On Hamilton Cycle Decompositions of the Tensor Product of Complete Graphs

R. Balakrishnan[a], J.-C. Bermond[b], P. Paulraja[a] and M.-L. Yu[c]

July 19, 2002

[a] *Dept. of Mathematics, Annamalai University, Annamalainagar - 608002, India*
email:mathbala@satyam.net.in
[b] *MASCOTTE, joint project I3S - CNRS/INRIA/Université de Nice - Sophia Antipolis, 2004 Route des Lucioles, BP 93, F-06902, Sophia Antipolis, France*
email:Jean-Claude.Bermond@sophia.inria.fr
[c] *Dept. of Mathematics and Statistics, University College of the Fraser Valley, Abbotsford, B.C., Canada, V2S 4N2*
email:yuj@ucfv.bc.ca

Abstract

In this paper, we show that the tensor product of complete graphs is hamilton cycle decomposable.

1 Introduction

Let G and H be two simple graphs. The *tensor product* (also called *direct product*) of the graphs G and H, $G \otimes H$, is the graph with the vertex set $V(G \otimes H) = V(G) \times V(H)$ and with the edge set $E(G \otimes H) = \{(u, x)(v, y), uv \in E(G)$ and $xy \in E(H)\}$. A k-regular multigraph G has a *hamilton cycle decomposition* if its edge set can be partitioned into $k/2$

hamilton cycles when k is even, or into $(k-1)/2$ hamilton cycles plus a one factor (or perfect matching), when k is odd. In this paper, we study the hamilton cycle decomposition of $K_r \otimes K_s$.

The problem of finding hamilton cycle decompositions of product graphs is not new. Hamilton cycle decompositions of various product graphs have been studied by many people (see the survey papers [1] and [5] or the book [6]). One interesting problem is to investigate whether the product graph of two hamilton cycle decomposable graphs is also hamilton cycle decomposable. Many results for various products have been obtained in the last few years [2,4,7,9,10,11]. Like other products, the tensor product graph has some interesting properties. For example, the tensor product of two hamilton cycle decomposable graphs may not be connected: consider the product of two even cycles. Jha [8] conjectured that if both G and H are hamilton cycle decomposable and $G \otimes H$ is connected, then the tensor product graph is also hamilton cycle decomposable. But this conjecture was disproved in [3]. It would be interesting to know what extra condition(s) should be added to G and H to ensure a hamilton cycle decomposition of $G \otimes H$.

The following result concerning the tensor product has been known for a long time.

Theorem 1.1. Let G and H be two even regular simple graphs. If both G and H are hamilton cycle decomposable and at least one of them has odd order, then $G \otimes H$ is hamilton cycle decomposable.

The above result can be obtained from the fact that $C_r \otimes C_s$ has a hamilton cycle decomposition if at least one of r and s is odd and that the tensor product is distributive over edge disjoint union of graphs. The next result follows immediately from Theorem 1.1.

Corollary 1.2. $K_r \otimes K_s$ has a hamilton cycle decomposition if both r and s are odd and $r, s \geq 3$.

The main result in this paper is Theorem 1.3.

Theorem 1.3. If $r, s \geq 3$, then $K_r \otimes K_s$ has a hamilton cycle decomposition.

Note that the case when at least one of r and s is less than three is trivial.

2 Proof of the main result

The proof of Theorem 1.3 depends on several relatively simple lemmas.

We will use λG (resp. λD) to denote the graph (digraph) obtained by replacing each edge (resp. arc) of G (resp. D) with λ edges (resp. arcs). Let ab denote the edge between the vertices a and b, and (a, b) denote the arc from a to b. A k-cycle is denoted by either $(v_1, v_2, ..., v_k, v_1)$ or $e_1e_2...e_k$ where $e_i = v_iv_{i+1}, i = 1, 2, ..., k-1$, and $e_k = v_kv_1$ and a k-path is denoted by $[v_1, v_2, ..., v_k]$.

Let $V(K_{2q}) = \{\infty, 1, 2, ..., 2q-1\}$, and

$$H^i = (\infty, 1+i, 2+i, 2q-1+i, 3+i, 2q-2+i, 4+i, ..., q+1+i, \infty),$$

where $0 \leq i \leq 2q-2$. (The arithmetic calculations are modulo $2q-1$ on the residues $1, 2, ..., 2q-1$.) Clearly, the H^i's are hamilton cycles of K_{2q}.

We will also denote by e^i_j, $1 \leq j \leq 2q$, the jth edge of H^i. We have

$$e^i_j = \begin{cases} \infty(1+i), & \text{if } j = 1 \\ (2+i-j/2)(1+i+j/2), & \text{if } j \text{ is even and } j \neq 2q \\ (i+(j+1)/2)(2+i-(j+1)/2), & \text{if } j \text{ is odd and } j \neq 1 \\ (q+1+i)\infty, & \text{if } j = 2q \end{cases}$$

Example: $q = 4$ and 5

$H^0 : \infty$ 1 2 7 3 6 4 5 ∞
$H^1 : \infty$ 2 3 1 4 7 5 6 ∞
$H^2 : \infty$ 3 4 2 5 1 6 7 ∞
$H^3 : \infty$ 4 5 3 6 2 7 1 ∞
$H^4 : \infty$ 5 6 4 7 3 1 2 ∞
$H^5 : \infty$ 6 7 5 1 4 2 3 ∞
$H^6 : \infty$ 7 1 6 2 5 3 4 ∞

$H^0 : \infty$ 1 2 9 3 8 4 7 5 6 ∞
$H^1 : \infty$ 2 3 1 4 9 5 8 6 7 ∞
$H^2 : \infty$ 3 4 2 5 1 6 9 7 8 ∞
$H^3 : \infty$ 4 5 3 6 2 7 1 8 9 ∞
$H^4 : \infty$ 5 6 4 7 3 8 2 9 1 ∞
$H^5 : \infty$ 6 7 5 8 4 9 3 1 2 ∞
$H^6 : \infty$ 7 8 6 9 5 1 4 2 3 ∞
$H^7 : \infty$ 8 9 7 1 6 2 5 3 4 ∞
$H^8 : \infty$ 9 1 8 2 7 3 6 4 5 ∞

The following results are some simple observations and they will be used extensively.

Lemma 2.1. (a) $\bigcup_{i=0}^{2q-2}\{e_j^i\} = \bigcup_{i=0}^{2q-2}\{e_{2q+1-j}^i\}$

(b) The graphs induced by the edge sets $\bigcup_{i=0}^{2q-2}\bigcup_{j=1}^{q}\{e_j^i\}$ and $\bigcup_{i=0}^{2q-2}\bigcup_{j=q+1}^{2q}\{e_j^i\}$ are K_{2q}.

(c) The H^i, $0 \leq i \leq 2q-2$, form a hamilton cycle decomposition of $2K_{2q}$.

(d) The H^i, $0 \leq i \leq q-2$, form a hamilton cycle decomposition of $K_{2q} \backslash F$, where F is the one factor $\bigcup_{j=1}^{q-1}\{(q-j)(q+j)\}\bigcup\{\infty, q\}$. ■

We remark that $K_r \otimes K_s$ can be obtained from K_s by replacing each vertex x of K_s by a set of r vertices V_x and each edge xy by a set of $r-1$ one factors between V_x and V_y. More precisely, we let $V_x = \{x_0, x_1, ..., x_{r-1}\}$ and denote the one factor of distance k ($k = 1, 2, ..., r-1$) from V_x to V_y, the set of edges $F_{x,y}^k = \{x_i y_{i+k}, i = 0, 1, ..., r-1\}$ (the addition on the subscripts is modulo r). Note that the order between x and y is important as a one factor of distance k from V_x to V_y is a one factor of distance $r-k$ ($\equiv -k (mod\ r)$) from V_y to V_x. If we fix an orientation for the edges of K_s and denote the resulting tournament by T_s, then we have

$V(K_r \otimes K_s) = \bigcup_{x \in V(T_s)} V_x$ and
$E(K_r \otimes K_s) = \bigcup_{(x,y) \in A(T_s)} \bigcup_{k=1}^{r-1} F_{x,y}^k$, where $A(T_s)$ is the arc set of T_s.

Assume that in $(r-1)T_s$, the $r-1$ arcs between any pair of vertices are labelled $1, 2, ..., r-1$. Then the arc (x, y) of $(r-1)T_s$ with label k can be associated to the set of edges $F_{x,y}^k$ in $K_r \otimes K_s$ and in fact this is a one to one correspondance relationship. We will construct a hamilton cycle decomposition of $K_r \otimes K_s$ from an oriented hamilton cycle decomposition of the labelled $(r-1)T_s$ by associating a suitable oriented (not directed) hamilton cycle of $(r-1)T_s$ with a hamilton cycle of $K_r \otimes K_s$ (see Lemma 2.3). For that we need the following definition.

Definition 2.2. Let $H = (v_1, v_2, ..., v_s, v_1)$ be an oriented (not necessary directed) hamilton cycle of $(r-1)T_s$. We can also write $H = a_1 a_2 ... a_s$, where a_i is the arc of $(r-1)T_s$ between v_i and v_{i+1}. We define the label of H, $\ell(H) = \sum_{i=1}^{s} \varepsilon_i \ell(a_i)$, where $\ell(a_i)$ is the label of the arc a_i and $\varepsilon_i = 1$ if $a_i = (v_i, v_{i+1})$, $\varepsilon_i = -1$ otherwise.

Lemma 2.3. Let $H = (v_1, v_2, ..., v_s, v_1) = a_1 a_2 ... a_s$ be an oriented hamilton cycle of $(r-1)T_s$. If $\ell(H)$ and r are relatively prime, then the edge set in $K_r \otimes K_s$ corresponding to H forms a hamilton cycle of $K_r \otimes K_s$.

Proof. The edge set $\bigcup_i F^{\varepsilon_i \ell(a_i)}_{v_i,v_{i+1}}$ forms a 2-factor of $K_r \otimes K_s$. Consider the subgraph of $K_r \otimes K_s$ induced by the edge set $\bigcup_{i\neq 1} F^{\varepsilon_i \ell(a_i)}_{v_i,v_{i+1}}$, it is a union of r paths of length $s-1$. When the edge set $F^{\varepsilon_1 \ell(a_1)}_{v_1,v_2}$ is added, the condition that $\ell(H)$ and r are relatively prime guarantees that the resulting 2-factor is a hamilton cycle of $K_r \otimes K_s$. ■

Corollary 2.4. If there exists a decomposition of $(r-1)T_s$ into oriented hamilton cycles, such that the label of each hamilton cycle is relatively prime to r, then there exists a hamilton cycle decomposition of $K_r \otimes K_s$.

Proof of Theorem 1.3:

The case where both r and s are odd was already covered by Corollary 1.2. So we can assume that at least one of r and s is even. As the tensor product is commutative, we will assume that s is even. Let $s = 2q$.

In what follows, we give a precise description of T_{2q} (or T_s) by giving a special orientation to K_{2q}. First we take the hamilton cycle decomposition of $2K_{2q}$, H^i, $i = 0, 1, ..., 2q-2$ as described in the beginning of this section. For $0 \le i \le 2q-2$, we orient the hamilton path $H^i \backslash e^i_{2q} = [\infty, 1+i, 2+i, 2q-1+i, 3+i, 2q-2+i, ..., q+1+i]$ into a directed path from ∞ to $q+1+i$. Now we justify that this orientation is well defined. By Lemma 2.1 (c), each edge except those in the form of ∞i has been oriented twice. From the definition of H^i, each of these edges appears in two different hamilton paths and the end vertices are in the same order. Let T_{2q} be the resulting tournament; then we have $A(T_{2q}) = \bigcup_{i=1}^{2q-1}\{(\infty, i), (i, i+\alpha)$ for α odd, $\alpha \le q-1$, $(i, i-\alpha)$ for α even, $\alpha \le q-1\}$.

In the remainder of the paper, we will attach the above orientation to H^i defined in the beginning of the section, and use H^i to denote the resulting oriented hamilton cycles. Hence H^i's form an oriented hamilton cycle decomposition of $2T_{2q}$. (Notice that H^i's are not directed hamilton cycles.) H^i can also be expressed in terms of the arcs in the form $a^i_1 a^i_2 ... a^i_{2q}$, where a^i_j is e^i_j with the above orientation. We have the following lemma.

Lemma 2.5. $\ell(H^i) = \sum_{j=1}^{2q-1} \ell(a_j^i) - \ell(a_{2q}^i)$.

Proof. Notice that in H^i, a_j^i's, $1 \leq j \leq 2q-1$, are in the same direction, but $a_{2q}^i = (\infty, q+1+i)$ is not. Therefore $\varepsilon_i = 1, 1 \leq i \leq 2q-1$ and $\varepsilon_{2q} = -1$.

We now divide the proof of Theorem 1.3 into two cases.

Case 1. r is odd.

Let $r = 2p+1$. To exhibit a decomposition of $K_{2p+1} \otimes K_{2q}$ into $p(2q-1)$ hamilton cycles, we will decompose the labelled $2pT_{2q}$ into $p(2q-1)$ oriented hamilton cycles, so that the label of each cycle is relatively prime to $2p+1$.

Case 1.1. p is even. We decompose $2pT_{2q}$ into the $p(2q-1)$ oriented hamilton cycles, H_x^i and H_{x+1}^i for $0 \leq i \leq 2q-2$ and $x = 0, 2, 4, ..., p-2$. (here H_x^i and H_{x+1}^i are the same as H^i defined before). We arrange the labels on the arcs as follows:

in H_x^i, let $\ell(a_1^i) = p - x$;
$\ell(a_j^i) = p + 2 + x$, $2 \leq j \leq q$;
$\ell(a_j^i) = p - 1 - x$, $q + 1 \leq j \leq 2q$, and
in H_{x+1}^i, let $\ell(a_1^i) = p + 2 + x$;
$\ell(a_j^i) = p - x$, $2 \leq j \leq q$;
$\ell(a_j^i) = p + 1 + x$, $q + 1 \leq j \leq 2q$.

Notice that $\{p-1-x, p-x, p+1+x, p+2+x\}$, $x = 0, 2, ..., p-2$, form a partition of $\{1, 2, ..., 2p\}$. By Lemma 2.1(c), $\bigcup_{i=0}^{2q-2} H_x^i \cup H_{x+1}^i$ is the union of four copies of T_{2q}. By Lemma 2.1(b) the subgraphs induced by $\bigcup_{i=0}^{2q-2} \bigcup_{j=1}^{q} \{a_j^i\}$ and $\bigcup_{i=0}^{2q-2} \bigcup_{j=q+1}^{2q} \{a_j^i\}$ are T_{2q}. For example, the label $p - x$ is assigned to a_1^i in H_x^i and to a_j^i for $2 \leq j \leq q$ in H_{x+1}^i. So we can conclude that one copy of T_{2q} is labeled $p - x$. Similarly, one copy of each is labelled $p - 1 - x$, $p + 1 + x$ and $p + 2 + x$.

Finally it remains to prove that the label of each oriented hamilton cycle is relatively prime to $2p + 1$. By Lemma 2.5,

$$\begin{aligned}\ell(H_x^i) &= (p-x) + (q-1)(p+2+x) + (q-1)(p-1-x) - (p-1-x) \\ &= q(2p+1) - 2p \equiv 1 (mod\ 2p+1), \text{ and}\end{aligned}$$

$$\begin{aligned}\ell(H_{x+1}^i) &= (p+2+x) + (q-1)(p-x) + (q-1)(p+1+x) - (p+1+x) \\ &= q(2p+1) - 2p \equiv 1 (mod\ 2p+1).\end{aligned}$$

In all the cases the labels of the hamilton cycles are relatively prime to $2p+1$. By Corollary 2.4, the above decomposition of $2pT_{2q}$ will produce a hamilton cycle decomposition of $K_{2p+1} \otimes K_{2q}$.

Case 1.2. p is odd. As in Case 1.1 we use the $(2q-1)(p-1)$ hamilton cycles H_x^i and H_{x+1}^i for $0 \le i \le 2q-2$ and $x = 0, 2, ..., p-3$. Now the 4-subsets $\{p-1-x, p-x, p+1+x, p+2+x\}$, $x = 0, 2, ..., p-3$, form a partition of $\{2, 3, ..., 2p-1\}$. Hence we decompose the arcs with labels in $\{2, 3, ..., 2p-1\}$ into $(2q-1)(p-1)$ hamilton cycles whose labels are relatively prime to $2p+1$.

It remains to decompose the arcs of $2T_{2q}$ with labels 1 and $2p$ into the $2q-1$ hamilton cycles, H_{p-1}^i for $0 \le i \le 2q-2$:
in H_{p-1}^i, $\ell(a_j^i) = 1$ for $1 \le j \le q$, and
$\ell(a_j^i) = 2p$ for $q+1 \le j \le 2q$.

$\ell(H_{p-1}^i) = q + (q-1)2p - 2p = (2p+1)(q-2) - 2 \equiv -2(mod\ 2p+1)$. $\ell(H_{p-1}^i)$ and $2p+1$ are relatively prime as $2p+1$ is odd.

So far we have shown that $K_r \otimes K_s$ has a hamilton cycle decomposition if one of r and s is odd.

Case 2. r is even.
Let $r = 2p$. We will show that $K_{2p} \otimes K_{2q}$ can be decomposed into $(p-1)(2q-1) + q - 1$ hamilton cycles and a one factor. Again we will exhibit a decomposition of $(2p-1)T_{2q}$ into oriented hamilton cycles with labels relatively prime to $r = 2p$.

Case 2.1. p is even. Note that the set of labels $\{1, 2, ..., 2p-1\}$ can be partitioned into $(p-2)/2$ 4-subsets,
$\{p-2-x, p-1-x, p+1+x, p+2+x\}$, $x = 1, 3, ..., p-3$ and a 3-subset $\{p-1, p, p+1\}$.
We first assign the labels to the arcs of the following $(p-2)(2q-1)$ hamilton cycles:
for $0 \le i \le 2q-2$ and $x = 1, 3, ..., p-3$,
in H_x^i, let $\ell(a_1^i) = p-1-x$;
$\ell(a_j^i) = p+2+x, 2 \le j \le q$;
$\ell(a_j^i) = p-2-x, q+1 \le j \le 2q$, and
in H_{x+1}^i, let $\ell(a_1^i) = p+2+x$;

$$\ell(a_j^i) = p-1-x, 2 \le j \le q;$$
$$\ell(a_j^i) = p+1+x,\ q+1 \le j \le 2q.$$

$$\ell(H_x^i) = p-1-x+(q-1)(p+2+x)+(q-1)(p-2-x)-(p-2-x)$$
$$= 2p(q-1)+1 \equiv 1(mod\ 2p)$$
$$\ell(H_{x+1}^i) = p+2+x+(q-1)(p-1-x)+(q-1)(p+1+x)-(p+1+x)$$
$$= 2p(q-1)+1 \equiv 1(mod\ 2p).$$

Hence the labels of the above hamilton cycles are relatively prime to $2p-1$. So far we have decomposed all the arcs of $(2p-1)T_{2q}$ with labels in $\{1,2,...,2p-1\}\backslash\{p-1,p,p+1\}$. It remains to partition the arcs of the three copies of T_{2q} with labels $p-1,p,p+1$ into $(2q-1)+(q-1)$ hamilton cycles and a one factor. We first assign the labels to the arcs of the $2q-1$ hamilton cycles, $H_{p-1}^i, 0 \le i \le 2q-2$:

in H_{p-1}^i, for $1 \le j \le q$ and $j \ne j_0$,
$\ell(a_j^i) = p+1$, and
for $j = j_0$,
$\ell(a_{j_0}^i) = p$, where
$j_0 = 1$, for $0 \le i \le q-2$ and
$i = q, q+2, ..., 2q-2$ if q is even, and
$i = q, q+2, ..., 2q-3$ if q is odd;
$j_0 = 2$, for $i = q-1, q+1, ..., 2q-3$ if q is even and
$i = q-1, q+1, ..., 2q-2$ if q is odd;
for $q+1 \le j \le 2q$,
$\ell(a_j^i) = p-1$.

Note that the label of each of these hamilton cycles is
$$\ell(H_{p-1}^i) = p+(q-1)(p+1)+(q-1)(p-1)-(p-1) \equiv 1(mod\ 2p).$$

Example: From the decompositions exhibited in the beginning of the section, it is easy to check that the arcs with label p are the following;

when $2q=8$, $a_1^0 = (\infty,1), a_1^1 = (\infty,2), a_1^2 = (\infty,3), a_1^4 = (\infty,5),$
$a_1^6 = (\infty,7), a_2^3 = (4,5), a_2^5 = (6,7)$
when $2q=10$, $a_1^0 = (\infty,1), a_1^1 = (\infty,2), a_1^2 = (\infty,3), a_1^3 = (\infty,4),$
$a_1^5 = (\infty,6), a_1^7 = (\infty,8), a_2^4 = (5,6), a_2^6 = (7,8),$
$a_2^8 = (9,1)$

Note that the arcs not used by the hamilton cycles above form a copy of T_{2q} and these arcs have the label p except for the arcs $a_{j_0}^i$ which have the label $p+1$. Therefore, the last step is to obtain $q-1$ hamilton cycles by decomposing the arcs of this T_{2q}. Using Lemma 2.1 (d) we take the decomposition of T_{2q} into the $q-1$ hamilton cycles H_p^i, for $0 \le i \le q-2$. The labels of each hamilton cycle are fixed and can be described precisely:

for $0 \le i \le q-2$, in H_p^i,

when i is even,

$\ell(a_j^i) = p$ if $1 \le j \le 2q$ and $j \ne 1, 2q-1$ and $2q$,

$\ell(a_1^i) = \ell(a_{2q-1}^i) = \ell(a_{2q}^i) = p+1$;

$(\ell(H_p^i) = p+1+(2q-3)p+p+1-(p+1) \equiv 1(mod\ 2p))$

when i is odd,

$\ell(a_1^i) = p+1$ and $\ell(a_j^i) = p$, $2 \le j \le 2q$.

$(\ell(H_p^i) = p+1+(2q-2)p-p \equiv 1(mod\ 2p))$

Hence the decomposition satisfies the conditions.

Example:

For $2q = 8$, the arcs with labels $p+1$ are
$a_1^0 = (\infty, 1), a_7^0 = (4, 5), a_8^0 = (\infty, 5)$ in H_p^0,
$a_1^2 = (\infty, 3), a_7^2 = (6, 7), a_8^2 = (\infty, 7)$ in H_p^2, and $a_1^1 = (\infty, 2)$ in H_p^1.
For $2q = 10$, the arcs with labels $p+1$ are
$a_1^0 = (\infty, 1), a_9^0 = (5, 6), a_{10}^0 = (\infty, 6)$ in H_p^0,
$a_1^2 = (\infty, 3), a_9^2 = (7, 8), a_{10}^2 = (\infty, 8)$ in H_p^2,
$a_1^1 = (\infty, 2)$ in H_p^1, and $a_1^3 = (\infty, 4)$ in H_p^3.

Notice that when q is odd, the last arc labelled $p+1$, namely $(9, 1)$ (in general $(2q-1, 1)$), appears in the remaining one factor.

The remaining edges form a one factor of T_{2q}. It is clear that this one factor can be used to construct a one factor in $K_{2p} \otimes K_{2q}$.

Case 2.2. p is odd. We can assume that q is odd ($q \ge 3$) (the case when q is even has been dealt with as in Case 2.1).

We partition the set of labels $\{1, 2, ...2p-1\}$ into $(p-3)/2$ 4-subsets $\{p-2-x, p-1-x, p+1+x, p+2+x\}$ for $x = 2, 4, ..., p-3$ and a 5-subset $\{p-2, p-1, p, p+1, p+2\}$. We deal with the arcs of $(2p-1)T_{2q}$ with labels other than $\{p-2, p-1, p, p+1, p+2\}$ in the same way as in Case 2.1.

The remaining edges are five copies of T_{2q} whose arcs are labelled $\{p-2, p-1, p, p+1, p+2\}$ and they are partitioned into $2(2q-1)+(q-1)$ oriented hamilton cycles as follows:

For $0 \le i \le 2q-2$, in H^i_{p-1},

$\ell(a^i_j) = p-2$, for $1 \le j \le q$ and $j \ne 3$;

$\ell(a^i_3) = p+1$;

$\ell(a^i_j) = p+2$, for $q+1 \le j \le 2q$.

So $\ell(H^i_{p-1}) = p+1+(q-1)(p-2)+(q-1)(p+2)-(p+2) \equiv -1(mod\ 2p)$.

For the other $2q-1$ hamilton cycles, we do the following.

For $0 \le i \le 2q-2$, in H^i_p,

for $1 \le j \le q$ and $j \ne 3, j_0$,

$\ell(a^i_j) = p+1$,

$\ell(a^i_3) = p-2$, and

$\ell(a^i_{j_0}) = p$, where

$j_0 = 1$, for $0 \le i \le q-2$ and $i = q, q+2, ..., 2q-3$, and

$j_0 = 2$, for $i = q-1, q+1, ..., 2q-2$;

for $q+1 \le j \le 2q$ and $j \ne 2q-2$,

$\ell(a^i_j) = p-1$, and

$\ell(a^i_{2q-2}) = p$.

$$\ell(H^i_p) = p+p-2+(q-2)(p+1)+(q-2)(p-1)+p-(p-1) \equiv -1(mod\ 2p).$$

Example: When $2q = 10$, from the listed decomposition we can check that the arcs with label p are the following;

$$a^0_1 = (\infty, 1), a^1_1 = (\infty, 2), a^2_1 = (\infty, 3), a^3_1 = (\infty, 4),$$
$$a^5_1 = (\infty, 6), a^7_1 = (\infty, 8), a^4_2 = (5, 6), a^6_2 = (7, 8), a^8_2 = (9, 1)$$

The arcs not used by the hamilton cycles form a copy of T_{2q} and they have label p except the arcs $a^i_{j_0}$ ($\ell(a^i_{j_0}) = p+1$) and a^i_{2q-2} ($\ell(a^i_{2q-2}) = p-1$). Like in the Case 2.1, using Lemma 2.1 (d), we partition these arcs into $q-1$ hamilton cycles $H^i_{p+1}, 0 \le i \le q-2$ and a one factor. The labels on the arcs are fixed as follows:

For $0 \le i \le q-2$, in H^i_{p+1},

when i is even,

$\ell(a^i_1) = \ell(a^i_{2q}) = \ell(a^i_{2q-1}) = p+1$,

$\ell(a^i_3) = \ell(a^i_{2q-2}) = p-1$, and otherwise

$\ell(a^i_j) = p$,
when i is odd,
$\ell(a^i_1) = p + 1$,
$\ell(a^i_3) = \ell(a^i_{2q-2}) = p - 1$, and otherwise
$\ell(a^i_j) = p$.

Example: For $2q = 10$, the arcs with labels $p + 1$ are
$a^0_1 = (\infty, 1), a^0_9 = (5, 6), a^0_{10} = (\infty, 6)$ in H^0_{p+1},
$a^2_1 = (\infty, 3), a^2_9 = (7, 8), a^2_{10} = (\infty, 8)$ in H^2_{p+1},
$a^1_1 = (\infty, 2)$ in H^1_{p+1}, and $a^3_1 = (\infty, 4)$ in H^3_{p+1}.

As in the previous case, the last arc labelled $p + 1$, namely, $(9, 1)$ (in general $(2q - 1, 1)$) appears in the remaining one factor.

When i is even,
$\ell(H^i_{p+1}) = (p + 1) + p + (p - 1) + (2q - 6)p + (p - 1) + (p + 1) - (p + 1)$
$\equiv -1(mod\ 2p)$.
When i is odd,
$\ell(H^i_{p+1}) = (p+1)+p+(p-1)+(2q-6)p+(p-1)+p-p \equiv -1(mod\ 2p)$.

So in all cases $\ell(H^i_{p+1})$ and $2p$ are relatively prime and we have the required hamilton cycle decomposition of $(r - 1)T_{2q}$. ∎

Aknowledgement. The first three authors thank the Indo-French Centre for the Promotion of Advanced Research (Centre Franco-Indien Pour la Promotion de la Recherche Avancee, New Delhi) for the support through Project 401-1. Part of the research was done while the last author was visiting SLOOP (now MASCOTTE), joint project I3S-CNRS/UNSA/INRIA and he was supported by the Natural Science and Engineering Research Council of Canada.

We would also like to thank the referees for some useful remarks.

Reference

1. B. Alspach, J.-C. Bermond and D. Sotteau, Decomposition into Cycles I: Hamilton decompositions in Cycles and Rays, G.Hahn et *al* eds (Kluwer Academic Press 1990) 9-18.

2. P. Aubert and B. Schneider, Decomposition de $K_m + K_n$ en cycles hamiltoniens, Discrete Math. 37 (1981) 19-27.

3. R. Balakrishnan and P. Paulraja, Hamilton cycles in tensor product of graphs, Discrete Math. 186 (1998) 1-13.

4. Z. Baranyai and G. Szasz, Hamiltonin decompositions of lexicographic product, J. Comb. Theory (B), 31 (1981), 253-261.

5. J.-C. Bermond, Hamiltonian decompositions of graphs, digraphs and hypergraphs, Annals of Discrete Math. 3 (1978) 21-28.

6. J. Bosak, Decompositions of Graphs, Kluwer Academic Press 1990.

7. C. Fan and J. Liu, Hamiltonian decompositions of strong products, J. of Graph Theory, 29 (1998) 45-55.

8. P. Jha, Hamilton decompositions of products of cycles, Indian J. Pure and Appl. Math. 23 (1992) 723-729.

9. P. Jha, N. Agnihotri and R. Kumar, Edge exchanges in Hamiltonian decompositions of Kronecker-product graphs, Computers Math. Applic. 31(2) (1996) 11-19.

10. R. Stong, Hamilton decompositions of cartesian product of graphs, Discrete Math. 90 (1991) 169-190.

11. M. Zhou, Decompositions of some product graphs into 1-factors and Hamiltoniian cycles, Ars Combinatoria, 28 (1989) 258-268.