On regular packings and coverings

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TO ALEX ROSA ON HIS FIFTIETH BIRTHDAY

ABSTRACT

Let us call a regular packing of K_n with K_k's a set of edge disjoint subgraphs of K_n isomorphic to K_k such that the partial graph G, called the leave of the packing, generated by the edges not covered is regular. Similarly, a regular covering of K_n with K_k's is a set of subgraphs of K_n isomorphic to K_k which cover all the edges of K_n at least once and such that the multigraph H, called the excess of the covering, obtained by deleting K_m from the union of the K_k's, is regular. Here we study maximum regular packings and minimum regular coverings. More exactly we determine the minimum degrees of the leaves and excesses in case of regular packings and coverings of K_n with K_3's and K_4's. We also exhibit minimum regular coverings containing maximum regular packings.

1. Introduction

A classical problem in design theory concerns the existence of (n,k,λ) BIB designs. It is well known that the existence of a $(n,k,1)$ BIB design is equivalent to the existence of a decomposition of the complete graph K_n into K_k's. It is also well known that such a decomposition can exist only if $n(n-1)\equiv 0\,(mod\,k(k-1))$ and $n-1\equiv 0\,(mod\,k-1)$. These conditions have been shown to be sufficient for any k and n large enough (Wilson [18]) and also for $k=3$ (for a survey on Steiner triple systems, see Doyen and Rosa [7]), $k=4$ and $k=5$ (see Hanani [8]). When n does not meet the necessary conditions one can either try to find the maximum number of K_k's, or the maximum cardinality of a packing of K_k's into K_n, (that is the maximum number of K_k's included in K_n), or the minimum number of K_k's (that is the minimum cardinality of a covering of K_n with K_k's (that is the edges of K_n at least once). Such values edge disjoint K_k's necessary to cover all the edges of K_n at least once). Such values have been completely determined for $k=3$ (see Hanani [8]) and for $k=4$ (see Brouwer [5] and Mills [12,13]).

Here we are interested in a slightly different problem that we call regular packing or covering of K_n with K_k's. This problem was initially motivated by a problem of interconnection networks, called bus networks of diameter 1, modeled by hypergraphs. In such a network there are n processors connected to buses. Each bus contains the same number k of processors. For every pair of processors there must exist at least one

bus containing them (diameter 1). These conditions correspond to the classical covering of K_n with K_k's. A further requirement (of regularity) is that each processor belongs to the same number of buses, which gives rise to the notion of regular covering (for details see Bermond, Bond and Sacle [3] and[2]).

A *regular packing* of K_n with K_k's is a set of edge disjoint subgraphs of K_n, isomorphic to K_k such that the partial graph G, generated by the edges not covered, is regular of degree d_G. In the literature G is also called the *leave* of the packing (see [6]). In other words we want to find a regular graph G such that $K_n - G$ can be decomposed into K_k's. Note that a regular packing of K_n with K_k's is just a packing with K_k's such that each vertex of K_n belongs to the same number of K_k's. We shall say that a regular packing is *maximum* if d_G is as small as possible.

Similarly, a *regular covering* of K_n with K_k's is a set of subgraphs of K_n, isomorphic to K_k which cover all the edges of K_n, at least once such that the multigraph H, obtained by deleting K_n from the union of the K_k's, is regular of degree d_H. The multigraph H is also called the *excess* of the covering. In other words we want to find a regular multigraph H of degree d_H such that $K_n + H$ can be decomposed into K_k's. We shall say that a regular covering is *minimum* if d_H is as small as possible.

Furthermore we are interested in finding, when it is possible, a minimum regular covering of K_n with K_k's containing a maximum regular packing. That is equivalent to find whether there exist graphs G, d_G-regular, and H, d_H-regular, with d_G and d_H minimum, associated respectively to the packing and covering such that the $(d_G + d_H)$-regular graph $G + H$ can itself be decomposed into K_k's. In other words, for that purpose, it is sufficient to find a maximum regular packing such that the d_G-regular graph G can be covered by K_k's in such a way that every vertex belongs to $(d_G + d_H)/(k-1)$ graphs K_k's (obviously a covering of a subgraph G of K_n with K_k's is a set of K_k's containing all the edges of G).

The following proposition is Immediate

Proposition 1.1

The degree d_G of the regular leave G of a regular packing of K_n with K_k's satisfies the following equalities:

$$n-1-d_G \equiv 0 \pmod{k-1}$$
$$n(n-1-d_G) \equiv 0 \pmod{k(k-1)}.$$

The degree d_H of the regular excess H of a regular covering of K_n with K_k's satisfies the following equalities:

$$n-1+d_H \equiv 0 \pmod{k-1}$$
$$n(n-1+d_H) \equiv 0 \pmod{k(k-1)}$$

From that proposition, one can easily deduce lower bounds on d_G and d_H.

Here we show that these bounds are always reached in the cases $k = 3$ or 4 by constructing maximum regular packings and minimum regular coverings of K_n with K_3's and K_4's for any n (except for $n = 8$ in case of a packing with K_4's). In most of the cases the covering constructed contains the packing.

Let us remark that in some cases the optimum packings or coverings given in the literature ([5,13,14]) are necessarily regular (by counting arguments). However as we want, if possible, to give a covering containing the packing, we give the construction again.

2. Notation and basic lemmas

• K_n will denote the complete graph on n vertices. Usually $V(K_n)$, the vertex set of K_n, will be Z_n, the group of integers modulo n.
• $[x_1, x_2, \ldots, x_r]$ will denote the complete graph on the vertices $x_1, x_2, \ldots, x_r$. In case $r = 2$ it denotes an edge.
• $[x_1, x_2, \ldots, x_r] \pmod n$ will denote $\{[x_1 + i, x_2 + i, \ldots, x_r + i] : i = 0, 1, \ldots, n-1\}$.
• $K_n - G$ denotes the partial graph obtained from K_n by deleting the edges of G from K_n.
• If G and H are graphs on the same set of vertices V, $G + H$ denotes the multigraph with vertex set V containing the edges of G plus the edges of H (with eventually multiple edges).
• K_{r_1, r_2} denotes the complete bipartite graph with vertex sets of cardinality r_1 and r_2.
• $K_{r_1, r_2, \ldots, r_h}$ denotes the complete multipartite graph. If $r_1 = r_2 = \ldots = r_h = r$ then we denote $K_{r_1, r_2, \ldots, r_h}$ by $K_{h \times r}$.
• Let G be a graph on k vertices. A *parallel class* of G's on $n = pk$ vertices is the vertex disjoint union of p copies of G.

It is well known that the existence of a decomposition of $K_{h \times r}$ into K_k's is equivalent to the existence of $h-2$ orthogonal Latin squares of order r (or also to what is called a $GD(k, 1, r; hr)$ or $T(1, r)$ design). Classical results are the following.

Lemma 2.1 (see Hanani [8])

$K_{h \times r}$ can be decomposed into K_3's if and only if
$$\begin{cases} (h-1)r \equiv 0 \pmod 2 \\ hr^2(h-1) \equiv 0 \pmod 6 \\ h \geq 3 \end{cases}$$

Lemma 2.2 (see Brouwer, Hanani, Schrijver [4])

$K_{h \times r}$ can be decomposed into K_4's if and only if
$$\begin{cases} (h-1)r \equiv 0 \pmod 3 \\ hr^2(h-1) \equiv 0 \pmod{12} \\ h \geq 4, (h, r) \neq (4,2) \text{ or } (4,6) \end{cases}$$

Lemma 2.3 (see Bermond, Huang, Rosa, Sotteau [1])

If $K_{r,r,r}$ and $K_{r,r,r,r}$ can be decomposed into graphs G, then $K_{pr,pr,pr,qr}$ can be decomposed into G for $p \neq 2, 6, 0 \leq q \leq p$ (and if $q = 0$ for any p).

If $K_{r,r,r,r}$ can be decomposed into graphs G, then $K_{pr,pr,pr,pr,qr}$ can be decomposed into G for $p \neq 2, 3, 6, 10$ and $0 \leq q \leq p$.

In [10] Huang, Mendelsohn and Rosa introduce a problem which can be formulated as that of the existence of a decomposition of K_n into graphs all isomorphic to K_k except one isomorphic to K_r. They give the necessary conditions.

The following results have been proved on this problem, which will be used in the proof of the main theorem.

Lemma 2.4 (Brouwer [5])

K_n can be decomposed into one K_7 and K_4's if and only if $n \equiv 7$ or $10 \pmod{12}$, $n \neq 10, 19$.

Lemma 2.5 (Bermond, Bond [3])

K_n can be decomposed into one K_{10} and K_4's if and only if $n \equiv 7$ or $10 \pmod{12}$, $n \neq 7, 19, 22$.

In the section 5 we study a more particular case of this problem, since we want to have the additional property that there exists, in the decomposition, a parallel class of K_k's on $n - r$ vertices.

3. Main results

Theorem 3.1

If $n \geq 3$ there exist a maximum regular packing and a minimum regular covering of K_n with K_3's where the graph G and multigraph H are as follows.

o)	if $n \equiv 0$	(mod 6),	G is 1-regular and H is 1-regular
i)	if $n \equiv 1$	(mod 6),	G and H are empty
ii)	if $n \equiv 2$	(mod 6),	G is 1-regular and H is 5-regular
iii)	if $n \equiv 3$	(mod 6),	G and H are empty
iv)	if $n \equiv 4$	(mod 6),	G is 3-regular and H is 3-regular
v)	if $n \equiv 5$	(mod 6),	G is 4-regular and H is 2-regular

Moreover there exists a minimum covering containing a maximum packing in all cases except case o)

Proof

First, from Proposition 1.1, the given values of the degrees of G and H are lower bounds. Now we will prove that these bound are reached.

Case o) ($n = 6t$, $t \geq 1$):
Since K_{6t+1} can be decomposed into K_3's, we get a packing of K_n by deleting a vertex and all the K_3's containing it. The leave of the packing is a perfect matching.

It is known (see Mills [14] for a survey on the subject) that K_n can be covered with

$$\left\lceil n/3 \left\lceil (n-1)/2 \right\rceil \right\rceil K_3's.$$

In this case the covering is regular and the excess H is a perfect matching.

Obviously in this case it is not possible that the minimum covering contains the maximum packing since G and H are necessarily perfect matchings and the sum of two perfect matchings cannot be decomposed into K_3's.

Case i) ($n = 6t+1$, $t \geq 1$) and **case iii)** ($n = 6t+3$, $t \geq 0$):
It is known that, in these cases, the graph K_n can be decomposed into K_3's.

Case ii) ($n = 6t+2$, $t \geq 1$):
The packing is obtained exactly as in o), and the leave G is a perfect matching. A regular covering of G with each vertex in three K_3's is obtained by Corollary 4.2 applied to the 3-regular graph formed by the sum of G and any 2-regular graph on the same set of vertices.

Case iv) ($n = 6t+4$, $t \geq 0$):
We distinguish two subcases:

i) $t = 2r$, $r \geq 0$
If $r = 0$ then $n = 4$, $G = K_4$ is itself 3-regular. We can take $H = K_4$ since the multigraph formed by two identical K_4's can be decomposed into K_3's, $[0,1,2] \pmod 3$.
If $r \geq 1$, K_{12r+4} is the edge disjoint union of $3r+1$ vertex disjoint graphs K_4's and a $K_{(3r+1)\times 4}$. By Lemma 2.1 $K_{(3r+1)\times 4}$ can be decomposed into K_3's. We shall take for the excess H the same graph. G consists of the $3r+1$ vertex disjoint K_4's. As the multigraph formed by two identical K_4's can be decomposed into K_3's, $G + H$ can also be decomposed into K_3's.

ii) $t = 2r+1$, $r \geq 0$
If $r = 0$ then $n = 10$. A maximum regular packing consists of the following K_3's:

[0,3,9], [1,4,5], [2,0,6], [3,1,7], [4,2,8],
[0,7,8], [1,8,9], [2,9,5], [3,5,6], [4,6,7].

The leave G is the Petersen graph which is 3-regular. From Corollary 4.2 it can be covered with each vertex in three K_3's. So we have a covering of K_{10} containing the maximum packing where the excess H is a 3-regular multigraph.
If $r = 1$ then $n = 22$. The packing is given by the K_3's:

[0,1,3], [0,4,10] and [0,5,13] (mod 22).

The leave G is 3-regular and consists of the edges [0,7] and [0,11] (mod 22). It is the union of a hamiltonian cycle and a perfect matching of the edges [0,7] and [0,11] (mod 22), it can be covered with each vertex in three K_3's. Thus we have a covering of K_{22} containing the packing with an excess H which is 3-regular.
If $r \geq 2$, K_{12r+10} can be decomposed into r graphs K_{12}'s on vertex sets X_i, $1 \leq i \leq r$, one K_{10} on vertex set Y and a $K_{12,\ldots,12,10}$ on vertex sets $\bigcup X_i \cup Y$.

From above the K_{10} can be decomposed into K_3's plus the 3-regular Petersen graph.

From case o) each K_{12} on vertex set X_i can be decomposed into K_3's plus a perfect matching M_i for $1 \leq i \leq r$.

Now $K_{12,...,12,10}$ can be obtained from the multipartite graph $K_{(r+1)\times12}$ by deleting two vertices in the same part. As from Lemma 2.1, $K_{(r+1)\times12}$ can be decomposed into K_3's, the graph $K_{12,...,12,10}$ can be decomposed into K_3's plus two perfect matchings M and M' on $\bigcup X_i$. Now we have a regular packing of K_{12r+10} where the leave G is the 3-regular graph on $12r+10$ vertices which consists of two parts: one is the Petersen graph on 10 vertices, the other one is on $12r$ vertices: $\{\bigcup M_i + M + M'\}$. According to Corollary 4.2, both parts can be covered by K_3's in such a way that each vertex belongs to three K_3's.

Thus we have a covering of K_{12r+10} containing the packing with an excess H which is 3-regular.

Case v) ($n=6t+5, t\geq0$): If $t=0$ then $n=5$. We have $G=K_5$ and $H=C_5$, by covering G with the K_3's [0,1,2] (mod 5). If $t=1$ then $n=11$. The packing is given by the K_3's [0,1,3] (mod 11). The leave G is the 4-regular formed by the edges [0,4] and [0,5] (mod 11). We obtain the covering by adding the K_3's [0,1,5] (mod 11). Note that the excess H is a hamiltonian cycle. If $t\geq2$, one K_5 is the edge disjoint union of t vertex disjoint K_6's, on vertex sets X_i, $1\leq i \leq t$, one K_5 on vertex set Y and a multipartite graph $K_{6,...,6,5}$. According to Lemma 2.1, there exists a decomposition of $K_{(t+1)\times6}$ into K_3's. By deleting one vertex we obtain a decomposition of $K_{6,...,6,5}$ into K_3's and a perfect matching M on the vertex set $\bigcup X_i$.

Each K_6 on vertex set X_i is the edge disjoint union of two vertex disjoint K_3's and a 3-regular graph which consists of a perfect matching M_i and a cycle of length six C_6^i. K_5 is itself a 4-regular graph on Y. So we have a maximum regular packing of K_{6t+5} with K_3's, where the leave G is the 4-regular graph which consists of two parts: one is $\{\bigcup M_i + M + \bigcup C_6^i\}$, on vertex set $\bigcup X_i$, the other one is a K_5, on vertex set Y. Both parts can be covered with each vertex in three K_3's, the K_5 as in case $n=5$ and the other one as shown in Corollary 4.3.

Thus we have a covering of K_{6t+5} containing the packing with an excess H which is 2-regular. $\square$

Remark 3.2

The referee suggested to mention the following conjecture which seems very difficult.

Conjecture

For every given degree d, all d-regular graphs meeting the necessary conditions are leaves of a packing of K_n with K_3's, with finitely many exceptions.

Note that the case of hamiltonian cycles was solved by E. Mendelsohn [11] and the general case of any 2-regular graph is a particular case of a result of Colbourn and Rosa [6] also obtained by Hilton and Rodgers [9].

Note that the above conjecture is in fact a special case of the following one of Nash-Williams [15].

Conjecture (Nash-Williams)

If G is a graph of order n, $n \geq 15$, such that its number of edges is a multiple of 3, and every vertex has an even degree at least $\frac{3n}{4}$, then G can be decomposed into K_3's.

Remark 3.3

K. Heinrich drew our attention to the following strong recent result of Rees [17] which can be restated as follows:

Theorem (Rees)

K_{6t} can be decomposed into α perfect matchings and β parallel classes of K_3's if and only if $\alpha+2\beta=6t-1$ except for $\alpha=1$ and $t=1$ or $t=2$ (corresponding to the non-existence of Nearly Kirkman triple systems of order 6 or 12).

Using this theorem we can give a shorter proof of the existence of maximum regular packings for the non immediate cases $n=6t+u$ ($u=4,5$) in the following way. Let $V(K_n) = X\bigcup Y$ with $|X|=6t$, $|Y|=u$. By the theorem above the K_{6t} on X can be decomposed into $2u-1$ perfect matchings on X and $3t-u$ parallel classes of K_3's. Then K_n can be decomposed into the K_u on Y, $u-1$ perfect matchings on X and the $3tu$ triples obtained by joining any vertex of Y to a perfect matching of X. Thus we have a regular packing of K_n with K_3's for which the leave G consists of the K_u on Y plus the $u-1$ perfect matchings on X and is clearly a regular graph of degree $u-1$.

In fact we used only a corollary of the result of Rees. We needed only to decompose K_{6t} into perfect matchings and triangles (Note that such a decomposition solves the conjecture mentioned in Remark 3.2, when the graph is the union of d perfect matchings).

Theorem 3.4

If $n \geq 4$, there exist a maximum regular packing and a minimum regular covering of K_n with K_4's where the leave G and excess H are as follows.

o)	if $n \equiv 0$	(mod 12),	G is 2-regular and H is 1-regular
i)	if $n \equiv 1$	(mod 12),	G and H are empty
ii)	if $n \equiv 2$	(mod 12),	G is 1-regular and H is 5-regular
iii)	if $n \equiv 3$	(mod 12),	G is 2-regular and H is 10-regular
iv)	if $n \equiv 4$	(mod 12),	G and H are empty
v)	if $n \equiv 5$	(mod 12),	G is 4-regular and H is 8-regular
vi)	if $n \equiv 6$	(mod 12),	G is 5-regular and H is 1-regular
vii)	if $n \equiv 7$	(mod 12),	G is 6-regular and H is 6-regular
viii)	if $n \equiv 8$	(mod 12),	G is 1-regular and H is 2-regular
ix)	if $n \equiv 9$	(mod 12),	G is 8-regular and H is 4-regular
x)	if $n \equiv 10$	(mod 12),	G is 3-regular and H is 3-regular
xi)	if $n \equiv 11$	(mod 12),	G is 10-regular and H is 2-regular

except for $n=8$ where G is 4-regular.

Moreover there exists a minimum regular covering which contains a maximum regular packing in all cases except o) and xi) and $n=18$ (where we don't know whether it exists).

Proof:

First, from Proposition 1.1, the given values of the degrees of G and H are lower bounds (except for $n=8$). Now we will prove that these bounds are reached.

case o) ($n=12t$, $t \geq 1$) :

We obtain a decomposition of K_{12t} into K_4's and a parallel class of K_3's by deleting one vertex and the edges containing it from a decomposition of K_{12t+1} into K_4's. Thus we have a maximum regular packing with a leave G which is 2-regular. A union of K_3's cannot be covered with K_4's with each vertex in only one K_4. In order to be able to find a minimum covering containing the minimum packing we would need to have a decomposition of K_{12t} into K_4's and a parallel class of cycles of length four. Such a decomposition can not exist for $n=12$. Maybe it does for bigger n, and we can state the following problem.

Problem : Does there exist a decomposition of K_{12t} into K_4's and a parallel class of C_4's for $t \geq 2$?

However it is known that there exists a maximum regular packing of K_{12t} with K_4's where the excess H is a perfect matching. Indeed it has been proved in [12] that there exists a covering of K_{12t} with $12t^2$ graphs K_4's. Since each vertex belongs to $4t$ K_4's, the excess H is a perfect matching and the covering is regular.

case i) ($n=12t+1$, $t \geq 1$) and **case iv)** ($n=12t+4$, $t \geq 0$) :

It is known that, in these cases, the graph K_n can be decomposed into K_4's.

case ii) ($n=12t+2$, $t \geq 1$) :

From Lemma 2.2, $K_{(6t+1)\times 2}$ is the edge disjoint union of a perfect matching M and a multipartite graph $K_{(6t+1)\times 2}$ can be decomposed into K_4's. Let us take

$G = M = \{[2i, 2i+1], 0 \leq i \leq 6t\}$. We have a covering of G with each vertex in two K_4's by taking $\{[2i, 2i+1, 2i+2, 2i+3], 0 \leq i \leq 6t\}$.
So we have a covering of K_{12t+2} containing the packing where the excess H is a 5-regular multigraph.

case iii) ($n=12t+3$, $t \geq 1$) :

K_{12t+3}. By Lemma 2.2, $K_{(4t+1)\times 3}$ can be decomposed into K_4's and a multipartite graph $K_{(4t+1)\times 3}$. By Lemma 2.2, $K_{(4t+1)\times 3}$ can be decomposed into K_4's. Let us take for G the parallel class of K_3's which is a 2-regular graph. By Corollary 4.4 in section 4, we know that any 6-regular graph that can be decomposed into K_3's can be covered by K_4's with each vertex in four K_4's. We apply this result with the 6-regular graph formed by G and any two other parallel classes of K_3's.
So we have a covering of K_{12t+3} containing the packing where the excess H is a 10-regular multigraph.

case v) ($n=12t+5$, $t \geq 0$) :

First we prove the result for $n=5$ and 17.
If $n=5$ then $G=K_5$ and H is the 8-regular multigraph formed by two copies of K_5.
If $n=17$, the packing is given by the following K_4's: $[0,1,4,6]$ (mod 17). The leave G is formed by the edges $[0,7]$ and $[0,8]$ (mod 17). It is the union of two hamiltonian cycles and can be covered with each vertex in four K_4's by $[0,2,5,9]$ (mod 17).
Now let $n=12t+5$, with $t \geq 2$. By Lemma 2.4, K_{12t+7} can be decomposed into one K_7 and K_4's, if $t \neq 1$. We take such a decomposition and delete two vertices of the K_7. We obtain a decomposition of K_{12t+5} into a K_5, two parallel classes of K_3's (on the K_5 and the 12t other vertices) and K_4's. So the leave G we obtain consists of two parts, one is a 4-regular graph on 12t vertices, which is the union of two hamiltonian cycles and can be covered with each vertex in four K_4's, the other one is the K_5. We apply this result to the 6-regular graph formed by the part of G on 12t vertices and any other parallel class of K_3's on the same set of vertices.
So we have a covering of K_{12t+5} containing the packing where the excess H is 8-regular multigraph.

case vi) ($n=12t+6$, $t \geq 0$) :

First we prove the result for $n=6$ and 18.
If $n=6$ then $G=K_6$ is itself 5-regular and H is a perfect matching. The covering is given by the K_4's: $[0,1,2,3]$, $[0,1,4,5]$, $[2,3,4,5]$.
If $n=18$, then a regular packing can be obtained by taking $[0,1,3,8]$ (mod 18). An optimal covering (which is regular in that case) is given in [12] by Mills. We note that this covering does not contain any maximum regular packing and we don't know if such a covering does exist.
Now let $n=12t+6$, with $t \geq 2$. The graph K_{12t+6} is the edge disjoint union of a parallel class of K_6 and a multipartite graph $K_{(2t+1)\times 6}$. By Lemma 2.2, $K_{(2t+1)\times 6}$ can be decomposed into K_4's. Therefore we obtain a leave G which is a parallel class of K_6's decomposed into K_4's as in the case $n=6$ and the excess H is a perfect matching.
We cover each K_6 of G as in the case $n=6$.

case vii) (n=12t+7, t≥0):

First we prove the result for the cases n=7, 19.

If n=7 then G=K_7 is itself a 6-regular graph. We can take H=K_7 since the multi-graph formed by two copies of the same K_7 can be decomposed into K_4's as follows: [0,1,2,4] (mod 7).

If n=19, a maximum packing is given by [0,1,3,8] (mod 19) and the leave is a 6-regular graph G which is an edge disjoint union of K_3's : [0,4,10] (mod 19). So by Corollary 4.4, G can be covered by K_4's with each vertex in four K_4's.

Now let n=12t+7, with t≥2. By Lemma 2.5, K_{12t+10} can be decomposed into one K_10 and K_7's, if t≠1. We take such a decomposition and delete three vertices of the K_10. We obtain a decomposition of K_{12t+7} into a K_7, three parallel classes of K_3's (on the union of a K_7 vertices) and K_4's. So the leave G we obtain consists of the vertex disjoint K_3's.

The K_7 can be covered as in the case n=7. By Corollary 4.4 in section 4, the part of G which is 6-regular on 12t vertices can be covered by K_4's, with each vertex in four K_4's.

So we have a covering of K_{12t+7} containing the packing where the excess H is a 6-regular multigraph.

case viii) (n=12t+8, t≥0):

First we prove the result for the case n=8.

If n=8 then there exists no regular packing of K_8 with K_4's with a leave G 1-regular. Indeed, if such a packing exists K_8 − G should be decomposed into four K_4's with each vertex in two K_4's which is easily seen to be impossible. A maximum regular packing of K_8 is obtained with a 4-regular leave G by taking two disjoint K_4's.

A minimum regular covering of K_8 with a 2-regular excess H is given by the following K_4's (found in [13]):

[1,2,3,4]|[1,2,5,6]|[1,2,7,8]|[3,4,5,6]|[3,4,7,8]|[5,6,7,8]

This covering contains the maximum regular packing since it contains two vertex disjoint K_4's.

Now let n=12t+8, with t≥1. The graph K_{12t+8} is the disjoint union of a perfect matching and a multipartite graph K_{(6t+4)×2}. From Lemma 2.2, if t≥1, K_{(6t+4)×2} can be decomposed into K_4's. Let G be the matching {[2i,2i+1] : i=0,1,...,6t+3}. It can be covered with each vertex in one K_4 with the following K_4's : {[4i, 4i+1, 4i+2, 4i+3] : i=0,1,...,3t+1}. Note that the excess H is the vertex disjoint union of cycles of length 4.

case ix) (n=12t+9, t≥0):

It is well known that K_{12t+13} can be decomposed into K_4's. If we choose a K_4 and delete its four vertices, we get a decomposition of K_{12t+9} into four parallel classes of K_3's and K_4's. So the leave G is the 8-regular graph formed by four parallel classes of K_3's.

From Corollary 4.5, we have a covering of G with each vertex in four K_4's.

So we have a covering of K_{12t+9} containing the packing where the excess H is a 4-reg-ular multigraph.

case x) (n=12t+10, t≥0):

If t=0 then n=10. The packing consists of the following K_4's:

[0,2,8,9], [1,3,9,5], [2,4,5,6], [3,0,6,7], [4,1,7,8]

The leave G is the Petersen graph which is a 3-regular graph.

In order to cover G with each vertex in two K_4's, we add the following K_4's:

[0,1,5,7], [1,2,6,8], [2,3,7,9], [3,4,8,5], [4,0,9,6]

If t=1 then n=22 . Let V(K_{22}) = Z_2 × Z_{11}. A maximum regular packing is given by the following K_4's:

$$\{[(0,i),(0,i+1),(0,i+3),(1,i)]|[(0,i+9),(1,i),(1,i+1),(1,i+5)],$$
$$[(0,i),(0,i+5),(1,i+6),(1,i+9)] : 0\leq i \leq 10\}.$$

where each vertex is in two K_4's.

If t≥2, the graph K_{12t+10} is the edge disjoint union of one K_10, 3t vertex disjoint K_4's and a multipartite graph K_{4,4,...,4,10}.

As t≥2, by Theorem 5.1, which will be proved in section 5, there exists a decomposition of the multipartite graph K_{4,4,...,4,10} into K_4's since the number of 4-sets is equal to 3t. From the case n=10, the graph K_10 is the edge disjoint union of a Petersen graph and a Petersen graph. So the leave G is the vertex disjoint union of a Petersen graph and a parallel class of K_4's.

In order to obtain a minimum regular covering (with an excess H 3-regular) containing the packing given above we add two parallel classes of K_4's on 12t points and the K_4's we used to cover the Petersen graph in K_10.

case xi) (n=12t+11 t≥0):

First we prove the result for t=0,1,2.

If t=0 then G=K_{11} is itself 10-regular. A minimum covering is given by [0,1,4,6] (mod 11). Note that the excess H is a hamiltonian cycle.

If t=1 then n=23. A maximum regular packing is obtained by taking the following K_4's : [0,1,7,21] (mod 23). In order to cover G with each vertex in four K_4's we add the following ones : [0,1,11,19] (mod 23). Note that the excess H is again a hamiltonian cycle.

If t=2 then n=35 . A maximum regular packing is given by : [0,1,4,14] and [0,7,9,15] (mod 35). A covering containing it is obtained by adding the following K_4's : [0,1,12,17] (mod 35). Note that the excess H is a hamiltonian cycle.

Now let n=12t+11, with t≥3, the graph K_{12t+11} is the edge disjoint union of t vertex disjoint K_{12}'s, one K_{11} and the multipartite graph K_{12,12,...,12,11}. From Lemma 2.2, the graph K_{(t+1)×12} can be decomposed into K_4's for t≥3. Therefore K_{12,12,...,12,11} can be decomposed into one parallel class of K_3's on 12t vertices and K_4's. Each K_{12} is the union of three vertex disjoint K_4's and an 8-regular graph. Let us take for leave G the 10-regular graph which consists of two parts, one is formed on 12t vertices by these 8-regular graphs plus the parallel class of K_3's, the other one is the K_{11}. Thus we have a maximum regular packing.

The existence of a minimum regular covering has been proved in [13] where it is proved that there exists a covering with (t+1)(12t+11) K_4's with each vertex in 4t+4

graphs K_4's. Therefore the excess H is 2-regular and the covering is regular minimum. Notice that in this last case we don't know whether there exists a minimum regular covering which contains a minimum regular packing. □

Remark 3.5

The proof of the theorem suggests a lot of decomposition problems. The most general one would be a problem analogous to the conjecture of remark 3.2, i.e. that all d-regular graphs meeting the necessary conditions are leaves (respectively excesses) of a regular packing (respectively covering) of K_n with K_4's. First interesting cases are when the leaves or excesses are hamiltonian cycles or more generally regular graphs of degree two, which generalize Mendelsohn's [11] or Colbourn and Rosa's [6] result. For example we state the following conjectures.

Conjectures

K_{12t} *minus a hamiltonian cycle*
K_{12t+3} *minus a hamiltonian cycle*   *can be decomposed into* K_4's
K_{12t+8} *plus a hamiltonian cycle*
K_{12t+11} *plus a hamiltonian cycle*

One can also ask for a generalization of Rees' theorem.

Conjecture

K_{12t} *can be decomposed into* α *parallel classes of* K_3's *and* β *parallel classes of* K_4's *if and only if* $2\alpha+3\beta=12t-1$ *(with a finite number of exceptions).*

If we don't want parallel classes of K_4's we obtain the problem introduced by Huang, Mendelsohn and Rosa [10]. A related problem is considered in section 5.

Note that the problem of decomposing K_n into α perfect matchings and β parallel classes of K_4's for $n\equiv 0 \pmod 4$ is easy to solve (essentially because a parallel class of K_4's is the union of 3 perfect matchings).

4. Some lemmas

We state here some lemmas used in the proofs of the main theorem. The results are not necessarily best possible, and they are given in the form used in section 3.

Lemma 4.1 :

i) *Let G be an $r(r-1)$-regular graph on n vertices that can be decomposed into K_r's. Then G can be covered with K_{r+1}'s, in such a way that each vertex belongs to $r+1$ graphs K_{r+1}'s.*

ii) *Let G be an $r(r-1)+1$-regular graph on n vertices that can be covered with K_{r+1}'s, in such a way that each vertex belongs to $r+1$ graphs K_{r+1}'s.*

iii) *Let G be an $r(r-1)+2$-regular graph on n vertices that can be decomposed into K_r's and a 2-regular graph. Then G can be covered with K_{r+1}'s, in such a way that each vertex belongs to $r+1$ graphs K_{r+1}'s.*

Proof

First we remark that in case iii) the 2-regular graph is the vertex disjoint union of cycles, and one can give an orientation to the cycles so that every vertex is the origin of only one arc. Any vertex of G is contained in r graphs K_r's of the decomposition, so the total number of K_r's is n. Let us add one vertex of G to each of the K_r's in order to transform it into a K_{r+1}, with each vertex of G added exactly once. Obviously after that transformation we have n graphs K_{r+1}'s, with each vertex of G belonging to $r+1$ of them. In order to do that and to have all the edges of G covered we need a bijection f between $V(G)$ and the set of the n K_r's, such that for any vertex x of G:

- in case i): x does not belong to $f(x)$.
- in case ii): if $[x,y]$ is an edge of the matching, then $y \in f(x)$.
- in case iii): if $[x,y]$ is the arc having x as origin in the 2-regular oriented partial graph of G, then $y \in f(x)$.

In order to show that such a bijection exists we define a bipartite graph, having as stable sets of vertices the K_r's and the vertices of G. We put an edge between a vertex of G and a K_r, if the vertex can be added to the K_r. The result is a regular bipartite graph (of degree $n-r$, r and r respectively in cases i), ii) and iii)). A well-known corollary of the König-Hall theorem states that a regular bipartite graph admits a perfect matching. The matching defines the bijection we need. □

We state now some corollaries of this lemma, which are used in section 3, in these terms.

Corollary 4.2 :

Let G be a 3-regular graph containing a perfect matching. Then G can be covered by K_3's, in such a way that each vertex belongs to three K_3's.

Corollary 4.3 :

Let G be a 4-regular graph containing two disjoint perfect matchings. Then G can be covered by K_3's, in such a way that each vertex belongs to three K_3's.

Corollary 4.4 :
Let G be a 6-regular graph that can be decomposed into K_3's. Then G can be covered by K_4's, in such a way that each vertex belongs to four K_4's.

Corollary 4.5 :
Let G be an 8-regular graph that can be decomposed into K_3's and a 2-regular graph. Then G can be covered by K_4's, in such a way that each vertex belongs to four K_4's.

5. A multipartite graph decomposition problem

We propose to study the following problem:

Problem
For what values of h and r can the multipartite graph $K_{4,4,...,4,r}$ on $4h+r$ vertices be decomposed into K_4's?

If the multipartite graph $K_{4,4,...,4,r}$ on $4h+r$ vertices can be decomposed into K_4's then necessarily

$$\begin{cases} h \equiv 0 \pmod 3 \\ r \equiv 1 \pmod 3 \\ h \geq r/2 + 1 \end{cases}$$

We obtain the first two necessary conditions by divisibility arguments and the third one by counting the edges covered by the K_4's containing an element of the r-set.

We can remark that Lemma 2.2 proves that the above necessary conditions are sufficient for the case r = 4 (this case can also be seen as a corollary of the well known theorem of Hanani [8], which says that K_n can be decomposed into parallel classes of K_4's if and only if $n \equiv 4 \pmod{12}$).

Here we will prove that the above necessary conditions are sufficient for the case r = 10.

We conjecture that the necessary conditions are sufficient in general, except eventually for a few values (for example it can be shown that $K_{4,4,1}$ cannot be decomposed into K_4's).

Theorem 5.1
$K_{4,4,...,4,10}$ can be decomposed into K_4's if and only if h, the number of 4-vertex parts, satisfies

$$\begin{cases} h \equiv 0 \pmod 3 \\ h \geq 6 \end{cases}$$

or equivalently

K_n can be decomposed into one K_{10}, a parallel class of K_4's on the other n−10 vertices and K_4's if and only if $n \equiv 10 \pmod{12}$, $n \neq 22$. or also,

K_n can be decomposed into a parallel class of K_4's, 10 parallel classes of K_3's and K_4's if and only if $n \equiv 0 \pmod{12}$, $n \neq 12$.

In what follows we will use the three alternative forms.
One can easily see, on the third formulation of the theorem, that it is a refinement of Lemma 2.5.

To prove the theorem we will need some composition lemmas, for which we will use the following remark:

Remark 5.2
$K_{t,t,t,t}$ can be decomposed into parallel classes of K_4's, for $t \neq 2,3,6,10$.

Proof
It is well known that $K_{t,t,t,t,t}$ can be decomposed into K_5's if and only if $t \neq 2,3,6,10$ (see [8]). Given such a decomposition we can delete the five vertices of a stable set of $K_{t,t,t,t,t}$. Each deleted vertex gives rise to a parallel class of K_4's in $K_{t,t,t,t}$. □

We have the following composition lemmas.

Lemma 5.3
Let $t \geq u$, if K_{12t} and K_{12u} can be decomposed into one parallel class of K_4's, 10 parallel classes of K_3's and K_4's then so do K_{48t} and $K_{48t+12u}$.

Proof
The graph $K_{48t+12u}$ is the edge disjoint union of four K_{12t}'s, one K_{12u} and the multipartite graph $K_{12t,12t,12t,12u}$. The K_{12t}'s and the K_{12u} can be decomposed into 10 parallel classes of K_3, one parallel class of K_4 and K_4's by hypothesis. By Lemma 2.2, $K_{4\times12}$ and $K_{5\times12}$ can be decomposed into K_4's. Therefore, according to Lemma 2.3, $K_{12t,12t,12t,12u}$ can be decomposed into K_4's for $u \leq t$. Therefore K_{48t+36} can be decomposed into 10 parallel classes of K_3's, one parallel class of K_4's, and K_4's. □

Lemma 5.4

If K_{12t+9} can be decomposed into 10 parallel classes of K_3's and K_4's then K_{48t+36} can be decomposed into a parallel class of K_4's, 10 parallel classes of K_3's and K_4's.

Proof

K_{48t+36} is the disjoint union of four K_{12t+9} and the multipartite graph $K_{12t+9,12t+9,12t+9,12t+9}$. By hypothesis the K_{12t+9} can be decomposed into 10 parallel classes of K_3's and K_4's. As $12t + 9 \neq 2,3,6,10$, by applying Remark 5.2, there exists a decomposition of $K_{12t+9,12t+9,12t+9,12t+9}$ into parallel classes of K_4's. □

Lemma 5.5

If K_{12t+3} can be decomposed into 7 parallel classes of K_3's and K_4's then K_{48t+12} can be decomposed into a parallel class of K_4's, 10 parallel classes of K_3's and K_4's.

Proof

Let $X = X_1 \cup X_2 \cup X_3 \cup X_4$ with $|X_i| = 12t+4$. By Remark 5.2 the multipartite graph $K_{12t+4,12t+4,12t+4,12t+4}$ on vertex set X can be decomposed into a parallel class of K_4's and K_4's as $12t+4 \neq 2,3,6,10$. Choose a particular K_4, $\{x_1,x_2,x_3,x_4\}$ with $x_i \in X_i$. Then the edges of the K_{48t+12} constructed on $X - \{x_1, x_2, x_3, x_4\}$ can be partitioned into the four K_{12t+3} on vertex set $X_i - x_i$, each decomposable into 7 parallel classes of K_3's and K_4's by hypothesis, the K_4's of $K_{12t+4,12t+4,12t+4,12t+4}$ not containing x_i for any i and the K_4's obtained from the K_4's containing one of the vertices x_i when deleting it. The K_3's of this decomposition can be partitioned into 10 parallel classes of K_3's on $48t+12$ vertices in the following way:

6 parallel classes, each of them being the union of parallel classes on each $X_i - x_i$,
4 parallel classes obtained for $i = 1, 2, 3, 4$ by taking, for each i, the parallel class still unused on $X_i - x_i$ and by adding the parallel class of K_3's on $\bigcup_{j \neq i}(X_j - x_j)$

obtained from the K_4's containing x_i after deletion of x_i. □

Proof of Theorem 5.1.

First the given conditions are necessary as we saw before. To prove that they are sufficient, by using Lemma 5.3, we only have to prove that K_n can be decomposed into a parallel class of K_3's and K_4's, 10 parallel classes of K_3's and K_4's, for $n=12t$ with $t = 2,3,4,5,6,7,9,11,13,17,21$ in order to finish the proof. From Lemma 5.4 and Lemma 2.5 we get the result with $t = 7,11$. From Lemma 5.5 and Lemma 2.4 we get the result with $t = 5,9,13,17,21$.

In what follows we will give the direct constructions for the remaining cases $(t = 2,3,4,6)$. □

The case $t = 2$:

For $n = 24$, the decomposition of K_{24} into a parallel class of K_4's and 10 parallel classes of K_3's is as follows. Let $V(K_{24}) = Z_6 \times Z_4$; the vertices are labeled (i,j) with $i = 0, 1,..., 5$ and $j = 0, 1, 2, 3$. The decomposition is given below:

Classes of K_3's:

[(0,j),(1,j),(2,j)]	[(3,j),(4,j+1),(5,j)]	for j = 0,1,2,3
[(0,j),(1,j+1),(3,j)]	[(2,j),(4,j),(5,j+1)]	for j = 0,1,2,3
[(0,j),(1,j+2),(4,j)]	[(2,j),(3,j+2),(5,j)]	for j = 0,1,2,3
[(0,j),(1,j+3),(5,j)]	[(2,j),(3,j+3),(4,j+2)]	for j = 0,1,2,3
[(0,j),(2,j+1),(4,j+1)]	[(1,j),(4,j+1),(5,j+3)]	for j = 0,1,2,3
[(0,j),(2,j+2),(3,j+1)]	[(1,j),(3,j+1),(5,j+2)]	for j = 0,1,2,3
[(0,j),(2,j+3),(5,j+1)]	[(1,j),(3,j+2),(4,j)]	for j = 0,1,2,3
[(0,j),(3,j+3),(5,j+2)]	[(1,j),(2,j+1),(4,j+3)]	for j = 0,1,2,3
[(0,j),(3,j+3),(5,j+2)]	[(1,j),(2,j+2),(4,j+3)]	for j = 0,1,2,3
[(0,j),(4,j+3),(5,j+3)]	[(1,j),(2,j+3),(3,j)]	for j = 0,1,2,3

Class of K_4's:

[(i,0)(i,1)(i,2)(i,3)]   for i = 0,...,5

The case $t = 3$:

For $n = 36$ the decomposition of K_{36} into 10 parallel classes of K_3's and 5 parallel classes of K_4's is as follows. Let $V(K_{36}) = Z_4 \times Z_9$; the vertices are labeled (i,j) with $i = 0, 1, 2, 3$ and $j = 0, 1,..., 8$. The decomposition uses the known decomposition of K_9 into 4 parallel classes of K_3's ([16]). We form 8 parallel classes in the following way: for each $t = 0, 1, 2, 3$ we take 2 parallel classes of K_3's constructed on the 27 vertices (i,j), $i \neq t$ with two parallel classes of the K_9 constructed on the vertices (t,j). The last two parallel classes are formed with the two unused classes on each K_9.

The case $t = 4$:

For $n = 48$ the decomposition of K_{48} into 10 parallel classes of K_3's, a parallel class of K_4's and K_4's is as follows. Let $V(K_{48}) = Z_{48}$.

Classes of K_3's:

[(0,j),(1,j),(2,j+3)]	for j = 0,...,8	and a class from (3,j)
[(0,j),(1,j+7),(2,j+3)]	for j = 0,...,8	and a class from (3,j)
[(0,j),(1,j+8),(2,j+6)]	for j = 0,...,8	and a class from (2,j)
[(0,j),(1,j+4),(3,j+2)]	for j = 0,...,8	and a class from (2,j)
[(0,j),(1,j+6),(3,j+7)]	for j = 0,...,8	and a class from (1,j)
[(0,j),(2,j+4),(3,j+1)]	for j = 0,...,8	and a class from (1,j)
[(0,j),(2,j+7),(3,j+6)]	for j = 0,...,8	and a class from (0,j)
[(1,j),(2,j+4),(3,j+6)]	for j = 0,...,8	and a class from (0,j)
[(1,j),(2,j+6),(3,j+4)]	for j = 0,1,2,3	and a class from (0,j)

Classes of K_4's:

[(0,j),(1,j),(2,j),(3,j)]	for j = 0,...,8
[(0,j),(1,j+1),(2,j+2),(3,j+3)]	for j = 0,...,8
[(0,j),(1,j+2),(2,j+1),(3,j+3)]	for j = 0,...,8
[(0,j),(1,j+3),(2,j+5),(3,j+5)]	for j = 0,...,8
[(0,j),(1,j+2),(2,j+5),(3,j+8)]	for j = 0,...,8
[(0,j),(1,j+5),(2,j+8),(3,j+4)]	for j = 0,...,8

For $v = 72$ the decomposition of K_{72} into 10 parallel classes of K_3's, a parallel class of K_4's and K_4's is as follows. Let $V(K_{72}) = Z_{72}$.

Classes of K_3's:

$[i, i+16, i+32]$ for $i = 0,...,15$
$[i, i+1, i+5]$ for $i \equiv 0 \pmod 3$
$[i, i+1, i+5]$ for $i \equiv 1 \pmod 3$
$[i, i+1, i+5]$ for $i \equiv 2 \pmod 3$
$[i, i+2, i+13]$ for $i \equiv 0 \pmod 3$
$[i, i+2, i+13]$ for $i \equiv 1 \pmod 3$
$[i, i+2, i+13]$ for $i \equiv 2 \pmod 3$
$[i, i+7, i+26]$ for $i \equiv 0 \pmod 3$
$[i, i+7, i+26]$ for $i \equiv 1 \pmod 3$
$[i, i+7, i+26]$ for $i \equiv 2 \pmod 3$

Class of K_4's:

$[i, i+12, i+24, i+36]$ for $i = 0,...,11$

K_4's:

$[i, i+3, i+20, i+30]$ for $i = 0,...,47$
$[i, i+6, i+14, i+39]$ for $i = 0,...,47$

The case $t = 6$:

Classes of K_3's:

$[i, i+24, i+48]$ for $i = 0,...,23$
$[i, i+1, i+5]$ for $i \equiv 0 \pmod 3$
$[i, i+1, i+5]$ for $i \equiv 1 \pmod 3$
$[i, i+1, i+5]$ for $i \equiv 2 \pmod 3$
$[i, i+2, i+10]$ for $i \equiv 0 \pmod 3$
$[i, i+2, i+10]$ for $i \equiv 1 \pmod 3$
$[i, i+2, i+10]$ for $i \equiv 2 \pmod 3$
$[i, i+7, i+20]$ for $i \equiv 0 \pmod 3$
$[i, i+7, i+20]$ for $i \equiv 1 \pmod 3$
$[i, i+7, i+20]$ for $i \equiv 2 \pmod 3$

Class of K_4's:

$[i, i+18, i+36, i+54]$ for $i = 0,...,17$

K_4's:

$[i, i+3, i+12, i+41]$ for $i = 0,...,71$
$[i, i+11, i+26, i+51]$ for $i = 0,...,71$
$[i, i+14, i+30, i+49]$ for $i = 0,...,71$
$[i, i+6, i+33, i+50]$ for $i = 0,...,71$

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