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Realization Theory of Nonlinear Hybrid Systems

Mihály Petreczky * — **Jean-Baptiste Pomet** **

* *Centrum voor Wiskunde en Informatica (CWI)*
P.O.Box 94079, 1090GB Amsterdam, The Netherlands
M.Petreczky@cwi.nl
** *INRIA Sophia Antipolis*
P.O.Box 93, 2004 Route des Lucioles, 6902
Jean-Baptiste.Pomet@sophia.inria.fr

ABSTRACT. *The paper investigates the realization problem for a class of analytic nonlinear hybrid systems without autonomous switching. Similarly to the classical nonlinear realization theory the realization problem for hybrid systems is translated to a formal realization problem of a class of abstract systems defined on rings of formal power series. Necessary conditions are presented for existence of a realization by such an abstract system and thus by a hybrid system. A notion analogous to the Lie-rank of nonlinear input-output maps is defined and the presented necessary condition involves a requirement that this generalised Lie-rank should be finite. We will also introduce the notion of strong Lie-rank and we will show that finiteness of the strong Lie-rank implies existence of a realization which is very close to the required hybrid system realization. Thus, finiteness of the strong Lie-rank can be seen as an "almost" sufficient condition. In the special case of nonlinear analytic systems both the finite Lie-rank and the finite strong Lie-rank condition presented in the paper reduces to the well-known finite Lie-rank condition. We will use theory of Sweedler-type coalgebras for studying the formal realization problem.*

RÉSUMÉ.

KEYWORDS: *hybrid systems, realization theory, Lie-rank, coalgebra, Hopf-algebra*

MOTS-CLÉS :

1. Introduction

Realization problem is one of the central problems of systems theory. Its aim is to find conditions under which an input-output map can be represented as' an input-output map of a certain system.

The aim of the paper is to investigate the realization problem for a class of hybrid systems which will be called *hybrid systems without guards*. That is, discrete events play the role of discrete inputs and a discrete event can be sent to the system at any time. Thus, one can trigger at any time a discrete state transition associated with a chosen discrete event.

In this paper we will address the following question. Consider an input-output map and formulate conditions for existence of a realization by a nonlinear hybrid system without guards.

The problem as it is stated above is quite difficult, therefore we will adopt a number of simplifications. First of all we will restrict ourselves to *analytic hybrid systems* , i.e. hybrid systems such that the underlying continuous control systems are analytic and the reset maps are analytic. To simplify the problem further, we will look only at local and formal realization. That is, we will try to find conditions with respect to which the input-output map coincides with the input-output map of a hybrid system locally, i.e. for small times. To facilitate the transition from global to the local problem we will introduce the concept of the hybrid Fliess-series expansion. Roughly speaking, an input-output map admits a hybrid Fliess-series expansion if its continuous-valued part can be represented as an infinite series of iterated integrals of the continuous inputs. The coefficients of these iterated integrals form a sequence which completely determines the input-output map locally. We will refer to this sequence as the *hybrid generating series associated with the input-output map*. Existence of a hybrid Fliess-series expansion is a necessary condition for existence of a local realization by an analytic hybrid system. The associated hybrid generating series can be thought of as a collection of high-order derivatives of the input-output map. It turns out that a necessary condition for existence of a hybrid system realization for an input-output map is that the corresponding generating series admits a representation of a particular form.

To be more precise, since the hybrid systems considered are analytic, we can associate with each underlying continuous system a formal power series ring, a finite family of continuous derivations and a formal power series. The formal power series ring corresponds to the ring of Taylor-series expansions of analytic functions around a point, the derivations are just the Taylor-series expansion of the vector fields of the system and the formal power series is just the Taylor series expansion of the readout map of the system. In the context of the transformation described above the analytic reset maps become continuous homomorphisms on formal power series rings, by taking the Taylor series expansion of each reset map around a suitably chosen point.

In this manner we get a construct which we will call a *formal hybrid system*.

The concept of formal hybrid system allows us to reformulate the necessary condition for existence of a hybrid system realization mentioned above. Namely, it turns out that existence of a realization by an analytic hybrid system implies that the generating series associated with the hybrid Fliess-series expansion of the input-output map *has a realization by a formal hybrid system*. Conversely, if we have a formal hybrid system such that the vector fields, reset maps and readout maps are in fact convergent formal power series, it will immediately yield us a hybrid system.

In fact, most of the paper is devoted to the realization problem for formal hybrid systems. That is, consider a map mapping sequences of discrete and continuous inputs symbols to discrete and continuous outputs. We would like to find necessary and sufficient conditions for existence of a formal hybrid system realizing this map. We will be able to present some necessary conditions and some results which indicate that these necessary conditions are very close to being sufficient ones.

The approach to realization theory of analytic hybrid systems sketched above is very similar to the classical approach to local realization theory of analytic nonlinear systems, [JAK 86, FLI 80].

In this paper we will use the theory of Sweedler-type coalgebras. Note that Sweedler-type coalgebras are not identical to coalgebras used by Jan Rutten ([RUT]). Although Sweedler-type coalgebra are a special case of the category theoretical coalgebras, they have much more structure. Roughly speaking a Sweedler-type coalgebra is a vector space on which a so called comultiplication and counit are defined. We will show that existence of a formal hybrid system realization is equivalent to existence of a realization by an abstract system of a certain type, which we will call *CCPI hybrid coalgebra systems*. Roughly speaking such a system is a system, state space of which is a coalgebra satisfying certain properties. Our efforts will be directed towards finding conditions for existence of such a hybrid coalgebra realization.

This paper is not the first attempt to use coalgebras for hybrid system. Already the paper by [GRO 95] advocated an approach based on coalgebras, and this paper uses similar ideas. Although the stated goal of the paper by Grossman and Larson was to use coalgebra theory for developing realization theory for hybrid systems, it just presented some reformulation of the already known results for finite-state automata and nonlinear control systems. It did not contain any new results for hybrid systems. The main contribution of the current paper when compared to the paper by Grossman and Larson is that it does present conditions for existence of a realization by hybrid systems. Moreover, the class of hybrid systems studied in this paper is more general and closer to what is generally understood as hybrid systems than the one in Grossman's and Larson's paper.

The approach to realization theory adopted in this paper bears resemblance to [GRU 94].

Realization of hybrid systems was addressed in a number of papers [PET 05b, PET , PET 05a]. In particular, [PET 05a] dealt with realization of bilinear and linear hybrid systems, i.e. hybrid systems without guards such that the continuous control systems are linear or bilinear and the reset maps are linear. In [PET 05a] necessary and sufficient conditions were derived. Note that linear and bilinear hybrid systems are special cases of analytic hybrid systems studied in this chapter. The conditions for existence of a linear (bilinear) hybrid system realization imply the conditions derived in this chapter, thus the results of the current chapter are consistent with the previous ones.

Let us present an informal summary of the main results of the paper.

– An input-output map has a realization by a hybrid system if and only if it has a hybrid Fliess-series expansion and the corresponding convergent generating series has a realization by a formal hybrid system such that all the readout maps and vector fields are convergent.

– A convergent generating series is a map, which maps sequences of discrete events and input symbols to continuous and discrete outputs. Such a map has a realization by a formal hybrid system, if it has a realization by a hybrid coalgebra system of a certain type (CCPI hybrid coalgebra system).

– We define the *Lie-rank* and *strong Lie-rank* of the input-output map. We will prove that if a map has a CCPI hybrid coalgebra system realization (equivalently it has a formal hybrid system realization), then its Lie-rank is finite. If its strong Lie-rank is finite, then it has a hybrid coalgebra realization which is very similar to a CCPI hybrid coalgebra realization We will prove that an input-output map cannot have a CCPI hybrid coalgebra realization (formal hybrid system realization), dimension of which is smaller than the Lie-rank of the map. We will also present a hybrid system, which can not be realized by a system, dimension of which equals the Lie-rank of the input-output map.

The outline of the chapter is the following. Section 2 settles the notation and terminology used in the paper. Section 3 presents the necessary results and terminology on formal power series and coalgebras. The reader might postpone reading this section until Section 7. Section 6 discusses the notion of hybrid Fliess-series expansion and characterises the input-output maps of hybrid systems in terms of Fliess-series expansion. Section 7 presents the relationship between local realization and formal realization problem. Section 8 presents the conditions for existence of a formal hybrid system realization.

A more detailed presentation of the results of this paper can be found in [PET 06].

2. Notation and terminology

For an interval $A \subseteq \mathbb{R}$ and for a suitable set X denote by $PC(A, X)$ the set of piecewise-continuous maps from A to X, i.e., maps which have at most finitely many points of discontinuity on any bounded interval and at any point of discontinuity the left-hand and the right-hand side limits exist and are finite. For a set Σ denote by Σ^* the set of finite strings of elements of Σ. For $w = a_1 a_2 \cdots a_k \in \Sigma^*$, $a_1, a_2, \ldots, a_k \in$

Σ the length of w is denoted by $|w|$, i.e. $|w| = k$. The empty sequence is denoted by ϵ. The length of ϵ is zero: $|\epsilon| = 0$. Let $\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$. The concatenation of two strings $v = v_1 \cdots v_k, w = w_1 \cdots w_m \in \Sigma^*$ is the string $vw = v_1 \cdots v_k w_1 \cdots w_m$. We denote by w^k the string $\underbrace{w \cdots w}_{k-times}$. The word w^0 is just the empty word ϵ. Denote by T the set $[0, +\infty) \subseteq \mathbb{R}$. Denote by $\mathbb{N}$ the set of natural numbers including 0. Denote by $F(A, B)$ the set of all functions from the set A to the set B. For any two sets A, B, define the functions $\Pi_A : A \times B \to A$ and $\Pi_B : A \times B \to B$ by $\Pi_A(a, b) = a$ and $\Pi_B(a, b) = b$. By abuse of notation we will denote any constant function $f : T \to A$ by its value. That is, if $f(t) = a \in A$ for all $t \in T$, then f will be denoted by a. For any function f the range of f will be denoted by $\mathrm{Im} f$. If A, B are two sets, then the set $(A \times B)^*$ will be identified with the set $\{(u, w) \in A^* \times B^* \mid |u| = |w|\}$. For any set A we will denote by $card(A)$ the cardinality of A.

For a finite set Σ denote by $\mathbb{R} < \Sigma^* >$ the set of all finite formal linear combinations of words over Σ. That is, a typical element of $\mathbb{R} < \Sigma^* >$ is of the form $\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_k w_k$, where $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ and $w_1, \ldots, w_k \in \Sigma^*$. It is easy to see that $\mathbb{R} < \Sigma^* >$ is a vector space. Moreover, we can define a linear associative multiplication on $\mathbb{R} < \Sigma^* >$, by $(\sum_{i=1}^N \alpha_i w_i)(\sum_{j=1}^M \beta_j v_j) = \sum_{i=1}^N \sum_{j=1}^M \alpha_i \beta_j w_i v_j$. The element ϵ which we will identify with 1 is the neutral element with respect to multiplication. It is easy to see that $R < \Sigma^* >$ is an algebra with the multiplication defined above.

3. Algebraic preliminaries

The goal of this section is to give a brief overview of the algebraic notions used in this chapter and to fix the notation and terminology. The material presented in this section is standard. The reader is strongly encouraged to consult the references provided in the text for further details. Subsection 3.1 presents a summary on formal power series in finitely many commuting variables. Subsection 3.2 presents the necessary preliminaries on Sweedler-type coalgebras. In this chapter in general, and throughout this section in particular we will assume that the reader is familiar with such basic algebraic notions as ring, algebra, ideal, module etc. The reader is referred to any textbook in this subject, for example [ZAR 75].

3.1. *Preliminaries on Formal Power Series*

The goal of this subsection is to present a very short overview of the main properties of formal power series in commuting variables. For a more detailed exposition the reader should consult [ZAR 75].

Consider the set $\mathbb{N}^n$ and define addition on this set as follows. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, then let $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n)$. The ring of formal power series $\mathbb{R}[[X_1, \ldots, X_n]]$ in commuting variables $X_1, X_2, \ldots, X_n$ is defined as the $\mathbb{R}$ vector space of formal infinite sums $S = \sum_{\alpha\in\mathbb{N}^n} S_\alpha X^\alpha$, where $X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$. Addition, multiplication are defined by $(\sum_{\alpha\in\mathbb{N}^n} S_\alpha X^\alpha) + (\sum_{\alpha\in\mathbb{N}^n} T_\alpha X^\alpha) = \sum_{\gamma\in\mathbb{N}^n} (S_\gamma + T_\gamma) X^\gamma$ and $(\sum_{\alpha\in\mathbb{N}^n} S_\alpha X^\alpha) \cdot (\sum_{\alpha\in\mathbb{N}^n} T_\alpha X^\alpha) = \sum_{\gamma\in\mathbb{N}^n} (\sum_{\alpha+\beta=\gamma} S_\alpha T_\beta) X^\gamma$ Multiplication by scalar is defined as $a(\sum_{\alpha\in\mathbb{N}^n} S_\alpha X^\alpha) = \sum_{\alpha\in\mathbb{N}^n} a S_\alpha X^\alpha$. The neutral element for addition is $\sum_{\alpha\in\mathbb{N}^n} S_\alpha X^\alpha$, with $S_\alpha = 0$ for all $\alpha \in \mathbb{N}^n$. The neutral element for multiplication is $\sum_{\alpha\in\mathbb{N}^n} S_\alpha X^\alpha$ with $S_{(0,0,\ldots,0)} = 1$ and $S_\alpha = 0$ for all other $\alpha \in \mathbb{N}$. The latter element will be denoted simply by 1. It is easy to see that $\mathbb{R}[[X_1, \ldots, X_n]]$ forms an algebra with the operations above. For each $\alpha \in \mathbb{N}^n$ let $deg(\alpha) = \sum_{j=1}^n \alpha_i$. For each $n \in \mathbb{N}$ define the ideal $I_n = \{\sum_{\alpha\in\mathbb{N}^n} S_\alpha X^\alpha \mid S_\alpha = 0 \text{ for all } \alpha \in \mathbb{N}^n, deg(\alpha) \leq n\}$. We define the Zariski topology on $\mathbb{R}[[X_1, \ldots, X_n]]$ as the topology generated by the open sets $f + I_n$ for $f \in \mathbb{R}[[X_1, \ldots, X_n]]$ and $n \in \mathbb{N}$. A map $D : \mathbb{R}[[X_1, \ldots, X_n]] \to \mathbb{R}[[Y_1, \ldots, Y_m]]$ is said to be continuous if it it continuous with respect to the Zariski topology. A map $D : \mathbb{R}[[X_1, \ldots, X_n]] \to \mathbb{R}$ is said to be continuous, if it is continuous as a map between topological spaces, where $\mathbb{R}[[X_1, \ldots, X_n]]$ is considered with the Zariski topology and $\mathbb{R}$ is considered with the discrete topology Recall that if A, B are two $\mathbb{R}$ algebras, then a linear map $f : A \to B$ is called a *derivation*, if the Leibniz-rule holds. That is, $f(ab) = af(b) + bf(a)$. If $f(ab) = f(a)f(b)$, then we will call f an algebra morphism. If $f : A \to B$ is a continuous algebra morphism, then it is uniquely determined by the values $f(X_i) \in B, i = 1, \ldots, n$.

Denote by A the ring $A = \mathbb{R}[[X_1, \ldots, X_n]]$. Denote by D_i, $i = 1, \ldots, n$ the continuous derivations $D_i : A \to \mathbb{R}$ such that $D_i(X_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. Denote by 1_A^* the map $1^* : A \to \mathbb{R}$ such that $1_A^*(\sum_{\alpha\in\mathbb{N}^n} a_\alpha X^\alpha) = a_{(0,0,\ldots,0)}$. It is well-known ([ZAR 75]) that 1_A^* is a continuous algebra morphism. Denote by $\frac{d}{dX_i}, i = 1, \ldots, n$ the ith partial derivative of the ring $A = \mathbb{R}[[X_1, \ldots, X_n]]$. That is, $\frac{d}{dX_i} : A \to A$ is a continuous derivation such that $\frac{d}{dX_i}(X_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$ The set of all continuous derivations $A \to A$ forms an A module and any continuous derivation $D : A \to A$ can be written as $D = \sum_{j=1}^n S_i \frac{d}{dX_i}$, where $S_i \in A$. Notice that for any continuous derivation $D : A \to A$ the map $1_A^* \circ D : A \to A$ defines a continuous derivation to $\mathbb{R}$. It is also well-known that $D_i = 1_A^* \circ \frac{d}{dX_i}$ for all $i = 1, \ldots, n$. For each $k \in \mathbb{N}$ denote by $\frac{d^k}{dX_i^k}$ the maps $\underbrace{\frac{d}{dX_i} \circ \cdots \circ \frac{d}{dX_i}}_{k-times} : A \to A$, If $k = 0$ the we assume that $\frac{d}{dX_i}^0(h) = h$, i.e., $\frac{d}{dX_i}^0$ is the identity map. For each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ define the map $\frac{d}{dX}^\alpha$ as $\frac{d}{dX_1}^{\alpha_1} \circ \frac{d}{dX_2}^{\alpha_2} \circ \cdots \circ \frac{d}{dX_n}^{\alpha_n} : A \to A$.

3.2. *Preliminaries on Sweedler-type Coalgebras*

The goal of this subsection is to give a very short introduction to the field of coalgebras, bialgebras. Readers for whom this is the first encounter with the field are strongly encouraged to consult the book [SWE 69].

Let k be a field of characteristic 0, for our purposes the reader can assume that $k = \mathbb{R}$. Recall the notion of *a tensor product of two vector spaces* and recall that the tensor product of A and B is denoted by $A \otimes B$. A tuple (C, δ, ϵ) is called a *coalgebra* if C is a k-vector space, $\delta : C \to C \otimes C$ and $\epsilon : C \to k$ are k-linear maps such that a number of properties hold. We require the following conditions to hold for coalgebras. For each $c \in C$, if $\delta(c) = \sum_{i=1}^{m} c_{i,1} \otimes c_{i,2}$, then $\sum_{i=1}^{m} c_{i,1} \otimes \delta(c_{i,2}) = \sum_{i=1}^{m} \delta(c_{i,1}) \otimes c_{i,2} \in C \otimes C \otimes C$ and $c = \sum_{i=1}^{m} \epsilon(c_{i,1})c_{i,2} = \sum_{i=1}^{m} \epsilon(c_{i,2})c_{i,1}$. The first condition is referred to as *coasscociativity*. The second condition says that ϵ has the *counit property*. If in addition, for each $c \in C$, $\delta(c) = \sum_{i=1}^{m} c_{i,1} \otimes c_{i,2} = \sum_{i=1}^{m} c_{i,2} \otimes c_{i,1}$, then we will say that (C, δ, ϵ) is *cocommutative*. The map δ will be referred to as the *comultiplication* and the map ϵ will be referred to as the *counit*. A map T is said to be a coalgebra map from coalgebra (C, δ, ϵ) to coalgebra $(B, \delta^{'}, \epsilon^{'})$ if $T : C \to B$ is a linear map such that $\epsilon^{'} = \epsilon \circ T$ and $(T \otimes T) \circ \delta = \delta^{'} \circ T$, where $T \otimes T : C \otimes C \ni c_1 \otimes c_2 \mapsto T(c_1) \otimes T(c_2)$. In the sequel we will denote a coalgebra (C, δ, ϵ) simply by C and if T is a coalgebra map from (C, δ, ϵ) to (B, δ, ϵ) we will write $T : C \to B$ and we will state that T is a coalgebra map.

Recall that a k-vector space A with k-linear maps $M : A \otimes A \to A$ and $u : k \to A$ is called an *algebra* if M defines an associative multiplication and $u(1)$ defines the unit element. That is, for each $a, b, c \in A$, $M(a, M(b, c)) = M(M(a, b), c)$ and $M(a, u(1)) = M(u(1), a) = a$. If in addition M defines a commutative multiplication, that is, $M(a, b) = M(b, a)$ for all $a, b \in A$, then we will say that A is a *commutative algebra*. As usual in mathematics, we will write ab instead of $M(a, b)$ and 1 instead of $u(1)$ if the maps M and u are clear from the context. All the notions we are going to use for algebras such as ideals, maximal ideals, etc. are the standard ones, the reader can consult [ZAR 75]. For any k-vector space V denote by V^* the linear dual of it, that is, $V^* = \{f : V \to k \mid f \text{ is a linear map}\}$ It is easy to see that if C is a coalgebra, then the vector space C^* is an algebra with the multiplication and unit defined as follows. For each $c_1^*, c_2^* \in C^*$ let $M(c_1^*, c_2^*)(c) = \sum_{i=1}^{m} c_1^*(c_{i,1})c_2^*(c_{i,2})$, where $\delta(c) = \sum_{i=1}^{m} c_{i,1} \otimes c_{i,2}$. Going back to defining the algebra structure on C^*, we will define the unit u as follows. For each $s \in k$ let $u(s)(c) = s\epsilon(c)$. It is not difficult to see that u can be identified with ϵ^* and $M = \delta^* \circ i$, where $i : C^* \otimes C^* \to (C \otimes C)^*$ is the natural inclusion defined by $i(c_1^* \otimes c_2^*)(c) = c_1^*(c)c_2^*(c)$ for all $c_1^*, c_2^* \in C^*$, $c \in C$. If C is a cocommutative coalgebra, then C^* is a commutative algebra. If $f : C \to D$ is a coalgebra map, then $f^* : D^* \to C^*$ is an algebra map, where $f^*(d^*)(c) = d^*(f(c))$ for all $d^* \in D^*$ and $c \in C$. That is, f^* is the usual dual map of f, as it is usually defined in linear algebra.

Notice that if $(C, \delta_C, \epsilon_D)$ and $(D, \delta_D, \epsilon_D)$ are coalgebras, then $C \otimes D$ has a natural coalgebra structure $(C \otimes D, \delta^{'}, \epsilon^{'})$, where $\delta^{'}(c \otimes d) = \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{i,1} \otimes d_{j,1}) \otimes$

$(c_{i,2} \otimes d_{j,2}) \in (C \otimes D) \otimes (C \otimes D)$ and $\epsilon^{'}(c \otimes d) = \epsilon_C(c)\epsilon_D(d)$. with the assumption that $c, d \in C$, $\delta_C(c) = \sum_{i=1}^{m} c_{i,1} \otimes c_{i,2}$ and $\delta_D(d) = \sum_{j=1}^{n} d_{j,1} \otimes d_{j,2}$. Similarly, if A is an algebra, then $A \otimes A$ has a natural algebra structure $(A \otimes A, M^{'}, u^{'})$ where $M^{'}((a \otimes b), (a^{'} \otimes b^{'}) = (aa^{'} \otimes bb^{'})$ and $u^{'}(1) = u(1) \otimes u(1)$. It is easy to see that the ground field k has a natural algebra and coalgebra structure.

We will say that $(C, \delta, \epsilon, M, u)$ is a *bialgebra* if (C, δ, ϵ) is a coalgebra, (C, M, u) is an algebra, δ, ϵ are algebra morphisms and M, u are coalgebra morphisms. Here, we assumed that $C \otimes C$ has the natural algebra and coalgebra structure inherited from C, see the discussion above.

If C is a coalgebra, then a subspace $J \subseteq C$ is called *coideal* if $\delta(J) = J \otimes C + C \otimes J$ and $J \subseteq \ker \epsilon$. A subspace $D \subseteq C$ is called *subcoalgebra* if $\delta(D) \subseteq D \otimes D$. If J is a coideal of C, the the quotient space C/J admits a natural coalgebra structure, such that the canonical projection $\pi : C \ni c \mapsto [c] \in C/J$ is a coalgebra map. Conversely, if $f : C \to D$ is a coalgebra map, then $\ker f$ is a coideal and $C/\ker f$ is isomorphic to $\mathrm{Im} f$ as a coalgebra. Recall the duality between algebras and coalgebras. For any coalgebra C and any subspace $D \subseteq C$, denote by $D^{\perp}$ the annihilator $D^{\perp} = \{c^* \in C^* \mid \forall d \in D : c^*(d) = 0\} \subseteq C^*$. Conversely, for any subspace $A \subseteq C^*$ denote by $A^{\perp} = \{c \in C \mid \forall a \in A : a(c) = 0\}$. Then it follows that for any subspace $D \subseteq C$, $(D^{\perp})^{\perp} = D$. If D is a subcoalgebra of C, then $D^{\perp}$ is an ideal in C^*. If $A \subseteq C^*$, then $A^{\perp}$ is a coideal in C. It is also easy to see that $(C/A^{\perp})^*$ is isomorphic to $A \subseteq (A^{\perp})^{\perp}$.

For a coalgebra C an element $g \in C$ such that $\delta(g) = g \otimes g$ and $\epsilon(g) = 1$ will be called *of group-like element of* C. The set of all group-like elements of C will be denoted by $G(C)$. An element $p \in C$ will be called *primitive* if $\delta(p) = g \otimes p + p \otimes g$ for some group-like element $g \in G(C)$ and $\epsilon(p) = 0$. The set of all primitive elements will be denoted by $P(C)$. A subcoalgebra $D \subseteq C$ is called *simple* if D does not contain any proper subcoalgebra, i.e. if $S \subseteq D$ is a subcoalgebra, then either $S = \{0\}$ or $S = D$. The coalgebra C is called *pointed* if every simple coalgebra D of C is of dimension one. That is, C is pointed if every simple subcoalgebra D of C is of the form $D = \{\alpha g \mid \alpha \in k\}$ for some group-like element $g \in G(C)$. A coalgebra C is called *irreducible*, if for every pair of subcoalgebras $S, D \subseteq C$, $S \cap D \neq \{0\}$, unless either $S = \{0\}$ or $D = \{0\}$. If C is pointed irreducible, then it follows that C has a unique group-like element g, i.e. $G(C) = \{g\}$ and for any subcoalgebra $\{0\} \neq D \subseteq C$, $g \in D$. If C is cocommutative, then $C = \bigoplus_{i \in I} C_i$ such that C_i is an irreducible subcoalgebra of C and there is no irreducible subcoalgebra of C properly containing C_i. Such C_is will be called *irreducible components of* C. Thus, an irreducible component of a coalgebra C is a subcoalgebra $D \subseteq C$ such that for each irreducible subcoalgebra $S \subseteq C$, if $D \subseteq S$, then $S = D$. If $f : C \to D$ is a algebra morphism, then $f(G(C)) \subseteq G(D)$ and $f(P(C)) \subseteq P(D)$. Moreover, if f is surjective, then $f(G(C)) = G(D)$. It also holds that if C is pointed irreducible, then $f(C)$ is pointed irreducible too.

Let A, B be algebras and let C be a coalgebra and consider a linear map $\psi : C \otimes A \to B$. We will say that ψ is a *measuring* , if for all $c \in C$, $a, b \in A$, $\psi(c \otimes ab) = \sum_{i=1}^{n} \psi(c_{i,1} \otimes a)\psi(c_{i,2} \otimes b)$ where $\delta(c) = \sum_{i=1}^{n} c_{i,1} \otimes c_{i,2}$.

Let V be a k-vector space and define the *cofree commutative pointed irreducible coalgebra* $B(V)$ as the cocommutative pointed irreducible coalgebra for which the following holds.

– There exists a linear map $\pi : B(V) \to V$

– If C is a cocommutative pointed irreducible coalgebra, $C^{+} = \ker \epsilon$ and $f : C^{+} \to V$ is a linear map, then there exists a unique coalgebra map $F : C \to B(V)$ such that $\pi \circ F|_{C^{+}} = f$.

It is known that $B(V)$ exists for each vector space V and $P(B(V)) = V$. Moreover, for each cocommutative pointed irreducible coalgebra C there exists a unique injective coalgebra $\pi : C \to B(P(C))$ such that $\pi|_{P(C)} : P(C) \to P(C)$ is the identity map. It is also known that if $k = \mathbb{R}$ and $\dim V = n < +\infty$ then the dual $B(V)^{*}$ of V is isomorphic to the algebra of formal power series $\mathbb{R}[[X_1, \ldots, X_n]]$ in n commuting variables (in fact, it holds for any field k of characteristic zero that $B(V)^{*} \cong k[[X_1, \ldots, X_n]]$).

4. Moore-automata

In this section we will give a brief overview of realization theory of finite Moore-automaton. The material is classical, see [G´ 72, EIL 74] for more on this topic. A *finite Moore-automaton* is a tuple $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ where Q, Γ are finite sets, $\delta : Q \times \Gamma \to Q$, $\lambda : Q \to O$. The set Q is called the *state-space*, O is called the *output space* and Γ is called the *input space*. The function δ is the *state-transition map*, λ is the *readout map*. Denote by $card(\mathcal{A})$ the cardinality of the state-space Q of $\mathcal{A}$, i.e. $card(\mathcal{A}) = card(Q)$.

Define the functions $\widetilde{\delta} : Q \times \Gamma^{*} \to Q$ and $\widetilde{\lambda} : Q \times \Gamma^{*} \to O$ as follows. Let $\widetilde{\delta}(q, \epsilon) = q$ and $\widetilde{\delta}(q, w\gamma) = \delta(\widetilde{\delta}(q, w), \gamma), w \in \Gamma^{*}, \gamma \in \Gamma$ Let $\widetilde{\lambda}(q, w) = \lambda(\widetilde{\delta}(q, w)), w \in \Gamma^{*}$. By abuse of notation we will denote $\widetilde{\delta}$ and $\widetilde{\lambda}$ simply by δ and λ respectively.

Let $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ and $q_0 \in Q$. The pair $(\mathcal{A}, q_0)$ is said to be *an automaton realization* of $\phi : \Gamma^{*} \to O$ if $\lambda(q_0, w) = \phi(w), \forall w \in \Gamma^{*}, j \in J$ An automaton $\mathcal{A}$ is said to be a realization of ϕ if there exists a $q_0 \in Q$ such that $(\mathcal{A}, q_0)$ is a realization of ϕ. Let $(\mathcal{A}, q_0)$ and $(\mathcal{A}^{'}, q_0^{'})$ be two automaton realizations. Assume that $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ and $\mathcal{A}^{'} = (Q^{'}, \Gamma, O, \delta^{'}, \lambda^{'})$. A map $S : Q \to Q^{'}$ is said to be an *automaton morphism* from $(\mathcal{A}, q_0)$ to $(\mathcal{A}^{'}, q_0^{'})$, denoted by $S : (\mathcal{A}, q_0) \to (\mathcal{A}^{'}, q_0^{'})$ if $S(\delta(q, \gamma)) = \delta^{'}(S(q), \gamma), \forall q \in Q, \gamma \in \Gamma$, $\lambda(q) = \lambda^{'}(S(q)), \forall q \in Q$, $S(q_0) = q_0^{'}$. An automaton realization $(\mathcal{A}, q_0)$ of $\phi : \Gamma^{*} \to O$ is called *minimal* if for each automaton realization $(\mathcal{A}^{'}, q_0^{'})$ of ϕ $card(\mathcal{A}) \leq card(\mathcal{A}^{'})$. Let $\phi : \Gamma^{*} \to O$. For every $w \in \Gamma^{*}$ define $w \circ \phi : \Gamma^{*} \to O$*–the left shift of* ϕ *by* w as $w \circ \phi(v) = \phi(wv)$. Define the set $W_{\phi} \subseteq F(\Gamma^{*}, O)$ by $W_{\phi} = \{w \circ \phi : \Gamma^{*} \to O \mid w \in \Gamma^{*}\}$. An automaton $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ is called *reachable from* $Q_0 \subseteq Q$, if $\forall q \in Q : \exists w \in \Gamma^*, q_0 \in Q_0 : q = \delta(q_0, w)$. A realization $(\mathcal{A}, q_0)$ is called reachable if $\mathcal{A}$ is reachable from $\{q_0\}$. A realization $(\mathcal{A}, q_0)$ is called *observable* or *reduced*, if $\forall q_1, q_2 \in Q : [\forall w \in \Gamma^* : \lambda(q_1, w) = \lambda(q_2, w)] \implies q_1 = q_2$. The following result is a simple reformulation of the well-known properties of realizations by automaton. For references see [EIL 74].

Theorem 1. *Let $\phi : \Gamma^* \to O$. ϕ has a realization by a finite Moore-automaton if and only if W_ϕ is finite. In this case a realization of ϕ is given by $(\mathcal{A}_{can}, \phi)$ where $\mathcal{A} = (W_\phi, \Gamma, O, L, T)$, and $L(\phi, \gamma) = \gamma \circ \phi$, $T(\phi) = \phi(\epsilon)$, $\phi \in W_{\mathcal{D}}, \gamma \in \Gamma$. The realization $(\mathcal{A}_{can}, \phi)$ is reachable and observable. A finite Moore-automaton $(\mathcal{A}, q_0)$ is minimal if and only if it is reachable and observable. All minimal realizations of ϕ are isomorphic*

5. Nonlinear Hybrid Systems

In this subsection we will present the formal definition and some elementary properties of *nicely analytic input-affine hybrid systems without guards*.

Definition 1. *A tuple*

$$H = (\mathcal{A}, (\mathcal{X}_q, g_{q,j}, h_{q,i})_{q \in Q, j=0,\dots,m, i=1,\dots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, \{x_q\}_{q \in Q})$$

is called a nicely analytic input-affine hybrid system (abbreviated as NHS) *if the following holds.*

– $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ – *is a Moore-automaton*

– $\mathcal{X}_q = \mathbb{R}^{n_q}$ *for some* $n_q \in \mathbb{N}$ *and* $\mathcal{X}_q$ *is viewed as a real analytic manifold.*

– *For each* $q \in Q$, $j = 1, \dots, m$, *the map* $g_{q,j} : \mathcal{X}_q \to \mathbb{R}^{n_q}$ *is a real analytic map. With the usual identification of* R^{n_q} *with the tangent space of* $\mathcal{X}_q = \mathbb{R}^{n_q}$ *at any point,* $g_{q,j}$ *can be viewed as a vector field.*

– *For each* $q \in Q$ *and* $i = 1, \dots, p$ *the map* $h_{q,i} : \mathcal{X}_q \to \mathbb{R}$ *is a real analytic map.*

– *For each* $q \in Q$, $\gamma \in \Gamma$, *the maps* $R_{\delta(q,\gamma),\gamma,q} : \mathcal{X}_q \to \mathcal{X}_{\delta(q,\gamma)}$ *are real analytic.*

– *There exists a collection* $\{x_q \in \mathcal{X}_q \mid q \in Q\}$ *of continuous states, such that for each* $q \in Q$

$$\forall \gamma \in \Gamma : R_{\delta(q,\gamma),\gamma,q}(x_q) = x_{\delta(q,\gamma)}$$

The set Q of states of $\mathcal{A}$ is called the *set discrete modes*, the input alphabet Γ of $\mathcal{A}$ is called the *set of discrete events*. The space $\mathcal{U} = \mathbb{R}^m$ will be viewed as the *space of continuous inputs* and the space $\mathcal{Y} = \mathbb{R}^p$ will be viewed as the *space of continuous outputs*. The vector fields $f_{q,j}$, $j = 1, \dots, p$ give rise to the following vector field $f_q : \mathcal{X}_q \times \mathcal{U} \to \mathbb{R}^{n_q}$ which depends on the continuous inputs from $\mathcal{U} = \mathbb{R}^m$. defined by $f_q(x, u) = g_{q,0}(x) + \sum_{j=1}^{m} g_{q,j}(x) u_j$. Here $u = (u_1, \dots, u_m)^T \in \mathbb{R}^m$. The maps $h_{q,i}, i = 1, \dots, p$ yield a map $h_q : \mathcal{X}_q \ni x \mapsto (h_{q,1}(x), \dots, h_{q,p}(x))^T \in \mathcal{Y} = \mathbb{R}^p$

Thus, for each discrete state $q \in Q$ the maps $f_{q,j}$, $j = 0, \ldots, m$ and $h_{q,i}$, $i = 0, \ldots p$ define the following analytic input-affine control system

$$\frac{d}{dt}x(t) = f_q(x(t), u(t)) = g_{q,0}(x) + \sum_{j=1}^{m} g_{q,j}(x(t))u_j(t), \quad y(t) = h_q(x(t))$$

That is, the tuple $(\mathcal{X}_q, f_q, h_q)$ can be viewed as the continuous input-affine control system associated with the discrete state $q \in Q$. In order to avoid technicalities concerning the existence and domain of definition of the solution of the differential equation $\frac{d}{dt}x(t) = f_q(x(t), u(t)$. we will assume that f_q, is *globally Lipschitz*. Thus, the solution of the differential equation $\frac{d}{dt}x(t) = f_q(x(t), u(t))$ is well-defined for all $t \in \mathbb{R}$ and u piecewise-continuous functions, i.e., $u \in PC(\mathbb{R}, \mathcal{U})$. When it is clear from the context, we will refer to nicely analytic input-affine hybrid systems simply as hybrid systems.

Denote by $\mathcal{H}_H = \bigcup_{q \in Q}\{q\} \times \mathcal{X}_q$ the *state space of the hybrid system* H. Denote by $\mathcal{X}_H = \bigcup_{q \in Q} \mathcal{X}_q$ the set of *continuous states of* H and denote by $\mathcal{A}_H = \mathcal{A}$ the *Moore automaton of the hybrid system* H. If it is clear from the context which hybrid system we mean, then for the sake of simplicity we will omit the subscript and we will write simply $\mathcal{H}$, $\mathcal{X}$ and $\mathcal{A}$.

The inputs of the hybrid system H are functions from $PC(T, \mathcal{U})$ and sequences from $(\Gamma \times T)^*$. The interpretation of a sequence $(\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*$ is the following. The event γ_i took place *after* the event γ_{i-1} and t_{i-1} is the *elapsed time between the arrival of* γ_{i-1} *and the arrival of* γ_i. That is, t_i is the difference of the arrival times of γ_i and γ_{i-1}. Consequently, $t_i \geq 0$ but we allow $t_i = 0$, that is, we allow γ_i to arrive instantly after γ_{i-1}. If $i = 1$, then t_1 is simply the time when the event γ_1 arrived.

The *state trajectory* of the system H is a map $\xi_H : \mathcal{H} \times PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T \to \mathcal{H}$ of the following form. For each $u \in PC(T, \mathcal{U})$, $w = (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*$, $t_{k+1} \in T$, $h_0 = (q_0, x_0) \in \mathcal{H}$ it holds that

$$\xi_H(h_0, u, w, t_{k+1}) = (\delta(q_0, \gamma_1 \cdots \gamma_k), x_H(h_0, u, w, t_{k+1}))$$

where the map $x : T \ni t \mapsto x_H(h_0, u, w, t) \in \mathcal{X}$ is the solution of the differential equation

$$\frac{d}{dt}x(t) = f_{q_k}(x(t), u(t + \sum_{1}^{k} t_j))$$

where $q_i = \delta(q_0, \gamma_1 \cdots \gamma_i)$, $i = 1, \ldots, k$ and

$$x(0) = x_H(h_0, u, w, 0) = R_{q_k, \gamma_k, q_{k-1}}(x_H(x_0, u, (\gamma_1, t_1) \ldots (\gamma_{k-1}, t_{k-1}), t_k))$$

if $k > 0$ and $x(0) = x_0$ if $k = 0$. Define the function $\upsilon_H : \mathcal{H} \times PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T \to O \times \mathcal{Y}$ by $\upsilon_H((q_0, x_0), u, (w, \tau), t) = (\lambda(q_0, w), h_q(x_H((q_0, x_0), u, (w, \tau), t)))$

where $q = \delta(q_0, w)$. For each $h \in \mathcal{H}$ the *input-output map of the system H induced by h* is the function

$$\upsilon_H(h,.) : PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T \ni\ni (u,(w,\tau),t) \mapsto \upsilon_H(h,u,(w,\tau),t) \in O \times \mathcal{Y}$$

We will denote the map $(u,s,t) \mapsto \Pi_{\mathcal{Y}} \circ \upsilon_H(h,u,s,t) \in \mathcal{Y}$ by $y_H(h,.)$ and we will denote $y_H(h,.)(u,s,t)$ simply by $y_H(h,u,s,t)$.

Let H be a hybrid system and let $q_0 \in Q$ be a discrete state of the hybrid system H. We will call the pair (H,q_0) a *realization* . The state q_0 just specifies the initial state (q_0, x_{q_0}) of the system. An input-output map $\phi \in F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y})$ is said to be *realized* by a hybrid realization (H,q_0) if $\upsilon_H((q_0,x_{q_0}),.) = \phi$. We will say that H *realizes* ϕ if there exists an initial discrete state $q_0 \in Q$ such that (H,q_0) realizes q_0. With slight abuse of terminology, sometimes we will call both H and (H,q_0) a *realization* of ϕ.

For a hybrid system H the dimension of H is defined as $\dim H = (card(Q), \sum_{q\in Q} \dim \mathcal{X}_q) \in \mathbb{N} \times \mathbb{N}$. The first component of $\dim H$ is the cardinality of the discrete state-space, the second component is the sum of dimensions of the continuous state-spaces. For each $(m,n),(p,q) \in \mathbb{N}\times\mathbb{N}$ define the partial order relation $(m,n) \leq (p,q)$, if $m \leq p$ and $n \leq q$. A realization H of a map ϕ is called a *minimal realization* of ϕ, if for any realization $H^{'}$ of ϕ: $\dim H \leq \dim H^{'}$.

Consider the set $PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T$ and define the topology generated by the following collection of open sets $\{V_K \mid K \in \mathbb{R}, K > 0\}$, where $V_K = \{(u,(\gamma_1,t_1)\cdots(\gamma_k,t_k),t_{k+1}) \mid (\sum_{j=1}^{k+1} t_j) \cdot ||u||_{\sum_{j=1}^{k+1} t_j,\infty} < K\}$. Notice that for any open subset U in this topology it holds that $(u,(\gamma_1,0)\cdots(\gamma_k,0),0) \in U$ for all $\gamma_1,\ldots,\gamma_k \in \Gamma, k \geq 0$. In the rest of the chapter we will tacitly assume that all topological statements about the set $PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T$ refer to the topology defined above.

We will say that the hybrid system H is *local realization* of an input-output map $f \in F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y})$ if there exist an open set $U \subseteq PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T$ such that for some discrete state $q \in Q$,

$$\forall (u,w,t) \in \mathcal{U} : f(u,w,t) = \upsilon_H((q,x_q),u,w,t)$$

Similarly to the global case, we will say that (H,q) is a local realization of f.

6. Input-Output Maps of Hybrid Systems

Recall from classical nonlinear systems theory [ISI 89, WAN 89] that state and output trajectories of nonlinear analytic input-affine control systems admit a representation in terms of iterated integrals. A similar statement remains true for hybrid systems too. In order to state the the existence of such a representation formally, we will need to introduce some additional notation and terminology.

We will start with defining the concept of hybrid convergent generating series and hybrid Fliess-series expansions.

6.1. *Hybrid Convergent Generating Series*

We will start with defining the notion of iterated integrals, see [ISI 89, WAN 89]. For each $u=(u_1,\dots,u_k)\in\mathcal{U}=\mathbb{R}^m$ denote $d\zeta_j[u]=u_j, j=1,2,\dots,m,\quad d\zeta_0[u]=1$. Denote the set $\{0,1,\dots,m\}$ by Z_m. For each $j_1\cdots j_k\in\mathrm{Z}_m^*, j_1,\cdots,j_k\in\mathrm{Z}_m$, $k\geq 0, t\in T, u\in PC(T,\mathcal{U})$ define $V_{j_1\cdots j_k}[u](t)=1$ if $k=0$ and for all $k>1$, let $V_{j_1\cdots j_k}[u](t)=\int_0^t d\zeta_{j_k}[u(\tau)]V_{j_1,\dots,j_{k-1}}[u](\tau)d\tau$. For each $w_1,\dots,w_k\in\mathrm{Z}_m^*$, $(t_1,\cdots,t_k)\in T^k, u\in PC(T,\mathcal{U})$ define

$$V_{w_1,\dots,w_k}[u](t_1,\dots,t_k)=V_{w_1}(t_1)[u]V_{w_2}(t_2)[\mathrm{Shift}_1(u)]\cdots V_{w_k}[\mathrm{Shift}_{k-1}(u)](t_k)$$

where $\mathrm{Shift}_i(u)=\mathrm{Shift}_{\sum_1^i t_i}(u), i=1,2,\dots,k-1$.

Assume that Z_m and Γ are disjoint sets. Denote by $\widetilde{\Gamma}$ the set $\Gamma\cup\mathrm{Z}_m$. Then any $w\in\widetilde{\Gamma}^*$ is of the form $w=w_1\gamma_1\cdots w_k\gamma_k w_{k+1}$, where $\gamma_1,\dots,\gamma_k\in\Gamma$, $w_1,\dots,w_{k+1}\in\mathrm{Z}_m^*, k\geq 0$.

A map $c:\widetilde{\Gamma}^*\to\mathcal{Y}$ is called a *hybrid generating convergent series on* $\widetilde{\Gamma}^*$ if there exists $K,M>0, K,M\in\mathbb{R}$ such that for each $w\in\widetilde{\Gamma}^*$,

$$||c(w)||<|w|!KM^{|w|}$$

where $||.||$ is some norm in $\mathcal{Y}=\mathbb{R}^p$. The notion of generating convergent series is related to the notion of convergent power series from [ISI 89, WAN 89].

Let $c:\widetilde{\Gamma}^*\to\mathcal{Y}$ be a generating convergent series. For each $u\in PC(T,\mathcal{U})$ and $s=(\gamma_1,t_1)\cdots(\gamma_k,t_k)\in(\Gamma\times T)^*, t_{k+1}\in T$ define the series

$$F_c(u,s,t_{k+1})=\sum_{w_1,\dots,w_{k+1}\in\mathrm{Z}_m^*}c(w_1\gamma_1\cdots\gamma_k w_{k+1})V_{w_1,\dots,w_{k+1}}[u](t_1,\dots,t_{k+1})$$

It is easy to see that for small enough $t_1,\dots,t_{k+1}\in T, u$ the series above is absolutely convergent. More precisely, let $T_s=\sum_{j=1}^{k+1}t_j$ and $||u||_{S,\infty}=\sup\{||u(t)||\mid t\in[0,S]\}$ It can be shown, that if $T_s\cdot||u||_{T_s,\infty}<(2M(1+m))^{-1}$, then $F_c(u,s,t_{k+1})$ is absolutely convergent. Define the set

$dom(F_c)=\{(u,s,t)\in PC(T,\mathcal{U})\times(\Gamma\times T)^*\times T\mid s=(\gamma_1,t_1)\cdot\cdots\cdot(\gamma_k,t_k)\in(\Gamma\times T)^*, k\geq 0, (t+\sum_{j=1}^k t_j)\cdot||u||_{t+\sum_{j=1}^k t_j,\infty}<(2M(1+m))^{-1}\}$.

Then for each $(u,s,t)\in dom(F_c)$ the series $F_c(u,s,t)$ is absolutely convergent and thus we can define the map $F_c: dom(F_c)\ni(u,s,t)\mapsto F_c(u,s,t)$ Recall the definition of the topology of $PC(T,\mathcal{U})\times(\Gamma\times T)^*\times T$ from Section 2. It is easy to see that for any hybrid convergent generating series c the set $dom(F_c)$ is open in that topology. It can be shown that c determines F_c locally uniquely. More precisely, if there exists a non-empty open subset of $U \subseteq domF_c \cap domF_d$, such that $\forall s \in U$: $F_c(s) = F_d(s)$, i.e. $F_c = F_d$ on the open set U, then $c = d$.

6.2. *Input-output Maps of Nonlinear Hybrid Systems*

Consider a hybrid system

$H = (\mathcal{A}, (\mathcal{X}_q, g_{q,j}, h_{q,i})_{q\in Q, j=0,\ldots,m, i=1,\ldots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, \{x_q\}_{q\in Q})$

For each $q \in Q$ denote by A_q the algebra of real-valued real analytic functions of $\mathcal{X}_q$, i.e. $A_q = C^{\omega}(X_q) = \{f : \mathcal{X}_q \to \mathbb{R} \mid f \text{ is real analytic } \}$. It is well-known that each vector field $X \in T\mathcal{X}_q$ induces a map $X : A_q \to A_q$, defined by $X(f)(x) = \sum_{j=1}^{n_q} X_j(x) \frac{df}{dx_j}(x)$, where X is assume to be of the form $X = \sum_{j=1}^{n} X_j \frac{d}{dx_j}$. In particular, each vector field $g_{q,j}$, $j \in \mathbb{Z}_m$ induces a map $g_{q,j} : A_q \to A_q$. Assume that $w = j_1 \cdots j_k \in \mathbb{Z}_m^*$, $j_1, \ldots, j_k \in \mathbb{Z}_m$, $k \geq 0$. Then define the map $g_{q,w} : A_q \to A_q$ by $g_{q,w} = g_{q,j_1} \circ g_{q,j_2} \circ \cdots \circ g_{q,j_k}$.

Notice that each reset map $R_{\delta(q,\gamma),\gamma,q}$ induces a map $R^*_{\delta(q,\gamma),\gamma,q} : A_{\delta(q,\gamma)} \to A_q$ defined by $R^*_{\delta(q,\gamma),\gamma,q}(f)(x) = f(R_{\delta(q,\gamma),\gamma,q}(x))$. Thus, for any $s = w_1\gamma_1 \cdots \gamma_k w_{k+1} \in \widetilde{\Gamma}^*$, such that $w_1, \ldots, w_k \in \mathbb{Z}_m^*$, $\gamma_1, \ldots, \gamma_k \in \Gamma$, we get that the map

$$G_{H,q,s} = g_{q_0,w_1} \circ R^*_{q_1,\gamma_1,q_0} g_{q_1,w_2} \circ \cdots \circ R^*_{q_{k+1},\gamma_k,q_k} \circ g_{q_k,w_{k+1}} : A_{q_k} \to A_q \quad (1)$$

is well-defined, where $q_i = \delta(q, \gamma_1 \cdots \gamma_i)$, $i = 0, \ldots, k$, $q_0 = q$. In particular, if $h \in A_{q_k}$, and $x \in \mathcal{X}_q$, then $G_{H,q,s}(h)(x) \in \mathbb{R}$.

Define for any $(q, x) \in \mathcal{H}$ define the generating series $c_{q,x} : \widetilde{\Gamma}^* \to \mathcal{Y}$, as follows, for each $s \in \widetilde{\Gamma}^*$, $s = w_1\gamma_1 \cdots w_k\gamma_k w_{k+1}$, $w_1, \ldots, w_{k+1} \in \mathbb{Z}_m^*$, $\gamma_1, \ldots, \gamma_k \in \Gamma$, $\delta(q, \gamma_1 \cdots \gamma_k) = q_k$, let $c_{q,x}(s) = G_{H,q,s}(h_{q_k})(x)$ It is easy to see that $c_{q,x}$ is a generating convergent power series. Using arguments similar to the standard ones for nonlinear state affine systems, one gets that for each $(u, (\gamma_1, t_1) \cdots (\gamma_k, t_k), t_{k+1}) \in dom(F_{c_{q,x}})$,

$$\begin{aligned} &y_H((q,x), u, (\gamma_1, t_1) \cdots (\gamma_k, t_k), t_{k+1}) = F_{c_{q,x}}(u, (\gamma_1, t_1) \cdots (\gamma_k, t_k), t_{k+1}) = \\ &= \sum_{w_1,\ldots,w_{k+1} \in \mathbb{Z}_m^*} c_{q,x}(w_1 z_1 \cdots w_k z_k w_{k+1}) V_{w_1,\ldots w_k}[u](t_1, \ldots, t_{k+1}) \end{aligned} \quad (2)$$

Let $f \in F(PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y})$ be an input-output map. Denote by f_D the map $\Pi_O \circ f$ and denote by f_C the map $\Pi_{\mathcal{Y}} \circ f$. We will say that f admits a *local hybrid Fliess-series expansion*, if and only if

– The map f_D depends only on Γ^*, that is,

$$f_D(u, (s, \underline{t}), t) = f_D(v, (s, \underline{\tau}), \tau)$$

for all $u, v \in PC(T, \mathcal{U})$, $\tau, t \in T$, $\underline{\tau}, \underline{t} \in T$, $s \in \Gamma^*$. Thus, the map f_D can be viewed as a map $f_D : \Gamma^* \to O$.

– There exists a generating convergent series $c_f : \widetilde{\Gamma}^* \to \mathcal{Y}$ and an open subset $U \subseteq dom(F_{c_f})$ such that

$$\forall (u,w,t) \in U : f_C(u,w,t) = F_{c_f}(u,w,t)$$

Theorem 2. *Let*

$$H = (\mathcal{A}, (\mathcal{X}_q, g_{q,j}, h_{q,i})_{q\in Q, j=0,\dots,m, i=1,\dots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, \{x_q\}_{q\in Q})$$

be a NHS and let $f \in F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y})$ be an input-output map. Then H is a local realization of f if and only if f has a hybrid Fliess-series expansion and there exists $q \in Q$ such that

– $\forall w \in \Gamma^* : f_D(w) = \lambda(q,w)$

– *For all $s = w_1\gamma_1 \cdots \gamma_k w_{k+1} \in \widetilde{\Gamma}^*$, $\gamma_1, \dots, \gamma_k \in \Gamma$, $w_1, \dots, w_{k+1} \in Z_m^*$, $k \geq 0$*

$$\begin{aligned} &c_f(w_1\gamma_1 w_2 \cdots \gamma_k w_{k+1}) = \\ &g_{q_0,w_1} \circ R^*_{q_1,\gamma_1,q_0} \circ g_{q_1,w_2} \cdots \circ R^*_{q_k,\gamma_k,q_{k-1}} \circ g_{q_k,w_{k+1}}(h_{q_k})(x_q) \end{aligned} \tag{3}$$

where $q_i = \delta(q, \gamma_1 \cdots \gamma_i)$, $i = 0, \dots, k$.

7. Formal Realization Problem For Hybrid Systems

Recall from Section 3.1 the notion of formal power series in commuting variables. As it was seen in the previous section, the local realization problem for nonlinear hybrid systems is equivalent to finding a particular representation for the hybrid convergent generating series corresponding to the input-output map. Notice that this representation was formulated completely in terms of reset maps and vector fields around a point and it is completely determined by the formal power series expansion of the analytic maps and vector fields involved. More precisely, consider a hybrid system

$$H = (\mathcal{A}, (\mathcal{X}_q, g_{q,j}, h_{q,i})_{q\in Q, j=0,\dots,m, i=1,\dots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, \{x_q\}_{q\in Q})$$

For each $q \in Q$, $j \in Z_m$ consider the formal power series expansion of $R_{\delta(q,\gamma),\gamma,q}$, $g_{q,j}$ and $h_{q,i}$. That is for each $q \in Q$ consider the ring of formal power series $A^f_q = \mathbb{R}[[X_1, \dots, X_{n_q}]]$ in commuting variables $X_1, \dots, X_{n_q}$. Then the formal power series expansion of $h_{q,i}(x) = \sum_{\alpha\in\mathbb{N}^{n_q}} h_{q,i,\alpha}(x - x_q)^\alpha$ for $i = 1, \dots, p$ around x_q results in a formal power series $h^f_{q,i} \in \mathbb{R}[[X_1, \dots, X_{n_q}]]$, defined by $h^f_{q,i} = \sum_{\alpha\in\mathbb{N}^{n_q}} h_{q,i,\alpha} X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_{n_q}^{\alpha_{n_q}}$. Similarly, if $g_{q,j} = \sum_{i=1}^{n_q} g_{q,j,i} \frac{d}{dx_i}$, then take the Taylor-series expansion of each $g_{q,j,i}$ around x_q, i.e. $g_{q,j,i}(x) = \sum_{\alpha\in\mathbb{N}} g_{q,j,i,\alpha}(x - x_q)^\alpha$ and define the following continuous derivation on $\mathbb{R}[[X_1, \dots, X_{n_q}]]$, $g^f_{q,j} = \sum_{i=1}^{n_q} g_{q,j,i} \frac{d}{dX_i}$ where $g_{q,j,i} = \sum_{\alpha\in\mathbb{N}^{n_q}} g_{q,j,i,\alpha} X^\alpha$. Finally, assume that $R_{\delta(q,\gamma),\gamma,q} - x_{\delta(q,\gamma)}$ is of the form $R_{\delta(q,\gamma),\gamma,q} - x_{\delta(q,\gamma)} = (R_{\delta(q,\gamma),\gamma,q,1}, \dots, R_{\delta(q,\gamma),\gamma,q,n_{\delta(q,\gamma)}})^T$ Each map $R_{\delta(q,\gamma),\gamma,q,i}$, $i = 1, \dots, n_{\delta(q,n)}$ is an analytic map with values in $\mathbb{R}$ and thus around x_q it admits a Taylor series expansion of the form $R_{\delta(q,\gamma),\gamma,q,i}(x) =$

$\sum_{\alpha\in\mathbb{N}^{n_q}} r_{\delta(q,\gamma),\gamma,q,i,\alpha}(x-x_q)^{\alpha}$. Notice that $R_{\delta(q,\gamma),\gamma,q}(x_q) - x_{\delta(q,\gamma)} = 0$ and thus $r_{\delta(q,\gamma),\gamma,q,i,(0,0,\ldots,0)} = R_{\delta(q,\gamma),\gamma,q,i}(x_q) = 0$. Define the formal power series $R^f_{\delta(q,\gamma),\gamma,q,i} = \sum_{\alpha\in\mathbb{N}^{n_q}} r_{\delta(q,\gamma),\gamma,q,i,\alpha}X^{\alpha}$ Let $r=\delta(q,\gamma)$ and $A_r=\mathbb{R}[[X_1,\ldots,X_{n_r}]]$ and define the continuous algebraic map $R^{f,*}_{r,\gamma,q}: A^f_r \to A^f_q$ by $R^{f,*}_{r,\gamma,q}(X_i) = R^f_{r,\gamma,q,i}$ for all $i=1,\ldots,n_r$. It is easy to see that $R^{f,*}_{r,\gamma,q}$ is indeed an algebra morphism.

The discussion above motivates the following definition. A tuple

$$F = (\mathcal{A}, (A_q, g_{q,j}, h_{q,i})_{q\in Q, j=0,\ldots,m, i=1,\ldots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q\in Q, \gamma\in\Gamma\}, q_0)$$

is called a *formal hybrid system*, where

– $\mathcal{A}=(Q,\Gamma,O,\delta,\lambda)$ is a Moore-automaton and $q_0\in Q$ is the initial state of $\mathcal{A}$.

– For each $q\in Q$, $A_q=\mathbb{R}[[X_1,\ldots,X_{n_q}]]$ is the ring of formal power series in commuting variable $X_q,\ldots,X_{n_q}$

– For each $q\in Q, j\in\mathcal{Z}_m$,

$$g_{q,j}: A_q \to A_q$$

defines a continuous derivation on A_q, i.e. $g_{q,j}=\sum_{i=1}^{n_q} g_{q,j,i}\frac{d}{dX_i}$, where $g_{q,j,i}\in A_q$, $i=1,\ldots n_q$.

– For each $q\in Q, i=1,\ldots,p, h_{q,i}\in A_q$

– For each $q\in Q$, $\gamma\in\Gamma$, $R_{\delta(q,\gamma),\gamma,q}: A_{\delta(q,\gamma)} \to A_q$ is a continuous algebra morphism, i.e. it is uniquely defined by its values $R_{\delta(q,\gamma),\gamma,q}(X_i)\in A_q$ and the free coefficient of $R_{\delta(q,\gamma),\gamma,q}(X_i)$ is zero, i.e. $1^*_{\mathbb{R}[[X_1,\ldots,X_{n_q}]]}(R_{\delta(q,\gamma),\gamma,q}(X_{i,\delta(q,\gamma)}))=0$

Let $q_0\in Q$. The discussion preceding the definition above yields that $F_{(H,q_0)}$ defined as

$$F_{(H,q_0)} = (\mathcal{A}, (A^f_q, g^f_{q,j}, h^f_{q,i})_{q\in Q, j\in \mathrm{Z}_m, i=1,\ldots,p}, \{R^{f,*}_{\delta(q,\gamma),\gamma,q} \mid q\in Q,\gamma\in\Gamma\}, q_0)$$

is a formal hybrid system. We will call F_H the *formal hybrid system associated with* (H,q_0). Let F be a formal hybrid system. The *dimension* of the formal hybrid system F is defined as $\dim F = (card(Q), \sum_{q\in Q} n_q)$.

Consider the formal hybrid system
$F = (\mathcal{A}, (A_q, g_{q,j}, h_{q,i})_{q\in Q, j=0,\ldots,m, i=1,\ldots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q\in Q,\gamma\in\Gamma\}, q_0)$

from the definition above. For each $q\in Q$, $w=j_1j_2\cdots j_l, j_1,\ldots,j_l\in \mathrm{Z}_m, l\geq 0$, denote by $g_{q,w}$ the following map $g_{q,w}=g_{q,j_1}\circ g_{q,j_2}\circ\cdots\circ g_{q,j_k}: A_q\to A_q$. For each $q\in Q$, $v=w_1\gamma_1 w_2\cdots\gamma_k w_{k+1}\in\widetilde{\Gamma}^*$, $k\geq 0$, $\gamma_1,\ldots,\gamma_k\in\Gamma$, $w_1,\ldots,w_{k+1}\in \mathrm{Z}^*_m$, denote by $G_{H,q,v}$ the map

$$G_{F,q,v} = g_{q_0,w_1}\circ R_{q_1,\gamma,q_0}\circ g_{q_1,w_2}\circ\cdots\circ R_{q_k,\gamma_k,q_{k-1}}\circ g_{q_k,w_{k+1}}: A_{q_k}\to A_q$$

where $q_i=\delta(q,\gamma_1\cdots\gamma_i)$, $i=0,\ldots,k$.

Consider the maps $f_c : \widetilde{\Gamma}^* \to \mathbb{R}^p$ and $f_d : \Gamma^* \to O$. We will say the the formal hybrid system F *is a realization of* (f_d, f_c) , if for all $s \in \widetilde{\Gamma}^*$:

$$\begin{aligned} \forall w \in \Gamma^* : f_d(w) = \lambda(q_0, w) \\ \forall v \in \widetilde{\Gamma}^* : f_c(v) = \phi_{q_0} \circ G_{F,q_0,v}(h_{q_e}) \end{aligned} \tag{4}$$

where $q_e = \delta(q_0, \gamma_1 \cdots \gamma_k)$ such that $v = w_1\gamma_1 \cdots \gamma_k w_{k+1}, \gamma_1, \dots, \gamma_k \in \Gamma$, $w_1, \dots, w_{k+1} \in \mathrm{Z}_m^*, k \geq 0$.

Theorem 2 has the following easy consequence

Lemma 1. *Let $f \in F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y})$ and assume that f has a hybrid Fliess-series expansion. Then (H, q_0) is a realization of f if and only if the formal hybrid system F_{H,q_0} is a realization of (f_D, c_f).*

Recall from Section 3 the notion of coalgebra. Recall that there exists a natural duality between algebras and coalgebras. We will exploit this duality by looking at formal hybrid systems defined on coalgebras. Recall from Section 3 that rings of formal power series in commuting variables have a natural characterisation as duals of certain coalgebras with very special property. This observation will enable us to use coalgebra theory for finding necessary and sufficient conditions for existence of a formal hybrid system realization. It will also enable us to place our results in the wider context of nonlinear realization theory. Below we will start with the definition of coalgebra systems and coalgebra hybrid systems. We will also discuss the relationship between coalgebra hybrid systems, formal hybrid systems and nonlinear hybrid systems.

Let H be a bialgebra, which will be referred to as the *bialgebra of inputs*.

A tuple $\Sigma = (C, H, \psi, \phi, J, \mu)$ is called a *control system on a coalgebra* if

– C is a cocommutative coalgebra.

– J is an arbitrary set.

– $\psi : C \otimes H \to C$ is a coalgebra map such that $\psi(a \otimes 1_H) = a$ and $\psi(a \otimes h_1 h_2) = \psi(\psi(a \otimes h_1) \otimes h_2)$ for all $h_1, h_2 \in H$, $a \in C$. Here 1_H denotes the unit element of H as an algebra.

– $\phi \in G(C)$, i.e. ϕ is a group-like element of C

– $\mu : J \to C^*$ is the family of readout maps.

We say that Σ *realizes a family of maps* $\Psi = \{y_j : H \to \mathbb{R} \mid j \in J\}$ if $\forall h \in H, \forall j \in J : y_j(h) = \mu(j)(\psi(\phi \otimes h))$.

Recall the notation from Section 2. Consider the set $\widetilde{\Gamma} = \Gamma \cup \mathrm{Z}_m$. The set $H = \mathbb{R} < \widetilde{\Gamma}^* >$ of all formal linear combinations words over $\widetilde{\Gamma}$ has a natural bialgebra structure defined by

$$\begin{aligned} \delta(\gamma) = \gamma \otimes \gamma \text{ for all } \gamma \in \Gamma \cup \{1\} \\ \delta(x) = 1 \otimes x + x \otimes 1 \text{ for all } x \in \mathrm{Z}_m \end{aligned} \tag{5}$$

$$\delta(w_1w_2\cdots w_k) = \delta(w_1)\delta(w_2)\cdots\delta(w_k) \text{ for all } w_1,\dots,w_k \in \widetilde{\Gamma}$$

$$\epsilon(x) = \begin{cases} 1 & \text{if } x \in \Gamma \cup \{1\} \\ 0 & \text{if } x \in \mathrm{Z}_m \end{cases} \tag{6}$$

$$\epsilon(w_1w_2\cdots w_k) = \epsilon(w_1)\epsilon(w_2)\cdots\epsilon(w_k) \text{ for all } w_1,\dots,w_k \in \widetilde{\Gamma}, k \geq 0$$

Although H is a bialgebra, it is not a Hopf-algebra. H as a coalgebra is cocommutative pointed coalgebra, but it is not irreducible. It is also easy to see that $G(H) = \{\gamma \in \Gamma \cup \{1\}\}$ is the set of group-like elements, and in fact $H = \bigoplus_{w\in\Gamma^*} H_w$, where for all $w = w_1\cdots w_k, k \geq 0, w_1,\dots,w_k \in \Gamma$,

$$H_w = \mathrm{Span}\{s_1w_1s_2\cdots w_ks_{k+1} \mid s_1,\dots,s_{k+1} \in \mathrm{Z}_m^*\}$$

It is easy to see that for each $w \in \Gamma^*$ the linear space H_w is in fact a subcoalgebra of H, moreover, H_w is pointed irreducible and cocommutative. It is also easy to see that the map $\psi : H_w \otimes \mathbb{R} < \mathrm{Z}_m^* > \to H_w, \psi(v \otimes s) = vs, s \in \mathrm{Z}_m^*, v \in H_w$ is well-defined and it is a coalgebra map. Similarly, for each $\gamma \in \Gamma$ the map $\psi_\gamma : H_w \ni s \mapsto s\gamma \in H_{w\gamma}$ is a well-defined coalgebra map.

From now one, unless stated otherwise, the symbol H will always refer to $R < \widetilde{\Gamma}^* >$ with the bialgebra structure defined above. Consider the pair of maps $f = (f_D, f_C)$, where $f_D : \Gamma^* \to O$ and $f_C : \widetilde{\Gamma}^* \to \mathbb{R}^p$. Consider the maps $f_{C,i} : \widetilde{\Gamma}^* \to \mathbb{R}$, where $f_C(w) = (f_{C,1}(w), f_{C,2}(w), \dots, f_{C,p}(w))^T$ for each $w \in \widetilde{\Gamma}^*$. Notice that each map $f_{C,i}$ can be uniquely extended to a linear map $\widetilde{f}_{C,i} : H \to \mathbb{R}$. In the sequel we will identify maps $f_{C,i}$ and linear maps $\widetilde{f}_{C,i}$ and we will denote both of them by $f_{C,i}$. Define the *family of input-output maps associated with* f as the following indexed set of maps $\Psi_f = \{f_{C,i} : H \to \mathbb{R} \mid i = 1,\dots,p\}$.

A *hybrid coalgebra system* is a tuple $HC = (\mathcal{A}, \Sigma, q_0)$, where

– $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ is a Moore-automata, $q_0 \in Q$,

– $\Sigma = (C, H, \psi, \phi, \mu)$ is a coalgebra system, such that

- $C = \bigoplus_{q\in Q} C_q$, where C_q is a subcoalgebra of C for each $q \in Q$ and C_q is pointed irreducible.

- $\phi \in C_{q_0}$

- For each $q \in Q, \forall w \in \mathrm{Z}_m^*, \forall z \in C_q : \psi(z \otimes w) \in C_q$ and $\forall \gamma \in \Gamma, \forall z \in C_q : \psi(z \otimes \gamma) \in C_{\delta(q,\gamma)}$

Since for each $q \in Q$, the coalgebra C_q is pointed irreducible, it has a unique group like element which we will denote by ϕ_q. It follows that $\phi = \phi_{q_0}$ and for each $w \in \Gamma$, $q \in Q$, $\phi(w \otimes \phi_q) = \phi_{\delta(q,w)}$. It also follows that C_q precisely coincides with the irreducible component of ϕ_q in C. We know that C is a direct sum of its irreducible components and it follows that C is pointed. Thus, it follows that there is a bijection between irreducible components of C and the coalgebras $C_q, q \in Q$.

A pair of maps $f = (f_D, f_C)$, where $f_D : \Gamma^* \to O$ and $f_C : \widetilde{\Gamma}^* \to \mathbb{R}^p$ is said to be *realized by a hybrid coalgebra system* $HC = (\mathcal{A}, \Sigma, q_0)$ if $(\mathcal{A}, q_0)$ is a realization of f_D and Σ is a realization of Ψ_f.

Recall from Subsection 3.2 that the ring of formal power series $\mathbb{R}[[X_1,\dots,X_n]]$ is isomorphic to the dual of of the cofree pointed irreducible cocommutative coalgebra $B(V)$, where V is any n-dimensional vector space. That is, $B(V)^* \cong \mathbb{R}[[X_1,\dots,X_n]]$. Below we will choose a particular V. Denote by A the ring $A = \mathbb{R}[[X_1,\dots,X_n]]$. Recall from Subsection 3.1 the definition and properties of continuous derivations on formal power series rings. Define the map $D_\alpha = 1_A \circ \frac{d}{dX}^\alpha$ for all $\alpha \in \mathbb{N}^n$. Define the set $\mathcal{D}_A^\infty = \mathrm{Span}\{D_\alpha \mid \alpha \in \mathbb{N}^n\}$. Notice that $\phi = D_{(0,0,\dots,0)} = 1_A^* \in \mathcal{D}_A^\infty$. Let $\mathcal{D}_A = \mathrm{Span}\{D_i \mid i = 1,\dots,n\}$. Define the linear maps $\epsilon : \mathcal{D}_A^\infty \to \mathbb{R}$ and $\delta : \mathcal{D}_A^\infty \to \mathcal{D}_A^\infty \otimes \mathcal{D}_A^\infty$ by $\epsilon(\phi) = 1$ and $\epsilon(D_\alpha) = 0$ if $\alpha \in \mathbb{N}^n, \alpha \neq (0,0,\dots,0)$. For each $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$ let $\delta(D_\alpha) = \sum_{\beta,\gamma\in\mathbb{N}^n,\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} D_\beta \otimes D_\gamma$ where $\beta+\gamma = (\beta_1+\gamma_1, \beta_2+\gamma_2,\dots,\beta_n+\gamma_n)$, $\beta = (\beta_1,\dots,\beta_n)$, $\gamma = (\gamma_1,\dots,\gamma_n)$ and $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$, $\beta! = \beta_1!\cdots\beta_n!$, $\gamma! = \gamma_1!\cdots\gamma_n!$. Define the multiplication $M : \mathcal{D}_A^\infty \otimes \mathcal{D}_A^\infty \to \mathcal{D}_A^\infty$ by $M(D_\alpha \otimes D_\beta) = D_{\alpha+\beta}$. Define the map $u : \mathbb{R} \to \mathcal{D}_A^\infty$ by $u(x) = x\phi$.

Lemma 2. *The tuple $(\mathcal{D}_A^\infty, \delta, \epsilon, M, u)$ is a bialgebra, moreover $\mathcal{D}_A^\infty$ is isomorphic as a bialgebra to the cofree pointed irreducible cocommutative coalgebra $B(\mathcal{D}_A)$ generated by $\mathcal{D}_A$.*

The lemma above implies that $(\mathcal{D}_A^\infty)^*$ is isomorphic to A. This algebra isomorphism is defined by $\psi_A : (\mathcal{D}_A^\infty)^* \ni S \mapsto \sum_{\alpha\in\mathbb{N}^n} S(\frac{1}{\alpha_1!}\frac{1}{\alpha_2!}\cdots\frac{1}{\alpha_n!}D_\alpha)X^\alpha$. The following lemma relates measuring of A and coalgebra maps of $\mathcal{D}_A^\infty$.

Lemma 3. *Let C be an coalgebra, let $A = \mathbb{R}[[X_1,\dots,X_n]]$ and $B = \mathbb{R}[[Y_1,\dots,Y_m]]$. Assume that $\psi : C \otimes A \to B$ is a measuring such that for each $c \in C$, the map $\psi_c : A \ni a \mapsto \psi(c \otimes a) \in B$ is a continuous map. Then $\eta_\psi : C \otimes \mathcal{D}_B^\infty \to \mathcal{D}_A^\infty$ is a coalgebra map, where $\eta_\psi(c \otimes D_\alpha)(a) = D_\alpha(\psi_c(a))$ for all $a \in A$.*

Conversely, assume that $\eta : C \otimes \mathcal{D}_B^\infty \to \mathcal{D}_A^\infty$ is a coalgebra map. Consider the map $\psi_\eta : C \otimes A \to B$, defined by $\psi_B^{-1} \circ \psi_\eta(c \otimes a)(D) = \eta(c \otimes D)(\psi_A^{-1}(a))$, for all $a \in A, c \in C$, $D \in \mathcal{D}_B^\infty$.Here ψ_A^{-1} and ψ_B^{-1} are the inverses of the algebra isomorphisms $\psi_A : (\mathcal{D}_A)^* \to A$ and $\psi_B : (\mathcal{D}_B)^* \to B$ respectively. Then ψ_η is a measuring such that for each $c \in C$ the map $\psi_{\eta,c} : A \ni a \mapsto \psi_\eta(c \otimes a) \in B$ is a continuous map.

In the sequel we will identify $\mathcal{D}_A^\infty$ and $B(\mathcal{D}_A)$ and we will identify their respective duals $(\mathcal{D}_A^\infty)^*$, $B(\mathcal{D}_A)^*$ with A. We will also identify $(B(V))^*$ with $A_V = \mathbb{R}[[X_1,\dots,X_n]]$ if $\dim V = n$.

Using Lemma 2 and Lemma 3 we can associate with each formal hybrid system a hybrid coalgebra system of a certain type and conversely, with each hybrid coalgebra system of a suitable type we can associate a formal hybrid system. Let HF be formal hybrid system of the form

$$HF = (\mathcal{A}, (A_q, g_{q,j}, h_{q,i})_{q\in Q, j=0,\dots,m, i=1,\dots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, q_0)$$

Define the hybrid coalgebra system HC_{HF} *associated with* HF as follows. $HC_{HF} = (\mathcal{A}, \Sigma_{HC}, q_0)$, where $\Sigma_{HC} = (C, H, \widetilde{\psi}, \widetilde{\phi}, \{1,\dots,p\}, \widetilde{\mu})$ such that

– $C = \bigoplus_{q\in Q} C_q$, and for all $q \in Q$, $C_q = B(\mathcal{D}_{A_q})$.

– $\widetilde{\psi} : C \otimes H \to C$, such that for all $w \in \widetilde{\Gamma}^*$, $D \in C_q$, $q \in Q$, $\widetilde{\psi}(D \otimes w) = D \circ G_{F,q,w}$, where D is viewed as a map $D : A_q \to \mathbb{R}$ and $G_{F,q,w}$ is viewed as a map $G_{F,q,w} : A_r \to A_q$, $w = s_1\gamma_1 \cdots \gamma_k s_{k+1}$, $\gamma_1, \ldots, \gamma_k \in \Gamma$, $s_1, \ldots, s_{k+1} \in \mathrm{Z}_m^*$, $r = \delta(q, \gamma_1 \cdots \gamma_k)$.

– $\widetilde{\phi} = 1_{q_0}$ where 1_{q_0} is the unique group-like element of C_q. Notice that $1_{q_0} = 1^*_{A_{q_0}}$ viewed as a map $A_q \to \mathbb{R}$.

– For all $j = 1, \ldots p$, $\widetilde{\mu}(j) \in C^*$, such that for each $q \in Q$, $D \in C_q$, $\widetilde{\mu}(j)(D) = D(h_{q,j})$.

It is an easy consequence of Lemma 2 and Lemma 3 that HC_{HF} is well-defined.

Conversely, let $HC = (\mathcal{A}, \Sigma, q_0)$ be a hybrid coalgebra system such that $\Sigma = (C, H, \psi, \phi, \{1, \ldots, p\}, \mu)$, $C = \bigoplus_{q\in Q} C_q$, $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ and $C_q = B(V_q)$, $\dim V_q = n_q$ for all $q \in Q$. We will call such hybrid coalgebra systems *CCPI hybrid coalgebra systems* (CCPI stands for cofree cocommutative pointed irreducible). Then using Lemma 2 and Lemma 3 and the conventions discussed after Lemma 3 we get that

$HF_{HC} = (\mathcal{A}, (A_q, g_{q,j}, h_{q,i})_{q\in Q, j=0,\ldots,m, i=1,\ldots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, q_0)$

is a well-defined formal hybrid system, where for all $q \in Q$, $A_q = C_q^*$, for all $j \in \mathrm{Z}_m$, $g_{q,j} : A_q \to A_q$, such that $g_{q,j}(h)(D) = h(\psi(D \otimes j))$ for all $D \in C_q$,$h \in A_q$, and $h_{q,i} \in A_q$ are such that $h_{q,i}(d) = \mu(i)(d)$ for all $d \in C_q$, $i = 1, \ldots, p$, and $R_{\delta(q,y),y,q}$, $y \in \Gamma$ are such that $R_{\delta(q,y),y,q}(h)(D) = \psi(D \otimes y)(h)$ for all $D \in C_q$, $h \in A_q$. It is also easy to see that HF_{HC} is well-defined and $HC_{HF_{HC}} = HC$.

It is also easy to see that HC is a realization of f if and only if HF_{HC} is a realizations of f. Conversely, HF is a realization of f if and only if HC_{HF} is a realization of f.

Combining the results above we arrive to the following important characterisation of existence of a formal hybrid system realization of f.

Theorem 3. *A pair of maps $f = (f_D, f_C)$, $f_D : \Gamma^* \to O$, $f_C : \widetilde{\Gamma}^* \to \mathbb{R}^p$ has a realization by a formal hybrid system if and only if it has a CCPI hybrid coalgebra system realization.*

In order to demonstrate the notions and results presented above, we will present below a concrete hybrid system and the formal hybrid realization and the CCPI hybrid coalgebra system associated with it.

Example 1. *Consider the following hybrid system*

$$H = (\mathcal{A}, (\mathcal{X}_q, g_{q,j}, h_{q,i})_{q\in Q, j=0,\ldots,m, i=1,\ldots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, \{x_q\}_{q\in Q})$$

where

– $p = 1, m = 1, \Gamma = \{a, b\}$,

– $Q = \{q_1, q_2\}, O = \{o\}, \mathcal{A} = (\{q_1, q_2\}, \{a, b\}, \{o\}, \delta, \lambda)$

$\delta(q_1, b) = q_2$, $\delta(q_1, a) = q_1$, $\delta(q_2, b) = q_2$, $\delta(q_2, a) = q_1$, $\lambda(q_1) = \lambda(q_2) = o$

– $\mathcal{X}_{q_1} = \mathbb{R}^2$, $g_{q_1,0}(x_1, x_2) = (1, 0)^T$, $g_{q_1,1}(x_1, x_2) = (0, 0)^T$*and* $h_{q_1}(x_1, x_2) = e^{x_1}$.

– $\mathcal{X}_{q_2} = \mathbb{R}^2$, $g_{q_2,0}(x_1, x_2) = (0, 1)^T$, $g_{q_2,1}(x_1, x_2) = (0, 0)^T$, $h_{q_2}(x_1, x_2) = x_2$.

– $R_{q_2,b,q_1}(x_1, x_2) = (x_1, x_1)^T$, $R_{q_1,a,q_2}(x_1, x_2) = (x_1 + x_2, 0)^T$, $R_{q_1,a,q_1}(x_1, x_2) = (x_1, x_2)^T$ *and* $R_{q_2,b,q_2}(x_1, x_2) = (x_1, x_2)^T$. *for all* $x_1, x_2 \in \mathbb{R}$.

– $x_{q_1} = x_{q_2} = (0, 0)^T$.

The formal realization associated with (H, q_1) *is of the following form*

$$F_{(H,q_0)} = (\mathcal{A}, (A_q, g_{q,j}, h_{q,i})_{q\in Q, j=0,\dots,m, i=1,\dots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, q_0)$$

where

– *Q,O,*Γ*,* p*,* m*,* $\mathcal{A}$ *are the same as above*

– $A_{q_1} = \mathbb{R}[[X_1, X_2]]$, $g_{q_1,0} = \frac{d}{dX_1}$, $g_{q_1,1} = 0$, $h_{q_1} = \sum_{n=1}^{\infty} \frac{1}{n!} X_1^n = \sum_{(a_1,a_2)\in\mathbb{N}^2} \beta_{(a_1,a_2)} X_1^{a_1} X_2^{a_2}$ *where* $\beta_{(a_1,a_2)} = 0$ *if* $a_2 > 0$.

– $A_{q_2} = \mathbb{R}[[X_3, X_4]]$, $g_{q_2,0} = \frac{d}{dX_4}$, $g_{q_2,1} = 0$. $h_{q_2} = X_3$.

– *It is enough to define the values of the maps* $R_{q_2,b,q_1} : A_{q_2} \to A_{q_1}$, $R_{q_1,a,q_2} : A_{q_1} \to A_{q_1}$ *for* X_3*,* X_4 *and* X_1, X_2 *respectively. Thus,* $R_{q_2,b,q_1}(X_3) = X_1$, $R_{q_2,b,q_1}(X_4) = X_1$ $R_{q_1,a,q_2}(X_1) = X_3 + X_4$, $R_{q_1,a,q_2}(X_2) = 0$ *The maps* $R_{q_1,a,q_1} : A_{q_1} \to A_{q_1}$, $R_{q_2,b,q_2} : A_{q_2} \to A_{q_2}$ *are the identity maps*.

The CCPI hybrid coalgebra representation associated with $HF = F_{H,q_0}$ *is of the following form,* $HC_{HF} = (\mathcal{A}, \Sigma)$*, where* $\mathcal{A}$ *is the Moore-automaton defined above and* $\Sigma = (C, H, \psi, \phi, \{1, \dots, p\}, \mu)$ *where*

– $C = C_{q_1} \oplus C_{q_2}$*, where* $C_{q_1} = \mathcal{D}^{\infty}_{A_{q_1}} = \mathrm{Span}\{D_{(\alpha,\beta)} \mid \alpha, \beta \in \mathbb{N}\}$ *and* $C_{q_2} = \mathcal{D}^{\infty}_{A_{q_2}} = \mathrm{Span}\{D_{(\alpha,\beta)} \mid \alpha, \beta \in \mathbb{N}\}$. *Denote the element* $\mathcal{D}^{\infty}_{A_{q_1}} \ni D_{(\alpha,\beta)} : A_{q_1} \to \mathbb{R}$ *by* $D^{(\alpha,\beta)}_{A_{q_1}}$. *Similarly, denote by* $\mathcal{D}^{(\alpha,\beta)}_{A_{q_2}}$ *the element* $D_{(\alpha,\beta)} : A_{q_2} \to \mathbb{R}$ *of* $\mathcal{D}^{\infty}_{A_{q_2}}$.

– *The map* $\psi : C \otimes H \to C$ *is of the following form. Notice that it is enough to define* $\psi(c \otimes x)$ *for* $x \in \{a, b, 0, 1\}$ *and that it is enough to define* $\psi(c \otimes x)$ *for* $c = D^{(\alpha,\beta)}_{A_{q_1}}$ *or* $c = D^{(\alpha,\beta)}_{A_{q_2}}$. *We define* $\psi(c \otimes x)$ *for the values of* c *and* x *above as follows.* $\psi(D^{(\alpha,\beta)}_{A_{q_1}} \otimes 0) = D^{(\alpha+1,\beta)}_{A_{q_1}}$, $\psi(D^{(\alpha,\beta)}_{A_{q_2}} \otimes 0) = D^{(\alpha,\beta+1)}_{A_{q_2}}$, $\psi(D^{(\alpha,\beta)}_{A_{q_1}} \otimes a) = D^{(\alpha,\beta)}_{A_{q_1}}$, $\psi(D^{(\alpha,\beta)}_{A_{q_2}} \otimes a) = D^{(\alpha+\beta,0)}_{A_{q_1}}$, $\psi(D^{(\alpha,\beta)}_{A_{q_1}} \otimes b) = \begin{cases} \sum_{j=1}^{\alpha} D^{(j,\alpha-j)}_{A_{q_2}} & \text{if } \beta = 0 \\ 0 & \text{if } \beta > 0 \end{cases}$, $\psi(D^{(\alpha,\beta)}_{A_{q_2}} \otimes b) = D^{(\alpha,\beta)}_{A_{q_2}}$. *Let* $\psi(c \otimes 1) = 0$ *for all* $c \in C$.

– *The map* $\mu(1) : C \to \mathbb{R}$ *is of the following form,* $\mu(1)(D^{(\alpha,\beta)}_{A_{q_1}} = \begin{cases} 1 & \text{if } \beta = 0 \\ 0 & \text{otherwise} \end{cases}$ *and* $\mu(1)(D^{(\alpha,\beta)}_{A_{q_2}}) = \begin{cases} 1 & \text{if } \beta = 0 \text{ and } \alpha = 1 \\ 0 & \text{otherwise} \end{cases}$

– The initial state is $\phi = 1^*_{A_{q_1}} = D_{(0,0)}$

Let $f = \upsilon_H((q_1, 0), .)$*, then the Fliess-series of* c_f *of* f *is of the form. For each* $s_1, \ldots, s_{k+1} \in \{0,1\}^*$, $\gamma_1, \ldots, \gamma_k \in \Gamma$,

$$c_f(s_1\gamma_1 \cdots s_k\gamma_k s_{k+1}) = \begin{cases} 1 & \textit{if } s_i \in \{0\}^* \textit{ for all } i = 1, \ldots, k+1, \textit{ and } \gamma_k = b \\ 1 & \textit{if } s_i \in \{0\}^* \textit{ for all } i = 1, \ldots, k+1 \textit{ and } \gamma_k = a \textit{ and} \\ & \sum_{i=1}^{k+1} |s_i| = 1 \\ 0 & \textit{otherwise} \end{cases}$$

The discrete valued part $f_D : \{a,b\}^* \to \{o\}$ *of* f *is the functions* $f_D(w) = o$ *for all* $w \in \{a,b\}^*$.

8. Main Result

In this section we will discuss criteria for existence of a realization by a hybrid coalgebra system, such that the coalgebras associated with each discrete state of the automaton are cofree cocommutative pointed irreducible with finite dimensional space of primitive elements. We will give a necessary condition and a condition which is an "almost" sufficient one. More precisely, the "almost" sufficient condition implies existence of a hybrid coalgebra system realization such that each coalgebra associated with some discrete state is pointed cocommutative irreducible with finite dimensional space of primitive elements. Such a hybrid coalgebra system is indeed very close to a CCPI hybrid coalgebra system. In fact, we conjecture that any such hybrid coalgebra system gives rise to a CCPI hybrid coalgebra system.

From Theorem 3 it follows that these criteria will give necessary and sufficient conditions for existence of a formal hybrid realization.

Let $\Sigma = (C, H, \psi, \phi, J, \mu)$ be a coalgebra system. Define the maps $R_\Sigma : H \to C$ by $R_\Sigma(h) = \psi(h \otimes \phi)$ for all $h \in H$. It is easy to see that R_Σ is a coalgebra map.

We will call C *reachable* if R_Σ is surjective. For each $h \in H, j \in J$ consider the map $O_{h,j} : C \ni c \mapsto \mu_j \circ \psi(c \otimes h) \in \mathbb{R}$. Notice that $O_{h,j} \in C^*$. Define the set $L_\Sigma = \{O_{h,j} \mid j \in J, h \in H\} \subseteq C^*$ and let $A_\Sigma = Alg(L_\Sigma)$ be the subalgebra of C^* generated by L_Σ (i.e., A_Σ is the smallest subalgebra of C^* which contains L_Σ). We will call L_Σ the *set of observables of* Σ and A_Σ the *algebra of observables of* Σ. Let $A_\Sigma^\perp = \{c \in C \mid \forall f \in A_\Sigma : f(c) = 0\}$. It follows that $A_\Sigma^\perp$ is a coideal. We will call Σ *observable* if $A_\Sigma^\perp = \{0\}$.

Let $\Sigma_1 = (C_1, H, \psi_1, \phi_1, J, \mu_1)$ and $\Sigma_2 = (C_2, H, \psi_2, \phi_2, J, \mu_2)$ be two coalgebra systems. A coalgebra map $T : C_1 \to C_2$ is called *coalgebra system morphism from* Σ_1 *to* Σ_2 and it is denoted by $T : \Sigma_1 \to \Sigma_2$, if $T(\phi_1) = \phi_2$, for each $c \in C_1, h \in H$, $T(\psi_1(c \otimes h)) = \psi_2(T(c) \otimes h)$ and for each $j \in J$, $c \in C_1$, $\mu_1(j)(c) = \mu_2(j)(T(c))$.

We will call a coalgebra system Σ_m realizing Ψ *a minimal realization* if for any reachable coalgebra system Σ realizing Ψ there exists a surjective coalgebra system morphism $T : \Sigma \rightarrow \Sigma_m$.

Denote by M the multiplication map on H. That is, $M : H \otimes H \rightarrow H, M(s \otimes v) = sv$. Since H is a bialgebra, the map M is a coalgebra map, moreover, $M(v, M(s,x)) = M(v, sx)$. Let $\Psi = \{f_j \in H^* \mid j \in J\}$ be an indexed set of elements of H^*. Define the map $\mu_\Psi : J \rightarrow H^*$ by $\mu_\Psi(j) = f_j$. Define the coalgebra control system

$$\Sigma_\Psi = (H, H, M, 1, J, \mu_\Psi)$$

It is easy to see that Σ_Ψ is indeed a coalgebra system, moreover, Σ_Ψ is a realization of Ψ, since $f_j(h) = f_j(M(1 \otimes h)) = \mu_\Psi(j) \circ M(1 \otimes h)$ for all $j \in J$. We will call Σ_Ψ the *cofree realization* of Ψ. We will denote the algebra of observables of Σ_Ψ by A_Ψ. That is, $A_{\Sigma_\Psi} = A_\Psi$. Notice that $A_\Psi \subseteq H^*$. It is easy to see that for Σ_Ψ the maps $O_{h,j}$ are of the form $O_{h,j}(v) = f_j(vh) = R_h f_j$. If $\Sigma = (C, H, \psi, \phi, J, \mu)$ is a realization of Ψ, then it is easy to see that $T_\Sigma : H \rightarrow C, T_\Sigma(h) = \psi(\phi \otimes h)$ defines a coalgebra system morphism $T_\Sigma : \Sigma \rightarrow \Sigma_\Psi$. Notice that $T_\Sigma = R_\Sigma$, i.e., T_Σ equals the reachability map.

Below we will state and prove that any set of input/output maps Ψ admits a minimal coalgebra realization.

Theorem 4. *(1)*
Let $\Psi = \{f_j \in H^* \mid j \in J\}$. *Then there always exists a minimal coalgebra system realization of* Ψ.

(2)
A coalgebra system realizing Ψ *is minimal if and only if it is reachable and observable.*

Proof. We will sketch the (easy) proof of (1) in order to present some constructions, which will be very useful later on. Take the cofree realization Σ_Ψ of Ψ. It is easy to see that Σ_Ψ is reachable. Consider the system $\Sigma_m = (H/A_\Psi^\perp, H, \widetilde{M}, [1], J, \widetilde{\mu}_\Psi)$ where $\widetilde{M}(h \times [k]) = [hk]$ and $\widetilde{\mu}_\Psi(j)([h]) = f_j(h)$, and $[h]$ denote the equivalence class generated by h with respect to the relation $[h] = [d] \iff h - d \in A_\Psi^\perp$. Then Σ_m is reachable and observable. If $\Sigma = (C, H, \psi, \phi, J, \mu)$ is reachable, then R_Σ is surjective and let $S : C \ni c \mapsto [h] \in H/A_\Psi^\perp$, where $R_\Sigma(h) = c$. It is easy to see that S is well-defined and it is a surjective coalgebra system morphism. □

We will call the minimal realization Σ_m from the above proof *canonical minimal realization* and we will denote it by $\Sigma_{\Psi,m}$.

Consider a pair of maps $f = (f_D, f_C)$, with $f_D : \Gamma^* \rightarrow O$ and $f_C : \widetilde{\Gamma}^* \rightarrow \mathbb{R}^p$. Recall the definition of the set $\Psi_f = \{f_{C,i} : H \rightarrow \mathbb{R} \mid i = 1, \ldots, p\}$ such that $f_C = (f_{C,1}, \ldots, f_{C,p})^T$. Recall that the maps $f_{C,i}$ are linear and thus belong to the dual H^* of H. Since $\Gamma \subseteq H$, we can define the map $L_w g$ for all $g \in H^*$ by $L_w g(h) = g(wh)$. Define the map $d_f : \Gamma^* \rightarrow O \times (H^*)^p$ as $\forall w \in \Gamma^* : d_f(w) = (f_D(w), (L_w f_{C,i})_{i=1,\ldots p})$. Denote by $\bar{O}$ the set $\bar{O} = O \times (H^*)^p$.

Assume that $HC = (\mathcal{A}, \Sigma, q_0)$ is a hybrid coalgebra system and assume that $\Sigma = (C, H, \psi, \phi, \{1, \dots, p\}, \mu)$ and $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$. Define the automaton $\bar{\mathcal{A}}_{HC} = (Q, \Gamma, \bar{O}, \delta, \bar{\lambda})$ as follows. Let $\bar{\lambda}(q) = (\lambda(q), (T_{q,j})_{j=1,\dots p})$, where $T_{q,j} \in H^*$ and $T_{q,j}(h) = \mu(j) \circ \psi(\phi_q \otimes h)$. Here ϕ_q denotes the unique group like element of C_q.

We get the following theorem, which gives a necessary and sufficient condition for HC to be a realization of f.

Theorem 5. *The hybrid coalgebra system $HC = (\mathcal{A}, \Sigma, q_0)$ is a realization of f if and only if $(\bar{\mathcal{A}}_{HC}, q_0)$ is a realization of d_f and Σ is a realization of Ψ_f.*

We will call a coalgebra system $\Sigma = (C, H, \psi, \phi, \{1, \dots, p\}, \mu)$ *a CCPI coalgebra system* if $C = \bigoplus_{i \in I} C_i$ such that I is finite, and for all $i \in I$, $C_i \cong B(V_i)$, $\dim V_i < +\infty$. Consequently, C is pointed and $G(C) = \{g_i \mid i \in I\}$, where g_i is the unique group-like element of C_i.

It is easy to see that Theorem 5 implies the following.

Theorem 6. *The pair $f = (f_D, f_C)$ admits a CCPI hybrid coalgebra system realization, only if d_f admits a Moore-automaton realization and Ψ_f admits a CCPI coalgebra system realization.*

We can also prove a result which is in some sense the converse of the theorem above.

Let $\Sigma = (C, H, \psi, \phi, \{1, \dots, p\}, \mu)$ be a coalgebra system such that C is pointed. We will say that Σ is *point-observable*, if $A_\Sigma^\perp \cap C_0 = \{0\}$, that is, if for some $g, h \in G(C)$, $g - h \in A_\Sigma^\perp$, then $g = h$. That is, the states belonging to $G(C)$ are distinguishable (observable). In particular, if Σ is observable, then it is point-observable.

Let $\Sigma = (C, H, \psi, \phi, \{1, \dots, p\}, \mu)$ be a point-observable coalgebra realization of Ψ_f, such that C is pointed. Let $\bar{\mathcal{A}} = (Q, \Gamma, \bar{O}, \delta, \bar{\lambda})$ be a Moore-automaton such that $(\bar{\mathcal{A}}, q_0)$ is a reachable realization of d_f. We can associate a hybrid coalgebra system $HC_{\bar{\mathcal{A}}, \Sigma, q_0}$ with $(\bar{\mathcal{A}}, q_0)$ and Σ. The construction goes as follows.

$$HC_{\bar{\mathcal{A}}, \Sigma, q_0} = (\mathcal{A}, \widetilde{\Sigma}, q_0)$$

where

– $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ where $\lambda(q) = o$ if $\bar{\lambda}(q) = (o, \bar{o})$.

– $\widetilde{\Sigma} = (\widetilde{C}, H, \widetilde{\psi}, \widetilde{\phi}, \{1, \dots, p\}, \widetilde{\mu})$ where

- $\widetilde{C} = \bigoplus_{q \in Q} C_q$, where for each $q \in Q$, C_q is the irreducible component of C with the unique group-like element ϕ_q defined by $\phi_q = \psi(w \otimes \phi)$, where $w \in \Gamma^*$ such that $\delta(q_0, w) = q$.

- With the notation above $\widetilde{\phi} = \phi_{q_0}$

- The map $\widetilde{\psi} : \widetilde{C} \otimes H \to \widetilde{C}$ is defined as follows. For each $q \in Q$,$c \in C_q$, $x \in \mathrm{Z}_m$, $\widetilde{\psi}(c \otimes x) = \psi(c \otimes x) \in C_q$. For each $q \in Q$, $c \in C_q$, $\gamma \in \Gamma$, $\widetilde{\psi}(c \otimes \gamma) = \psi(c \otimes \gamma) \in C_{\delta(q,\gamma)}$.

- For all $j \in J$, the map $\widetilde{\mu}(j) \in \widetilde{C}^*$ is such that for all $q \in Q$, $c \in C_q$, $\widetilde{\mu}(j)(c) = \mu(j)(c)$.

Lemma 4. *With the notation and assumptions above $HC = HC_{\bar{\mathcal{A}},\Sigma,q_0}$ is a well-defined hybrid coalgebra system which realizes f. If Σ is a CCPI coalgebra system then HC is a CCPI hybrid coalgebra system.*

Thus, we get the following characterisation of existence of a realization by a CCPI hybrid coalgebra system

Theorem 7. *The pair $f = (f_D, f_C)$ admits a CCPI hybrid coalgebra system realization, if d_f admits a Moore-automaton realization and Ψ_f admits a point-observable CCPI coalgebra system realization.*

It follows from the standard theory of Moore-automata that d_f had a Moore-automaton realization if and only if $W_{d_f} = \{w \circ d_f \mid w \in \Gamma^*\}$ is a finite set. Define the sets $D_f = \{w \circ f_D \mid w \in \Gamma^*\}$ and $K_f = \{(L_w f_{C,j})_{j=1,\dots,p} \in (H^*)^p \mid w \in \Gamma^*\}$.

Lemma 5. *With the notation above W_{d_f} is finite if and only if K_f is finite and D_f is finite. That is, d_f has a realization by a Moore-automaton if and only if f_D has a realization by a Moore-automaton and K_f is finite.*

Assume that K_f is finite, more precisely, let $K_f = \{q_i = (L_{w_i} f_{C,j})_{j=1,\dots,p} \mid i = 1,\dots N\}$. For each $q_i \in K_f$ define the set $H_{q_i} = \bigoplus_{w \in \Gamma^*, (L_w f_{C,j})_{j=1,\dots,p} = q_i} H_w$. It is easy to see that $H = \bigoplus_{i=1}^N H_{q_i}$. Consider the cofree realization Σ_{Ψ_f} and the minimal coalgebra realization $\Sigma_{\Psi_f,m} = (D, H, \psi, \phi, \{1,\dots,p\}, \mu)$ of Ψ_f where $D = H/A^{\perp}_{\Psi_f}$. There exists a canonical morphism $\pi : H \to D$ which defines a coalgebra system morphism $\pi : \Sigma_{\Psi_f} \to \Sigma_{\Psi_f,m}$. Since π is surjective and H is pointed, it follows that D is pointed. Moreover, it follows that $\Sigma_{\Psi_f,m}$ is observable. In fact, the following holds.

Lemma 6. *With the notation above $D = \bigoplus_{i=1}^N \pi(H_{q_i})$, and $\pi(H_{q_i})$ is pointed irreducible.*

That is, if $(\bar{\mathcal{A}}, q_0)$ is a minimal realization d_f and $\Sigma_{\Psi_f,m}$ is the canonical minimal realization of Ψ_f, then $HC_{\bar{\mathcal{A}},\Sigma_{\Psi_f,m},q_0}$ is a well-defined hybrid coalgebra system realization.

That is, we can formulate the following theorem.

Theorem 8. *The pair $f = (f_C, f_D)$, $f_C : \widetilde{\Gamma}^* \to \mathbb{R}^p$ and $f_D : \Gamma^* \to O$ has a realization by a hybrid coalgebra system, if and only if $card(K_f) < +\infty$ and $card(D_f) < +\infty$. If $(\bar{\mathcal{A}}, q_0)$ is a minimal Moore-automaton realization of d_f and*

$\Sigma_{\Psi,m}$ is the canonical minimal coalgebra system realization of Ψ_f, then $HC_{f,m} = HC_{\bar{\mathcal{A}},\Sigma_{\Psi_f,m},q_0}$ is a hybrid coalgebra system realization of f.

Below we will formulate necessary conditions for existence of a realization by a hybrid coalgebra systems. These conditions will involve finiteness requirements. That is, they will require that a certain infinite matrix has a finite rank and that certain sets are finite. Although such conditions are difficult to check, yet they are more informative than requiring that there exists a realization by a coalgebra system of a certain class. The obtained rank condition is similar to the classical Lie-rank condition for existence of a realization by a nonlinear system [ISI 89, FLI 80, JAK 86].

Consider the set $P(H) \subseteq H$ of primitive elements of H. It is easy to see that

$$P(H) = \{wPv \mid w, v \in \Gamma^*, P \in Lie < \mathrm{Z}_m^* >\}$$

where $Lie < \mathrm{Z}_m^* >$ denotes the set of all Lie-polynomials over Z_m. That is, $Lie < \mathrm{Z}_m^* >$ is the smallest subset of the set of all polynomials $\mathbb{R} < \mathrm{Z}_m^* >$ such that

– For all $x \in \mathrm{Z}_m$, $x \in Lie < \mathrm{Z}_m^* >$

– If $P_1, P_2 \in Lie < \mathrm{Z}_m^* >$, then $P_1P_2 - P_2P_1 \in Lie < \mathrm{Z}_m^* >$.

Define the *Lie-rank of f* as follows. Let $\widetilde{P}(H) = \mathrm{Span}\{h \in H \mid h \in P(H)\}$ and let

$$\mathrm{rank}\,_L f = \dim \widetilde{P}(H)/(A_{\Psi_f}^{\perp} \cap \widetilde{P}(H))$$

The notion of Lie-rank can be reformulated as follows. Consider the natural projection $\pi : H \ni h \mapsto [h] \in D = H/A_{\Psi_f}^{\perp}$. Then it is easy to see that $\mathrm{rank}\,_L f = \dim(\sum_{w\in\Gamma^*} \pi(P(H_w))$.

Let $HC = (\mathcal{A}, \Sigma, q_0)$ be a CCPI hybrid coalgebra system, Assume that $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$, $\Sigma = (C, H, \psi, \phi, \{1, \ldots, p\}, \mu)$ and $C = \bigoplus_{q\in Q} C_q$. Define the dimension of HC as $\dim HC = (card(Q), \sum_{q\in Q} \dim P(C_q))$. It is easy to see that if HF is a formal hybrid system realization of f, then $\dim HC_{HF} = \dim HF$. Conversely, if HF_{HC} is the formal hybrid system associated with HC, then $\dim HF_{HC} = \dim HC$.

Using the notation and terminology above, we get the following necessary condition for existence of a CCPI hybrid coalgebra system realization

Theorem 9. *The pair $f = (f_D, f_C)$, $f_D : \Gamma^* \to O$, $f_C : \widetilde{\Gamma}^* \to \mathbb{R}^p$ has a realization by a CCPI hybrid coalgebra system only if $\mathrm{rank}\,_L f < +\infty$, $card(K_f) < +\infty$ and $card(D_f) < +\infty$. For any CCPI hybrid coalgebra system realization HC of f, $(card(W_{d_f}, \mathrm{rank}\,_L f) \leq \dim HC$. That is, if $\dim HC = (p, q)$, then $card(W_{d_f}) \leq p$ and $\mathrm{rank}\,_L f \leq q$.*

Sketch. Let $\Sigma = (C, H, \psi, \phi, \{1, \ldots, p\}, \mu)$ be a CCPI coalgebra realization of Ψ_f. Assume that $C = \bigoplus_{i\in I} B(V_i)$, where I is finite. Define the set $\widetilde{P}(C) = \bigoplus_{i\in I} V_i$.

It is easy to see that $\widetilde{P}(C)$ is finite dimensional. Consider the coalgebra map $T_{\Sigma}: H \to C$. It is easy to see that $T_{\Sigma}(\widetilde{P}(H)) \subseteq \widetilde{P}(C)$ and thus $\widetilde{P}(H)/\widetilde{P}(H) \cap \ker T_{\Sigma} \cong T_{\Sigma}(\widetilde{P}(H))$.

Recall that $\ker T_{\Sigma} \subseteq A_{\Psi_f}^{\perp}$, where A_{Ψ_f} is the algebra generated by $R_h f$, $h \in H$ and $A_{\Psi_f}^{\perp} = \{h \in H \mid \forall g \in A_{\Psi_f}, g(h) = 0\}$. Since $\widetilde{P}(H) \cap \ker T_{\Sigma} \subseteq A_{\Psi_f}^{\perp} \cap \widetilde{P}(H)$ we get that $+\infty > \dim \widetilde{P}(C) \geq \dim \widetilde{P}(H)/\widetilde{P}(H) \cap \ker T_{\Sigma} \geq \dim \widetilde{P}(H)/\widetilde{P}(H) \cap A_{\Psi_f}^{\perp}$ □

Taking into account that f has a realization by a CCPI hybrid coalgebra system if and only if it has a realization by a formal hybrid system we get the main result of the chapter.

Theorem 10. *The pair $f = (f_D, f_C)$, $f_D : \Gamma^* \to O$, $f_C : \widetilde{\Gamma}^* \to \mathbb{R}^p$ has a realization by a formal hybrid system only if* $\text{rank}_L f < +\infty$, $card(K_f) < +\infty$ *and* $card(D_f) < +\infty$. *For any formal hybrid system realization HF of f,*
$(card(W_{d_f}), \text{rank}_L f) \leq \dim HF$.

That is, $\text{rank}_L f$ *gives a lower bound on the dimension of the continuous state space (number of variables) for each formal hybrid realization of f.*

Consider the canonical minimal coalgebra system $\Sigma_{\Psi_f,m}$ realization of f. Recall that $\Sigma_{\Psi_f,m} = (D, H, \psi, \phi, \{1, \dots, m\}, \mu)$ where $D = H/A_{\Psi_f}^{\perp}$. Define the vector space $\widetilde{P}(D) = \text{Span}\{d \in D \mid d \in P(D)\}$. Define the *strong Lie-rank of f* as

$$\text{rank}_{L,S} f = \dim \widetilde{P}(D)$$

It is easy to see that $\text{rank}_{L,S} f \leq \text{rank}_L f$. The difference between the Lie-rank and strong Lie-rank is highlighted by the following theorem.

Theorem 11. *With the notation above the following holds.*

(a) If $card(K_f) < +\infty$, $card(D_f) < +\infty$ *and* $\text{rank}_L f < +\infty$, *then there exists a hybrid coalgebra system realization HC of f such that $HC = (\mathcal{A}, \Sigma, q_0)$, $\Sigma = (C, H, \psi, \phi, J, \mu)$, $C = \bigoplus_{q \in Q} C_q$ and for each $q \in Q$, C_q is pointed irreducible and* $\dim T_{\Sigma}(P(H)) \cap P(C_q) < +\infty$,*where $q_i = L_{w_i} f \in K_f$, $\delta(q_0, w_i) = q$ and $T_{\Sigma} : H \ni h \mapsto \psi(\phi_{q_0} \otimes h)$ is the canonical map $T_{\Sigma} : \Sigma_{\Psi_f} \to \Sigma$.*

(b) If $card(K_f) < +\infty$, $card(D_f) < +\infty$ *and* $\text{rank}_{L,S} f < +\infty$ *then f has a realization by a hybrid coalgebra system $HC = (\mathcal{A}, \Sigma, q_0)$ such that $\Sigma = (C, H, \psi, \phi, J, \mu)$, $C = \bigoplus_{q \in Q} C_q$ and for each $q \in Q$ C_q is pointed irreducible and* $\dim P(C_q) < +\infty$.

Sketch. Assume that $card(K_f) < +\infty$ and $card(D_f) < +\infty$. Consider the minimal canonical coalgebra system realization
$\Sigma_{\Psi_f,m} = (D, H, \psi, \phi, \{1, \dots, p\}, \mu)$. Recall from Lemma 6 that $D = \bigoplus_{i=1}^{N} \pi(H_{q_i})$ where $H_{q_i} = \bigoplus_{(L_w f_{C,j})_{j=1,\dots,p} = q_i} H_w$, $K_f = \{q_1, \dots, q_N\}$ and $\pi : H \to D =$

$H/A_{\Psi_f}^{\perp}$ is the canonical projection $\pi(x) = [x]$. That is, each irreducible component of D is of the form $\pi(H_{q_i})$ for some $q_i \in K$.

Let $(\bar{\mathcal{A}}, q_0)$ be a minimal Moore-automaton realization of d_f. Recall the construction of $HC_{f,m} = HC_{\bar{\mathcal{A}},\Sigma_{\Psi,m},q_0}$ Recall that $HC_{f,m} = (\mathcal{A}, \widetilde{\Sigma}, q_0)$, such that $\bar{\mathcal{A}} = (Q, \Gamma, \bar{O}, \delta, \bar{\lambda})$, $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$,

$$\widetilde{\Sigma} = (\widetilde{C}, H, \widetilde{\psi}, \widetilde{\phi}, \{1, \ldots, p\}, \mu)$$

such that $\widetilde{C} = \bigoplus_{q \in Q} \widetilde{C}_q$ and $\widetilde{C}_q = \pi(H_{q_i})$, for $q_i = \Pi_{(H^*)^p}(\bar{\lambda}(q))$. It is known that HC is a hybrid coalgebra system realization of f.

Assume that $\mathrm{rank}\,_L f < +\infty$. Then for each $q \in Q$, such that $\delta(q_0, w) = q$, the following holds. $\widetilde{C}_q = \pi(H_{q_i})$, where $L_w f = q_i$ and $T_\Sigma(\widetilde{P}(H)) \cap P(C_q) = \pi(\widetilde{P}(H)) \cap P(\pi(H_{q_i})) = \mathrm{Span}\{h \in \widetilde{C}_q \mid h \in \pi(P(H_{q_i}))\}$. Since $\pi(\widetilde{P}(H)) = \widetilde{P}(H)/A_{\Psi_f}^{\perp}$ and $\mathrm{rank}\,_L f = \dim \pi(\widetilde{P}(H)) < +\infty$, we get that $\dim T_\Sigma(\widetilde{P}(H)) \cap P(C_q) < +\infty$. Thus, by taking HC part (a) of the theorem is proven.

Assume that $\mathrm{rank}\,_{L,S} f < +\infty$. Consider the hybrid coalgebra system HC. Then for all $q \in Q$, $\widetilde{C}_q = \pi(H_{q_i})$, such that $q_i = L_w f$ and $\delta(q_0, w) = f$. Since $\widetilde{P}(D) = \mathrm{rank}\,_{L,S} f < +\infty$ and $P(\widetilde{C}_q) = P(\pi(H_q)) \subseteq \widetilde{P}(D)$, we get that $\dim P(\widetilde{C}_q)) < \mathrm{rank}\,_{L,S} f < +\infty$. Thus, part (b) of the theorem is proven.

□

Let us try to find interpretation of the results of the theorem above. Part (a) of the theorem above says that the subspace of each C_q spanned by the elements of $Lie < \mathrm{Z}_m^* >$ and their translates by $\psi(. \otimes \gamma) : C \ni c \mapsto \psi(c \otimes \gamma)$, $\gamma \in \Gamma$ is finite dimensional.

Part (b) implies that for each $q \in Q$, C_q is pointed, irreducible and $n_q = \dim P(C_q) < +\infty$. But this implies that for each q, there exists an injective $S_q : C_q \to B(V_q)$, where $V_q = P(C_q)$. That is, there exists an algebra map

$$S_q^* : \mathbb{R}[[X_1, \ldots, X_{n_q}]] \to C_q^*$$

such that $(\mathrm{Im} S_q^*)^\perp = \{0\}$, i.e. for all $c \in C_q$ and $g \in C_q^*$ there exists some $Z \in \mathbb{R}[[X_1, \ldots, X_{n_q}]]$ such that $S_q^*(Z)(c) = g(c)$. That is, S_q^* is "almost" surjective. Thus, $\dim P(D) < +\infty$ implies existence of an "almost" formal hybrid system realization.

Thus, finiteness of $\mathrm{rank}\,_{L,S} f$ is a stronger requirement than finiteness of $\mathrm{rank}\,_L f$. As we have seen, if $\mathrm{rank}\,_{L,S} f < +\infty$, then there exists an "almost CCPI" realization of f, i.e. f can be realized by a *hybrid system with finite state space of some sort*.

In fact, we can give the following necessary condition for finiteness of $\mathrm{rank}\,_L f$. Define the following space

$$H_{L,f} = \{(L_P f_{C,i})_{i=1,\ldots,p} \mid P \in \widetilde{P}(H)\}$$

It is easy to see that $\dim H_{L,f} \leq \text{rank}\,_L f$. Thus, if $\text{rank}\,_L f < +\infty$, then $\dim H_{L,f} < +\infty$. Below we will present an example, which demonstrates that the Lie-rank might simply be not enough to capture all the necessary dimensions.

Example 2. *Consider the following hybrid system* $H = (\mathcal{A}, (\mathcal{X}_q, g_{q,j}, h_{q,i})_{q\in Q, j=0,\ldots,m, i=1,\ldots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, \{x_q\}_{q\in Q})$ *such that*

– $\Gamma = \{\gamma\}$, $\mathcal{A} = (\{q_1, q_2\}, \{\gamma\}, \{o\}, \delta, \lambda)$, *where* $\delta(q_1, \gamma) = q_2$, $\delta(q_2, \gamma) = q_2$, $\lambda(q_i) = o$, $i = 1, 2$.

– $U = \mathbb{R}$, $\mathcal{Y} = \mathbb{R}$,

– $\mathcal{X}_{q_1} = \mathcal{X}_{q_2} = \mathbb{R}$,

– $g_{q_1,0}(x) = 0$, $g_{q_1,1} = 1$ $h_{q_1}(x) = 0$ *and* $R_{q_2,\gamma,q_2}(x) = x^2$*, for all* $x \in X_{q_1}$, $u \in \mathcal{U}$,

– $h_{q_2}(x) = x$, $q_{q_2,0} = 0 = g_{q_2,1}$ *and* $R_{q_2,\gamma,q_2}(x) = x$ *for all* $x \in X_{q_2}$, $u \in \mathcal{U}$.

Consider the input-output map $f = \upsilon_H((q_1, 0), .)$. *Consider the pair* $\widetilde{f} = (f_D, c_f)$, *where* c_f *is a generating convergent series such that* $F_{c_f} = f_C$. *It is easy to see that* $\text{rank}\,_L \widetilde{f} = 0$. *On the other hand, it can be shown that* $\text{rank}\,_{L,S} \widetilde{f} \geq 1$. *It is easy to see that* $card(K_{\widetilde{f}}) = 2 = card(W_{d_{\tilde{f}}})$, *thus one needs at least two discrete states to realize* f. *Hence, unless we allow for zero dimensional continuos spaces, no realization can be of dimension smaller than* $(2, 2)$.

9. Conclusions

We have presented conditions for existence of a realization by a nonlinear hybrid system. The presented conditions are only necessary but there is a strong indication that the presented approach might lead to sufficient conditions as well. The presented conditions are consistent with the earlier results on hybrid systems and classical nonlinear systems. The main tool for developing the obtained results was the theory of coalgebras. Future research will be directed towards developing realization theory for polynomial and rational hybrid systems without guards and towards finding sufficient and necessary conditions for existence of a nonlinear hybrid system realization.

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