On Hamilton Cycle Decompositions of the Tensor Product of Complete Graphs

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Abstract

In this paper, we show that the tensor product of complete graphs is hamilton cycle decomposable.

1 Introduction

Let $G$ and $H$ be two simple graphs. The tensor product (also called direct product) of the graphs $G$ and $H$, $G \otimes H$, is the graph with the vertex set $V(G \otimes H) = V(G) \times V(H)$ and with the edge set $E(G \otimes H) = \{(u, x)(v, y), uv \in E(G) \text{ and } xy \in E(H)\}$. A $k$-regular multigraph $G$ has a hamilton cycle decomposition if its edge set can be partitioned into $k/2$
hamilton cycles when $k$ is even, or into $(k - 1)/2$ hamilton cycles plus a one factor (or perfect matching), when $k$ is odd. In this paper, we study the hamilton cycle decomposition of $K_r \otimes K_s$.

The problem of finding hamilton cycle decompositions of product graphs is not new. Hamilton cycle decompositions of various product graphs have been studied by many people (see the survey papers [1] and [5] or the book [6]). One interesting problem is to investigate whether the product graph of two hamilton cycle decomposable graphs is also hamilton cycle decomposable. Many results for various products have been obtained in the last few years [2,4,7,9,10,11]. Like other products, the tensor product graph has some interesting properties. For example, the tensor product of two hamilton cycle decomposable graphs may not be connected: consider the product of two even cycles. Jha [8] conjectured that if both $G$ and $H$ are hamilton cycle decomposable and $G \otimes H$ is connected, then the tensor product graph is also hamilton cycle decomposable. But this conjecture was disproved in [3]. It would be interesting to know what extra condition(s) should be added to $G$ and $H$ to ensure a hamilton cycle decomposition of $G \otimes H$.

The following result concerning the tensor product has been known for a long time.

**Theorem 1.1.** Let $G$ and $H$ be two even regular simple graphs. If both $G$ and $H$ are hamilton cycle decomposable and at least one of them has odd order, then $G \otimes H$ is hamilton cycle decomposable.

The above result can be obtained from the fact that $C_r \otimes C_s$ has a hamilton cycle decomposition if at least one of $r$ and $s$ is odd and that the tensor product is distributive over edge disjoint union of graphs. The next result follows immediately from Theorem 1.1.

**Corollary 1.2.** $K_r \otimes K_s$ has a hamilton cycle decomposition if both $r$ and $s$ are odd and $r, s \geq 3$.

The main result in this paper is Theorem 1.3.

**Theorem 1.3.** If $r, s \geq 3$, then $K_r \otimes K_s$ has a hamilton cycle decomposition.

Note that the case when at least one of $r$ and $s$ is less than three is trivial.
2 Proof of the main result

The proof of Theorem 1.3 depends on several relatively simple lemmas.

We will use $\lambda G$ (resp. $\lambda D$) to denote the graph (digraph) obtained by replacing each edge (resp. arc) of $G$ (resp. $D$) with $\lambda$ edges (resp. arcs). Let $ab$ denote the edge between the vertices $a$ and $b$, and $(a, b)$ denote the arc from $a$ to $b$. A $k$-cycle is denoted by either $(v_1, v_2, ..., v_k, v_1)$ or $e_1 e_2 ... e_k$ where $e_i = v_i v_{i+1}, i = 1, 2, ..., k - 1$, and $e_k = v_k v_1$ and a $k$-path is denoted by $[v_1, v_2, ..., v_k]$.

Let $V(K_{2q}) = \{\infty, 1, 2, ..., 2q - 1\}$, and

$$H^i = (\infty, 1 + i, 2 + i, 2q - 1 + i, 3 + i, 2q - 2 + i, 4 + i, ..., q + 1 + i, \infty),$$

where $0 \leq i \leq 2q - 2$. (The arithmetic calculations are modulo $2q - 1$ on the residues $1, 2, ..., 2q - 1$.) Clearly, the $H^i$'s are hamilton cycles of $K_{2q}$.

We will also denote by $e^i_j, 1 \leq j \leq 2q$, the $j$th edge of $H^i$. We have

$$e^i_j = \begin{cases} 
\infty(1 + i), & \text{if } j = 1 \\
(2 + i - j/2)(1 + i + j/2), & \text{if } j \text{ is even and } j \neq 2q \\
(i + (j + 1)/2)(2 + i - (j + 1)/2), & \text{if } j \text{ is odd and } j \neq 1 \\
(q + 1 + i)\infty, & \text{if } j = 2q 
\end{cases}$$

Example: $q = 4$ and 5

$H^0 : \infty 1 2 7 3 6 4 5 \infty \quad H^0 : \infty 1 2 9 3 8 4 7 5 6 \infty$
$H^1 : \infty 2 3 1 4 7 5 6 \infty \quad H^1 : \infty 2 3 1 4 9 5 8 6 7 \infty$
$H^2 : \infty 3 4 2 5 1 6 7 \infty \quad H^2 : \infty 3 4 2 5 1 6 9 7 8 \infty$
$H^3 : \infty 4 5 3 6 2 7 1 \infty \quad H^3 : \infty 4 5 3 6 2 7 1 8 9 \infty$
$H^4 : \infty 5 6 4 7 3 1 2 \infty \quad H^4 : \infty 5 6 4 7 3 8 2 9 1 \infty$
$H^5 : \infty 6 7 5 1 4 2 3 \infty \quad H^5 : \infty 6 7 5 8 4 9 3 1 2 \infty$
$H^6 : \infty 7 1 6 2 5 3 4 \infty \quad H^6 : \infty 7 8 6 9 5 1 4 2 3 \infty$
$H^7 : \infty 8 9 7 1 6 2 5 3 4 \infty \quad H^7 : \infty 8 9 7 1 6 2 5 3 4 \infty$
$H^8 : \infty 9 1 8 2 7 3 6 4 5 \infty \quad H^8 : \infty 9 1 8 2 7 3 6 4 5 \infty$

The following results are some simple observations and they will be used extensively.
Lemma 2.1. (a) \( \bigcup_{i=0}^{2q-2} \{ e^i_j \} = \bigcup_{i=0}^{2q-2} \{ e^i_{2q+1-j} \} \)

(b) The graphs induced by the edge sets \( \bigcup_{i=0}^{2q-2} \bigcup_{j=1}^{q} \{ e^i_j \} \) and \( \bigcup_{i=0}^{2q-2} \bigcup_{j=q+1}^{2q} \{ e^i_j \} \) are \( K_{2q} \).

(c) The \( H^i, 0 \leq i \leq 2q-2 \), form a hamilton cycle decomposition of \( 2K_{2q} \).

(d) The \( H^i, 0 \leq i \leq q-2 \), form a hamilton cycle decomposition of \( K_{2q} \setminus F \), where \( F \) is the one factor \( \bigcup_{j=1}^{q-1} \{(q-j)(q+j)\} \cup \{\infty, q\}. \)

We remark that \( K_r \otimes K_s \) can be obtained from \( K_s \) by replacing each vertex \( x \) of \( K_s \) by a set of \( r \) vertices \( V_x \) and each edge \( xy \) by a set of \( r-1 \) one factors between \( V_x \) and \( V_y \). More precisely, we let \( V_x = \{x_0, x_1, ..., x_{r-1}\} \) and denote the one factor of distance \( k \) \( (k = 1, 2, ..., r-1) \) from \( V_x \) to \( V_y \), the set of edges \( F^k_{x,y} = \{x_iy_{i+k}, i = 0, 1, ..., r-1\} \) (the addition on the subscripts is modulo \( r \)). Note that the order between \( x \) and \( y \) is important as a one factor of distance \( k \) from \( V_x \) to \( V_y \) is a one factor of distance \( r-k \) \( (\equiv -k \text{ (mod } r)\) from \( V_y \) to \( V_x \). If we fix an orientation for the edges of \( K_s \), and denote the resulting tournament by \( T_s \), then we have

\[
V(K_r \otimes K_s) = \bigcup_{x \in V(T_s)} V_x
\]

\[
E(K_r \otimes K_s) = \bigcup_{(x,y) \in A(T_s)} \bigcup_{k=1}^{r-1} F^k_{x,y}, \text{ where } A(T_s) \text{ is the arc set of } T_s.
\]

Assume that in \( (r-1)T_s \), the \( r-1 \) arcs between any pair of vertices are labelled \( 1, 2, ..., r-1 \). Then the arc \( (x,y) \) of \( (r-1)T_s \), with label \( k \) can be associated to the set of edges \( F^k_{x,y} \) in \( K_r \otimes K_s \), and in fact this is a one to one correspondence relationship. We will construct a hamilton cycle decomposition of \( K_r \otimes K_s \) from an oriented hamilton cycle decomposition of the labelled \( (r-1)T_s \), by associating a suitable oriented (not directed) hamilton cycle of \( (r-1)T_s \) with a hamilton cycle of \( K_r \otimes K_s \) (see Lemma 2.3). For that we need the following definition.

**Definition 2.2.** Let \( H = (v_1, v_2, ..., v_s, v_1) \) be an oriented (not necessary directed) hamilton cycle of \( (r-1)T_s \). We can also write \( H = a_1a_2...a_s \), where \( a_i \) is the arc of \( (r-1)T_s \) between \( v_i \) and \( v_{i+1} \). We define the label of \( H \), \( \ell(H) = \sum_{i=1}^{s} \varepsilon_i \ell(a_i) \), where \( \ell(a_i) \) is the label of the arc \( a_i \) and \( \varepsilon_i = 1 \) if \( a_i = (v_i, v_{i+1}) \), \( \varepsilon_i = -1 \) otherwise.

**Lemma 2.3.** Let \( H = (v_1, v_2, ..., v_s, v_1) = a_1a_2...a_s \) be an oriented hamilton cycle of \( (r-1)T_s \). If \( \ell(H) \) and \( r \) are relatively prime, then the edge set in \( K_r \otimes K_s \) corresponding to \( H \) forms a hamilton cycle of \( K_r \otimes K_s \).
Proof. The edge set $\bigcup_i F^e_i(e^{(a_i)})$ forms a 2-factor of $K_r \otimes K_s$. Consider the subgraph of $K_r \otimes K_s$ induced by the edge set $\bigcup_{i \neq 1} F^e_i(e^{(a_i)})$, it is a union of $r$ paths of length $s - 1$. When the edge set $F^e_1(e^{(a_1)})$ is added, the condition that $\ell(H)$ and $r$ are relatively prime guarantees that the resulting 2-factor is a hamilton cycle of $K_r \otimes K_s$. ■

Corollary 2.4. If there exists a decomposition of $(r - 1)T_e$ into oriented hamilton cycles, such that the label of each hamilton cycle is relatively prime to $r$, then there exists a hamilton cycle decomposition of $K_r \otimes K_s$.

Proof of Theorem 1.3:

The case where both $r$ and $s$ are odd was already covered by Corollary 1.2. So we can assume that at least one of $r$ and $s$ is even. As the tensor product is commutative, we will assume that $s$ is even. Let $s = 2q$.

In what follows, we give a precise description of $T_{2q}$ (or $T_s$) by giving a special orientation to $K_{2q}$. First we take the hamilton cycle decomposition of $2K_{2q}$, $H_i$, $i = 0, 1, ..., 2q - 2$ as described in the beginning of this section. For $0 \leq i \leq 2q - 2$, we orient the hamilton path $H_i \setminus e^{(2q)}_i = [\infty, 1 + i, 2 + i, 2q - 1 + i, 3 + i, 2q - 2 + i, ..., q + 1 + i]$ into a directed path from $\infty$ to $q + 1 + i$. Now we justify that this orientation is well defined. By Lemma 2.1 (c), each edge except those in the form of $\infty i$ has been oriented twice. From the definition of $H^i$, each of these edges appears in two different hamilton paths and the end vertices are in the same order. Let $T_{2q}$ be the resulting tournament; then we have $A(T_{2q}) = \bigcup_{i=1}^{2q-1} \{ (\infty, i), (i, i + \alpha) \}$ for $\alpha$ odd, $\alpha \leq q - 1$, $(i, i - \alpha)$ for $\alpha$ even, $\alpha \leq q - 1$.

In the remainder of the paper, we will attach the above orientation to $H^i$ defined in the beginning of the section, and use $H^i$ to denote the resulting oriented hamilton cycles. Hence $H^i$’s form an oriented hamilton cycle decomposition of $2T_{2q}$. (Notice that $H^i$’s are not directed hamilton cycles.) $H^i$ can also be expressed in terms of the arcs in the form $a^i_1a^i_2...a^i_{2q}$, where $a^i_j$ is $e^{\alpha}_j$ with the above orientation. We have the following lemma.
Lemma 2.5.  \( \ell(H^i) = \sum_{j=1}^{2q-1} \ell(a^i_j) - \ell(a^i_{2q}) \).

**Proof.** Notice that in \( H^i, a^i_j \)'s, \( 1 \leq j \leq 2q-1 \), are in the same direction, but \( a^i_{2q} = (\infty, q+1+i) \) is not. Therefore \( \varepsilon_i = 1, 1 \leq i \leq 2q-1 \) and \( \varepsilon_{2q} = -1 \).

We now divide the proof of Theorem 1.3 into two cases.

**Case 1.** \( r \) is odd.

Let \( r = 2p + 1 \). To exhibit a decomposition of \( K_{2p+1} \otimes K_2 \) into \( p(2q-1) \) hamilton cycles, we will decompose the labelled \( 2pT_{2q} \) into \( p(2q-1) \) oriented hamilton cycles, so that the label of each cycle is relatively prime to \( 2p+1 \).

**Case 1.1.** \( p \) is even. We decompose \( 2pT_{2q} \) into the \( p(2q-1) \) oriented hamilton cycles, \( H^i_x \) and \( H^i_{x+1} \) for \( 0 \leq i \leq 2q-2 \) and \( x = 0, 2, 4, \ldots, p-2 \). (here \( H^i_x \) and \( H^i_{x+1} \) are the same as \( H^i \) defined before). We arrange the labels on the arcs as follows:

in \( H^i_x \), let \( \ell(a^i_1) = p-x; \)
\( \ell(a^i_j) = p + 2 + x, \quad 2 \leq j \leq q; \)
\( \ell(a^i_{q+1}) = p - 1 - x, \quad q+1 \leq j \leq 2q, \) and

in \( H^i_{x+1} \), let \( \ell(a^i_1) = p + 2 + x; \)
\( \ell(a^i_j) = p - x, \quad 2 \leq j \leq q; \)
\( \ell(a^i_{q+1}) = p + 1 + x, \quad q+1 \leq j \leq 2q. \)

Notice that \( \{p - 1 - x, p - x, p + 1 + x, p + 2 + x\}, \quad x = 0, 2, \ldots, p-2 \), form a partition of \( \{1, 2, \ldots, 2p\} \). By Lemma 2.1(c), \( \bigcup_{i=0}^{2q-2} H^i_x \cup H^i_{x+1} \) is the union of four copies of \( T_{2q} \). By Lemma 2.1(b) the subgraphs induced by \( \bigcup_{i=0}^{2q-2} \bigcup_{j=1}^{q} \{a^i_j\} \) and \( \bigcup_{i=0}^{2q-2} \bigcup_{j=q+1}^{2q} \{a^i_j\} \) are \( T_{2q} \). For example, the label \( p - x \) is assigned to \( a^i_1 \) in \( H^i_x \) and to \( a^i_j \) for \( 2 \leq j \leq q \) in \( H^i_{x+1} \). So we can conclude that one copy of \( T_{2q} \) is labelled \( p - x \). Similarly, one copy of each is labelled \( p - 1 - x, p + 1 + x \) and \( p + 2 + x \).

Finally it remains to prove that the label of each oriented hamilton cycle is relatively prime to \( 2p+1 \). By Lemma 2.5,

\[
\ell(H^i_x) = (p-x) + (q-1)(p+2+x) + (q-1)(p-1-x) - (p-1-x) \\
= q(2p+1) - 2p \equiv 1 \pmod{2p+1}, \quad \text{and}
\]

\[
\ell(H^i_{x+1}) = (p+2+x) + (q-1)(p-x) + (q-1)(p+1+x) - (p+1+x) \\
= q(2p+1) - 2p \equiv 1 \pmod{2p+1}.
\]
In all the cases the labels of the Hamilton cycles are relatively prime to \(2p + 1\). By Corollary 2.4, the above decomposition of \(2pT_{2q}\) will produce a Hamilton cycle decomposition of \(K_{2p+1} \otimes K_{2q}\).

**Case 1.2. \(p\) is odd.** As in Case 1.1 we use the \((2q-1)(p-1)\) Hamilton cycles \(H_i^j\) and \(H_{x+1}^i\) for \(0 \leq i \leq 2q-2\) and \(x = 0, 2, ..., p-3\). Now the 4-subsets \(\{p-1-x, p-x, p+1+x, p+2+x\}\), \(x = 0, 2, ..., p-3\), form a partition of \(\{2, 3, ..., 2p-1\}\). Hence we decompose the arcs with labels in \(\{2, 3, ..., 2p-1\}\) into \((2q-1)(p-1)\) Hamilton cycles whose labels are relatively prime to \(2p+1\).

It remains to decompose the arcs of \(2T_{2q}\) with labels 1 and 2p into the \(2q - 1\) Hamilton cycles, \(H_{p-1}^i\) for \(0 \leq i \leq 2q-2\):
- in \(H_{p-1}^i\), \(\ell(a_j^i) = 1\) for \(1 \leq j \leq q\), and
- \(\ell(a_j^i) = 2p\) for \(q+1 \leq j \leq 2q\).

\[
\ell(H_{p-1}^i) = q + (q-1)2p - 2p = (2p+1)(q-2) - 2 \equiv -2 \pmod{2p+1}.
\]
\[
\ell(H_{p-1}^i)\) and \(2p+1\) are relatively prime as \(2p+1\) is odd.

So far we have shown that \(K_r \otimes K_s\) has a Hamilton cycle decomposition if one of \(r\) and \(s\) is odd.

**Case 2. \(r\) is even.**

Let \(r = 2p\). We will show that \(K_{2p} \otimes K_{2q}\) can be decomposed into \((p-1)(2q-1) + q - 1\) Hamilton cycles and a one factor. Again we will exhibit a decomposition of \((2p-1)T_{2q}\) into oriented Hamilton cycles with labels relatively prime to \(r = 2p\).

**Case 2.1. \(p\) is even.** Note that the set of labels \(\{1, 2, ..., 2p-1\}\) can be partitioned into \((p-2)/2\) 4-subsets,
- \(\{p-2-x, p-1-x, p+1+x, p+2+x\}\), \(x = 1, 3, ..., p-3\) and a 3-subset \(\{p-1, p, p+1\}\).

We first assign the labels to the arcs of the following \((p-2)(2q-1)\) Hamilton cycles:
- for \(0 \leq i \leq 2q-2\) and \(x = 1, 3, ..., p-3\),
  - in \(H_x^i\), let \(\ell(a_j^i) = p-1-x\);
  - \(\ell(a_j^i) = p+2+x\), \(2 \leq j \leq q\);
  - \(\ell(a_j^i) = p-2-x\), \(q+1 \leq j \leq 2q\), and
- in \(H_x^i\), let \(\ell(a_j^i) = p+2+x\);
\[\ell(a_{ij}^i) = p - 1 - x, \quad 2 \leq j \leq q;\]
\[\ell(a_{ij}^p) = p + 1 + x, \quad q + 1 \leq j \leq 2q.\]

\[\ell(H_{x}^i) = p - 1 - x + (q - 1)(p + 2 + x) + (q - 1)(p - 2 - x) - (p - 2 - x)\]
\[= 2p(q - 1) + 1 \equiv 1(mod\, 2p)\]
\[\ell(H_{x+1}^i) = p + 2 + x + (q - 1)(p - 1 - x) + (q - 1)(p + 1 + x) - (p + 1 + x)\]
\[= 2p(q - 1) + 1 \equiv 1(mod\, 2p).\]

Hence the labels of the above hamilton cycles are relatively prime to \(2p - 1\). So far we have decomposed all the arcs of \((2p - 1)T_{2q}\) with labels in \(\{1, 2, ..., 2p - 1\}\setminus\{p - 1, p, p + 1\}\). It remains to partition the arcs of the three copies of \(T_{2q}\) with labels \(p - 1, p, p + 1\) into \((2q - 1) + (q - 1)\) hamilton cycles and a one factor. We first assign the labels to the arcs of the \(2q - 1\) hamilton cycles, \(H_{x+1}^i, 0 \leq i \leq 2q - 2:\)

in \(H_{x+1}^i, \) for \(1 \leq j \leq q\) and \(j \neq j_0,\)
\[\ell(a_{ij}^i) = p + 1, \text{ and}\]
for \(j = j_0,\)
\[\ell(a_{ij_0}^i) = p, \text{ where}\]
\[j_0 = 1, \text{ for } 0 \leq i \leq q - 2 \text{ and}\]
\[i = q, q + 2, ..., 2q - 3 \text{ if } q \text{ is even, and}\]
\[i = q, q + 2, ..., 2q - 3 \text{ if } q \text{ is odd;}\]
\[j_0 = 2, \text{ for } i = q - 1, q + 1, ..., 2q - 2 \text{ if } q \text{ is even and}\]
\[i = q - 1, q + 1, ..., 2q - 2 \text{ if } q \text{ is odd;}\]

for \(q + 1 \leq j \leq 2q,\)
\[\ell(a_{ij}^i) = p - 1.\]

Note that the label of each of these hamilton cycles is
\[\ell(H_{x+1}^i) = p + (q - 1)(p + 1) + (q - 1)(p - 1) - (p - 1) \equiv 1(mod\, 2p).\]

**Example:** From the decompositions exhibited in the beginning of the section, it is easy to check that the arcs with label \(p\) are the following;

when \(2q = 8, a_0^1 = (\infty, 1), a_1^1 = (\infty, 2), a_2^1 = (\infty, 3), a_3^1 = (\infty, 5),\)
\[a_4^1 = (\infty, 7), a_5^1 = (4, 5), a_6^1 = (6, 7)\]
when \(2q = 10, a_0^1 = (\infty, 1), a_1^1 = (\infty, 2), a_2^1 = (\infty, 3), a_3^1 = (\infty, 4),\)
\[a_4^1 = (\infty, 6), a_5^1 = (\infty, 8), a_6^1 = (5, 6), a_7^1 = (7, 8),\]
\[a_8^1 = (9, 1)\]

Note that the arcs not used by the hamilton cycles above form a copy of \(T_{2q}\) and these arcs have the label \(p\) except for the arcs \(a_{ij}^i\) which have
the label $p + 1$. Therefore, the last step is to obtain $q - 1$ hamilton cycles by decomposing the arcs of this $T_{2q}$. Using Lemma 2.1 (d) we take the decomposition of $T_{2q}$ into the $q - 1$ hamilton cycles $H_p^i$, for $0 \leq i \leq q - 2$. The labels of each hamilton cycle are fixed and can be described precisely:

for $0 \leq i \leq q - 2$, in $H_p^i$,

when $i$ is even,

\[
\ell(a_j^i) = p \text{ if } 1 \leq j \leq 2q \text{ and } j \neq 1, 2q - 1 \text{ and } 2q,
\]

\[
\ell(a_1^i) = \ell(a_{2q-1}^i) = p + 1;
\]

\[
(\ell(H_p^i) = p + 1 + (2q - 3)p + p + 1 - (p + 1) \equiv 1 (mod 2p))
\]

when $i$ is odd,

\[
\ell(a_j^i) = p + 1 \text{ and } \ell(a_j^i) = p, 2 \leq j \leq 2q.
\]

\[
(\ell(H_p^i) = p + 1 + (2q - 2)p - p \equiv 1 (mod 2p))
\]

Hence the decomposition satisfies the conditions.

**Example:**

For $2q = 8$, the arcs with labels $p + 1$ are

$a_0^1 = (\infty, 1), a_7^0 = (4, 5), a_8^0 = (\infty, 5)$ in $H_8^0$,

$a_7^1 = (\infty, 3), a_2^7 = (6, 7), a_3^7 = (\infty, 7)$ in $H_8^1$, and $a_1^1 = (\infty, 2)$ in $H_8^1$.

For $2q = 10$, the arcs with labels $p + 1$ are

$a_0^1 = (\infty, 1), a_9^0 = (5, 6), a_0^9 = (\infty, 6)$ in $H_{10}^0$,

$a_2^1 = (\infty, 3), a_7^2 = (7, 8), a_3^7 = (\infty, 8)$ in $H_{10}^1$,

$a_1^1 = (\infty, 2)$ in $H_{10}^1$, and $a_1^1 = (\infty, 4)$ in $H_{10}^1$.

Notice that when $q$ is odd, the last arc labelled $p + 1$, namely $(9, 1)$ (in general $(2q - 1, 1)$), appears in the remaining one factor.

The remaining edges form a one factor of $T_{2q}$. It is clear that this one factor can be used to construct a one factor in $K_{2p} \otimes K_{2q}$.

**Case 2.2. $p$ is odd.** We can assume that $q$ is odd ($q \geq 3$) (the case when $q$ is even has been dealt with as in Case 2.1).

We partition the set of labels $\{1, 2,...,2p - 1\}$ into $(p - 3)/2$ 4-subsets $\{p - 2 - x, p - 1 - x, p + 1 + x, p + 2 + x\}$ for $x = 2, 4,..., p - 3$ and a 5-subset $\{p - 2, p - 1, p, p + 1, p + 2\}$. We deal with the arcs of $(2p - 1)T_{2q}$ with labels other than $\{p - 2, p - 1, p, p + 1, p + 2\}$ in the same way as in Case 2.1.
The remaining edges are five copies of $T_{2q}$ whose arcs are labelled \{\(p - 2, p - 1, p, p + 1, p + 2\)\} and they are partitioned into $2(2q - 1) + (q - 1)$ oriented hamilton cycles as follows:

For $0 \leq i \leq 2q - 2$, in $H_{p-1}^i$,
\[
\ell(a_j^i) = \begin{cases} p - 2, & \text{for } 1 \leq j \leq q \text{ and } j \neq 3; \\ p + 1, & \ell(a_3^i) = p + 1; \\ p + 2, & \text{for } q + 1 \leq j \leq 2q. \end{cases}
\]
So $\ell(H_{p-1}^i) = p + 1 + (q - 1)(p - 2) + (q - 1)(p + 2) - (p + 1) - (q - 1) \equiv -1 (mod \ 2p)$.

For the other $2q - 1$ hamilton cycles, we do the following.
For $0 \leq i \leq 2q - 2$, in $H_p^i$,
\[
\ell(a_j^i) = \begin{cases} p + 1, & \text{for } 1 \leq j \leq q \text{ and } j \neq 3, j_0, \\ p - 2, & \ell(a_3^i) = p - 2, \text{ and } \\ p, & \ell(a_0^i) = p, \text{ where } \\ j_0 = 1, \text{ for } 0 \leq i \leq q - 2 \text{ and } i = q, q + 2, ..., 2q - 3, \text{ and } \\ j_0 = 2, \text{ for } i = q - 1, q + 1, ..., 2q - 2; \\ p - 1, & \text{for } q + 1 \leq j \leq 2q \text{ and } j \neq 2q - 2, \\ p, & \ell(a_2^i) = p, \text{ and } \\ p, & \ell(a_{2q-2}^i) = p. \end{cases}
\]

$\ell(H_p^i) = p + p - 2 + (q - 2)(p + 1) + (q - 2)(p - 1) + p - (p - 1) \equiv -1 (mod \ 2p)$.

**Example:** When $2q = 10$, from the listed decomposition we can check that the arcs with label $p$ are the following:

\[
a_1^0 = (\infty, 1), a_1^1 = (\infty, 2), a_1^2 = (\infty, 3), a_1^3 = (\infty, 4), \\
a_1^5 = (\infty, 6), a_1^7 = (\infty, 8), a_2^4 = (5, 6), a_2^6 = (7, 8), a_3^5 = (9, 1)
\]

The arcs not used by the hamilton cycles form a copy of $T_{2q}$ and they have label $p$ except the arcs $a_{1}^{i_p} (\ell(a_{1}^{i_p}) = p + 1)$ and $a_{2q-2}^{i_{2q-2}} (\ell(a_{2q-2}^{i_{2q-2}}) = p - 1)$. Like in the Case 2.1, using Lemma 2.1 (d), we partition these arcs into $q - 1$ hamilton cycles $H_{p+1}^i, 0 \leq i \leq q - 2$ and a one factor. The labels on the arcs are fixed as follows:

For $0 \leq i \leq q - 2$, in $H_{p+1}^i$,
\[
\ell(a_1^i) = \ell(a_{2q}^i) = \ell(a_{2q-1}^i) = p + 1, \\
\ell(a_3^i) = \ell(a_{2q-2}^i) = p - 1, \text{ and otherwise}
\]
\( \ell(a_j^i) = p, \)
when \( i \) is odd,
\( \ell(a_j^i) = p + 1, \)
\( \ell(a_3^i) = \ell(a_{2q-2}^i) = p - 1, \) and otherwise
\( \ell(a_j^i) = p. \)

**Example:** For \( 2q = 10, \) the arcs with labels \( p + 1 \) are
\( a_1^0 = (\infty, 1), a_9^0 = (5, 6), a_{10}^0 = (\infty, 6) \) in \( H_{p+1}^0, \)
\( a_1^1 = (\infty, 3), a_9^1 = (7, 8), a_{10}^1 = (\infty, 8) \) in \( H_{p+1}^2, \)
\( a_1^2 = (\infty, 2) \) in \( H_{p+1}^1, \) and \( a_3^3 = (\infty, 4) \) in \( H_{p+1}^3. \)

As in the previous case, the last arc labelled \( p + 1, \) namely, (9,1) (in
general \( (2q-1, 1) \)) appears in the remaining one factor.

When \( i \) is even,
\[ \ell(H_{p+1}^i) = (p + 1) + p + (p - 1) + (2q - 6)p + (p - 1) + (p + 1) - (p + 1) \equiv -1 (mod \ 2p). \]
When \( i \) is odd,
\[ \ell(H_{p+1}^i) = (p + 1) + p + (p - 1) + (2q - 6)p + (p - 1) + p - p \equiv -1 (mod \ 2p). \]

So in all cases \( \ell(H_{p+1}^i) \) and \( 2p \) are relatively prime and we have the re-
quired hamilton cycle decomposition of \( (r - 1)T_{2q} \).

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Reference


