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$d \setminus D$	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	3	5	7	9	11	13	15	17	19	21	23	25
2	1	3	5	9	15	21	27	35	45	55	65	77	91
3	1	3	7	15	25	41	65	91	129	175	231	295	377
4	1	3	9	21	41	75	129	201	321	459	681	921	1289
5	1	3	11	27	61	123	231	387	681	1029	1683	2391	3653
6	1	3	13	33	85	183	377	693	1289	2055	3653	5421	8989
7	1	3	15	39	113	255	575	1131	2241	3867	7183	11173	19825
8	1	3	17	45	145	339	833	1725	3649	6723	13073	21549	40081
9	1	3	19	51	181	435	1159	2499	5641	10947	22363	39219	75517

Table 1: Values of $N_d^{HBN}(D)$

$D \setminus w$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	3											
2	1	3	5										
3	1	3	7	9									
4	1	3	9	15	17								
5	1	3	9	19	25	27							
6	1	3	9	27	45	50	53						
7	1	3	9	27	55	73	79	81					
8	1	3	9	27	81	135	153	159	161				
9	1	3	9	27	81	163	217	235	241	243			
10	1	3	9	27	81	243	405	459	476	482	485		
11	1	3	9	27	81	243	487	649	703	721	727	729	
12	1	3	9	27	81	243	729	1150	1313	1367	1384	1391	1393

Table 2: Values of $e(6, D, w)$

$D \setminus w$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	3											
2	1	3	5										
3	1	3	7	9									
4	1	3	9	15	17								
5	1	3	9	19	25	27							
6	1	3	9	27	45	50	53						
7	1	3	9	27	55	73	79	81					
8	1	3	9	27	81	135	153	159	161				
9	1	3	9	27	81	163	217	235	241	243			
10	1	3	9	27	81	243	405	459	476	482	485		
11	1	3	9	27	81	243	487	649	703	721	727	729	
12	1	3	9	27	81	243	729	1215	1377	1430	1448	1455	1457

Table 3: Values of $e(7, D, w)$

$d \setminus D$	0	1	2	3	4	5	6	7	8	9	10	11	12
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	0	0	2	3	4	4	5	6	7	7	8	9
3	0	0	0	0	2	3	3	4	4	6	5	6	6
4	0	0	0	0	0	0	3	4	4	5	5	6	6
5	0	0	0	0	0	0	0	0	4	5	5	10	6
6	0	0	0	0	0	0	0	0	0	0	5	6	6
7	0	0	0	0	0	0	0	0	0	0	0	0	6
8	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 4: Values of w for which $N_d^{HRN}(D) = e(d, D, w) + e(d-1, D, D-w) - 1$

$d \setminus D$	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	3	5	7	9	11	13	15	17	19	21	23	25
2	1	3	5	9	15	21	27	35	45	55	65	77	91
3	1	3	5	9	17	27	45	65	97	133	181	233	305
4	1	3	5	9	17	27	53	81	145	211	341	473	705
5	1	3	5	9	17	27	53	81	161	243	453	665	1137
6	1	3	5	9	17	27	53	81	161	243	485	729	1393
7	1	3	5	9	17	27	53	81	161	243	485	729	1457
8	1	3	5	9	17	27	53	81	161	243	485	729	1457
9	1	3	5	9	17	27	53	81	161	243	485	729	1457

Table 5: Values of $N_d^{HRN}(D)$