Presented at 17th Int. Sympos. on Math. Theory of Networks & Systems (MTNS)
Kyoto, July, 2006.

On the average control system

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***Abstract*— Considering a control system with fast and slow variations, the purpose of this note is to propose a notion of "average control system" that does not depend on how fast or slow the controls will be. This differs from the more usual use of averaging in control where the fast or slow variations of the control have to be specified before performing averaging.**

We also present an application to low thrust orbit transfer.

I. Introduction

A. Averaging for differential equations

Consider the differential equation

$$dx/dt = F(x, \theta)\,, \qquad d\theta/dt = \omega(x)/\varepsilon\,, \tag{1}$$

where x lives in $\mathbb{R}^n$, θ in $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and $\varepsilon > 0$ is a small parameter; where we assume that ω does not vanish:

$$\omega(x) \geq 1\,,\ \forall x\,, \tag{2}$$

so that, when ε is small, θ is a fast variable (its derivative is of the oreder of $1/\varepsilon$) while x is "slow", in the sense that $\dot{x}$ is bounded, irrespective of ε.

Note that a simple change of time ($ds/dt = \omega(x)$) turns the above into:

$$\frac{dx}{ds} = \frac{F}{\omega}(x, \theta_0 + \frac{s}{\varepsilon})\,, \quad x \in \mathbb{R}^n\,,$$

where F/ω is smooth and 2π-periodic with respect to its last argument.

The well known and very powerful principle of averaging says that, if ε is small enough, the x component of a solution of equation (1) is close to a solution of the average equation $d\bar{x}/dt = \overline{F}(\bar{x})$ where $\overline{F}$ is the average on one period of the right hand side of (1):

$$\overline{F}(x) = \frac{1}{2\pi} \oint_{\theta \in S^1} F(x, \theta)\, d\theta \tag{3}$$

(this does not depend on ω; it would if ω would depend on θ)

Theorem 1 ([1]): If $t \mapsto \bar{x}(t)$ is a solution of $d\bar{x}/dt = \overline{F}(\bar{x})$ defined on the time interval $[0, 1]$, and that remains in the interior of a compact set $\mathcal{K}$, then there exists a constant k such that the solution x^ε of (1) with $x^\varepsilon(0) = \bar{x}(0)$ satisfies $\|x^\varepsilon(t) - \bar{x}(t)\| < k\varepsilon$, for ε small enough. The constant k depends only on the map F on the compact $\mathcal{K}$.

Averaging is used in many instance, for example when dealing with small perturbations of integrable Hamiltonian systems [2], or in nonlinear oscillations, see [10].

This work was partially supported, and motivated, by *Alcatel Alenia Space, Cannes, France*, under a contractual research project.

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B. Averaging in Control theory

This powerful tool was also used in control theory because one encounters many systems, to be controlled, that exhibit oscillations at a fast speed compared to the rest of the dynamics, or it may be interesting to use a control that has slow-varying and fast-varying components, even if the original control system only displays one time scale. Let us mention some contributions, with no claim to be exhaustive.

In [8] the following optimal problem is considered

$$\begin{cases} \dot{x} = f(x, u(t), t, t/\varepsilon) \\ \min \int_0^T L(x, u)dt \quad , \end{cases} \tag{4}$$

and it is shown, under some assumptions, that the solutions of these problems (4) with fast variable are ε close in L^∞ norm to the solution of the limit problem (5) defined as follow

$$\begin{cases} \dot{x} = \frac{1}{\omega} \int_0^\omega f(x, u(t, \theta), t, \theta) d\theta \\ \min \int_0^T \left(\frac{1}{\omega} \int_0^\omega L(x, u(t, \theta)) d\theta \right) dt \end{cases} \tag{5}$$

This was applied to low thrust orbital transfer in [9]. Using this approach, it is shown in [4], [5] that the average equation of the Pontryagin maximum principle (for minimizing energy) on the cotangent bundle is the geodesic flow of some Riemannian metric

Also, in vibrational control [11], even for a control system that does not present fast oscillation, one adds oscillatory components to the control, and then uses averaging to analyze the resulting stability.

In all these approaches, the variation of the control, in particular its fast or slow dependence, on time and/or on the state (maybe an extended state, like when using Pontryagin Maximum principle) is prescribed *before* performing any average. Indeed, consider, instead of a differential equation (1) (without control), a control system

$$\begin{aligned} dx/dt &= F(x, \theta, u) \\ d\theta/dt &= \omega(x)/\varepsilon\,, \end{aligned} \tag{6}$$

where $u \in \mathbb{R}^m$ is a control, $(x, \theta) \in \mathbb{R}^n \times S^1$ is the state and F is smooth. Considering θ in S^1 means that F is 2π-periodic with respect to its last argument. One cannot apply directly an averaging formula like (3) when u is not prescribed: for instance, the average of $u \sin \theta$ (u scalar) is zero if u is constant or varies "slowly", but it is $\frac{1}{2}$ if $u = \sin \theta$.

In the above mentioned papers in optimal control, one applies first the maximum principle to obtains a a differential equation on the cotangent bundle that no longer contains any control and then uses averaging. In vibrational control, one decides to take a feedback of a special form, say (this is an extreme simplification) $u_1(x)+u_2(x)\sin\theta$, and then again it is possible to use averaging, but the obtained average system is valid only if u_1 and u_2 are functions of x (or at least vary slowly).

The contribution of the present note is to define directly an *average control system* for a control system like (6), *without* prescribing in advance the slow or fast variations of the control. We then illustrate it on an academic example (section III) and apply it to give a partial answer to an open question (section IV).

II. The average control system

A. Control systems with fast oscillations

We consider a control system of the following special form, where the state space is $M\times S^1$. M is a manifold of dimension n and $S^1=\mathbb{R}/2\pi\mathbb{Z}$ the circle (this is to emphasize that everything is periodic with respect to the variable θ). The control $u=(u_1,\dots,u_m)$ is m-dimensional and is bound to stay in the unit closed ball B_1 of the Euclidean space $\mathbb{R}^m$.

$$\begin{aligned}\dot I &= G(I,\theta)\,u \triangleq \sum_{k=1}^m u_k\,g_k(I,\theta)\\ \dot\theta &= 1/\varepsilon\\ u &\in B_1 \qquad \text{i.e. } \|u\|\le 1\ .\end{aligned} \tag{7}$$

A more general class of systems is (still with $u\in B_1$):

$$\begin{aligned}\dot I &= F(I,\theta)\,u\;+\;\varepsilon\,b(I,\theta,u,\varepsilon)\\ \dot\theta &= \omega(I,\theta)/\varepsilon+c(I,\theta,u,\varepsilon)\end{aligned} \tag{8}$$

with smooth bounded b and c and the assumption that the fast motion is indeed fast:

$$\omega(I,\theta)+\varepsilon\,c(I,\theta,u,\varepsilon)\ge 1 \tag{9}$$

for all $(I,\theta,u,\varepsilon)\in M\times S^1\times B_1\times[0,\varepsilon_0]$ ($\varepsilon_0>0$).

Via a change of time ($ds/dt=\omega(I,\theta)+\varepsilon c(I,\theta,u,\varepsilon)$), it can be turned into

$$\mathrm{d}I/\mathrm{d}s\;=\;G(I,\theta)\,u+\varepsilon h(I,\theta,u,\varepsilon)\ ,\quad \mathrm{d}\theta/\mathrm{d}s\;=\;1/\varepsilon$$

Obviously, $G=H/\omega$, and the expression of h follows. This allows to treat (8) with the results established for (7): that system was exactly the one above with $\varepsilon\,h=0$, but this term that uniformly goes to zero does not affect any of the results obtained for (7).

B. Definition of the average system

For fixed I, consider the following subset of T_IM:

$$\mathcal{E}(I)=\{\frac{1}{2\pi}\oint_{S^1}G(I,\theta)u(\theta)\mathrm{d}\theta,\ u(.)\in L^1(S^1,B_1)\}, \tag{10}$$

i.e. all the possible average values of $\dot I$ (as given by (7)) for fixed I and for *all* possible variation of u.

We call *average control system* of (7) the following differential inclusion on M:

$$\dot I\;\in\;\mathcal{E}(I)\ . \tag{11}$$

A parameterization of the set $\mathcal{E}(I)$ by independent variables ("controls") would provide a control system strictly speaking; we shall discuss this further when more properties of the set $\mathcal{E}(I)$ are established. For now, it is simpler to just consider the differential inclusion.

A *solution* of (11) on the time-interval $[0,1]$ is an absolutely continuous $I:[0,1]\to M$, where the measurable map $\dot I$ satisfies $\dot I(t)\in\mathcal{E}(I(t))$ for almost all t. From (10), this implies that, for almost all t, there exists a measurable map $\widehat{u}_t:S^1\to B_1$ such that

$$\dot I(t)\;=\;\frac{1}{2\pi}\oint_{S^1}G(I,\theta)\widetilde{u}_t(\theta)\mathrm{d}\theta\ .$$

The right-hand side is measurable w.r.t. t, but this does not tell how $\widetilde{u}_t(\theta)$ itself depends on t; in fact, joint measurability with respect to t,θ may always be granted:

Lemma 2: A map $I:[0,1]\to M$ is a solution of (11) if and only if there exists $\widehat{u}\in L^1([0,1]\times S^1)$ such that

$$I(t)=I(0)+\int_0^t\left(\frac{1}{2\pi}\oint_{S^1}G(I(t),\theta)\widehat{u}(t,\theta)\mathrm{d}\theta\right)\mathrm{d}\tau \tag{12}$$

for all t in $[0,1]$.

Proof: The minimum of the map $L^1([0,1]\times S^1)\to\mathbb{R}$ defined by

$$u\mapsto\int_0^1\left(\int_o^{2\pi}G(I(t),\theta)(u(t,\theta)-\widetilde{u}_t(\theta))\mathrm{d}\theta\right)^2\mathrm{d}t$$

must be zero. ■

C. The averaging result

Let us now state how this system is the "limit" control system of (7) when ε goes to zero.

Theorem 3 (averaging): Fix $(I_0,\theta_0)\in M\times S^1$. Let $\mathcal{K}$ be[1] a compact subset of M containing I such that any solution $t\mapsto(I(t),\theta(t))$ of (7) (with the bound $\|u\|\le 1$ on the control) such that $I(0)=I_0$ remains in the interior of $\mathcal{K}\times S^1$ for all time $t\in[0,1]$.

Let $t\mapsto I(t)$ be a solution of (11) defined for $t\in[0,1]$, with $I(0)=I_0$. Then there is a family of measurable controls, namely $u_\varepsilon(.)\in L^1([0,1],B_1)$ for all positive ε such that the solutions $t\mapsto I_\varepsilon(t)$ of (7) with $u=u_\varepsilon(t)$ and $(I_\varepsilon(0),\theta(0))=(I(0),\theta_0)$ converges uniformly on $[0,1]$ to $I(.)$ as ε tends to zero. The distance is bounded by $k\,\varepsilon$ for some constant k, that depends only on $\mathcal{K}$ and the map G.

Conversely, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers that tends to zero, and, for each n, $u_n(.)\in L^1([0,1],B_1)$ be a control and $I_n(.)$ be the solution of (7) with $\varepsilon=\varepsilon_n$, $u=u_n(t)$ and $(I_n(0),\theta(0))=(I_0,\theta_0)$. If $I_n(.)$ converges uniformly on $[0,1]$ to a map $t\mapsto I^\star(t)$, then $I^\star$ is a solution of (11).

Proof: In this proof $\operatorname{Lip}G$ stands for a Lipschitz constant of G on $\mathcal{K}$ and $\sup G$ for a bound of $\|G\|$ on $\mathcal{K}$.

[1] Existence of such a compact set has to do with forward completeness of the vector fields: if $M=\mathbb{R}^n$, it is sufficient that $\|g_k\|$ grows no more than linearly at infinity.

If $I : [0,1] \to M$ is a solution of (11), there exists, according to lemma 2, some $\widehat{u} \in L^1([0,1] \times S^1, B_1)$ such that (12) holds for all t. Then define, for $\varepsilon > 0$,

$$u_\varepsilon(t) = \frac{1}{2\pi} \int_0^{2\pi} \widehat{u}(t + \varepsilon\,\psi, \theta_0 + \frac{t}{\varepsilon})\,\mathrm{d}\psi\ . \qquad (13)$$

Recall that $\widehat{u}$ is 2π-periodic with respect to its second argument; by convention, and because values outside $[0,1]$ appear in the above formula, it can can be prolonged periodically with respect to the second argument as well: $\widehat{u}(t+1,\theta) = \widehat{u}(t,\theta)$ for all t and θ. Let I_ε be the solution of (7) with $u = u_\varepsilon(t)$ and $I_\varepsilon(0) = I_0$. A computation shows that $s\pi\,(I_\varepsilon(t) - I_0)$ can be written as

$$\int_0^t \int_0^{2\pi} G(I_\varepsilon(\tau),\varphi)\,\widehat{u}(\tau,\varphi)\,\mathrm{d}\varphi\,\mathrm{d}\tau\ + \Delta_\varepsilon(t)$$

with $\|\Delta_\varepsilon(t)\| \leq k_1\varepsilon$ and the constant $k_1 = \pi(2 + \operatorname{Lip} G)\sup G$ depends only on the compact $\mathcal{K}$. Since I satisfies (12), one has

$$\|I_\varepsilon(t) - I(t)\| \leq k_1\,\varepsilon + \operatorname{Lip} G \int_0^t \|I_\varepsilon(\tau) - I(\tau)\|\,\mathrm{d}\tau$$

and finally, by Gronwall Lemma, this implies

$$\|I_\varepsilon(t) - I(t)\| \leq k\,\varepsilon \quad \text{with} \quad k = k_1\left(e^{\operatorname{Lip} G} - 1\right)\ ,$$

hence the first part of the theorem.

To prove the converse, assume that $I_n(.)$ converges uniformly on $[0,1]$: for all $t \in [0,1]$,

$$\|I_n(t) - I^\star(t)\| \leq \alpha_n \qquad \text{with} \qquad \lim_{n\to\infty} \alpha_n = 0\ . \quad (14)$$

Define $\widehat{u}_n \in L^1([0,1] \times S^1, B_1)$ by

$$\widehat{u}_n(\tau,\varphi) = u_n(\frac{\tau}{\varepsilon_n} + \theta - \theta_0),\ \begin{cases} \frac{\tau}{\varepsilon_n} + \theta \equiv \varphi \mod 2\pi\ , \\ 0 \leq \theta < 2\pi\ , \end{cases}$$

Obviously, each $\widehat{u}_n$ is also in L^∞, and even in the unit ball of L^∞, hence there is a subsequence that converges weakly to some $\widehat{u}^\star$; we still denoted the subsequence by $\widehat{u}_n$ to avoid double indexes.

We shall prove that

$$I^\star(t) - I_0 = \int_0^t \left(\frac{1}{2\pi}\int_{S^1} G(I,\theta)\widehat{u}^\star(t,\theta)\mathrm{d}\theta\right)\mathrm{d}\tau\ . \quad (15)$$

This implies that $I^\star$ is a solution of (11) and will end the proof of the theorem.

Writing $2\pi\,(I_n(t) - I_0)$ as

$$\int_0^{2\pi}\int_{-\varepsilon_n\theta}^{t-\varepsilon_n\theta} G(I_n(\tau + \varepsilon_n\theta), \theta_0 + \frac{\tau}{\varepsilon_n} + \theta)\,u_n(\tau + \varepsilon_n\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta\ ,$$

it can be transformed into

$$I_n(t) - I_0 = \frac{1}{2\pi}\int_0^t\int_0^{2\pi} G(I^\star(\tau),\varphi)\,\widehat{u}_n(\tau,\varphi)\,\mathrm{d}\varphi\mathrm{d}\tau + \Delta_n(t)$$

where $\|\Delta_n(t)\| \leq k_1\varepsilon_n + k_2\alpha_n$ for some constants k_1, k_2. This proves that $\Delta_n(t)$ tends to zero; the left hand side tends to $I^\star(t) - I_0$ (uniformly) by assumption, and weak convergence of the sequence $(\widehat{u}_n)$ implies that the integral in the right-hand side converges to the one in (15). This does prove (15). ∎

D. Properties of the average system

We have defined the average system as the differential inclusion (11) with the set $\mathcal{E}(I)$ defined by (10). That set enjoys the following properties (the proof is rather elementary, and is skipped).

Theorem 4: For all I, the set $\mathcal{E}(I)$ is convex, closed, bounded, symmetric with respect to the origin and contains the origin. Furthermore, if

$$\operatorname{Rank}\left\{\frac{\partial^j g_k}{\partial\theta^j}(I,\theta)\right\}_{\substack{k\in\{1,\ldots,m\}\\ j\in\mathbb{N}}} = n \qquad (16)$$

for all θ, then the interior of $\mathcal{E}(I)$ is nonempty.

Remark that (16) is equivalent to controllability of the linear approximation of the control system (7) on $M \times S^1$ Under that controllability assumption, the fact that the interior of $\mathcal{E}(I)$ is nonempty means that the underlying control system has n controls, i.e. as many controls as states, whereas the original system had only $m < n$ controls.

Let us comment further on the properties of $\mathcal{E}(I)$ in the case where (16) is met. Then it is well known that each set $\mathcal{E}(I)$ has all the needed properties to be the unit ball for a uniquely defined (not necessarily Euclidean) norm on the vector space $T_I M$ (for any $v \in T_I M$, $\|v\|$ is the unique $\lambda > 0$ such that $v/\lambda \in \partial\mathcal{E}(I)$). This endows the manifold M with a *Finsler metric* (provided some smoothness properties with respect to I are checked). Note that the only additionnal requirement to get a *Riemannian metric* would be that these norms be Euclidean, i.e. that all the sets $\mathcal{E}(I)$ be ellipsoids. Finsler geometry [3] is more intricate than Riemannian geometry , but in a sense much simpler than general optimal control (or sub-Riemannian geometry) (also called Carnot-Caratheodory) because at each point, curves with all tengent directions are allowed.

One may view (7) as a sub-Riemannian geometry on M, parameterized by a fast varying parameter θ, ad we showed that, under an assumption on its dependence on $theta$ (16) (different from the ones regarding the disptribution spanned by the g_k's for fixed θ that are relevent in sub-Riemannian geometry), this sub-Riemannian geometry can be averaged to a Finsler geometry.

It is interesting to note that the authors of [4], [5] obtained, by a completely different argument, a Riemannian metric as the limit of what would be here the problem of minimizing energy (the integral of $\|u\|^2$) for the original system (7) with no constraint on u. The method differs a lot because it consists in apllying averaging for ODEs to the equation on the cotangent bundle ontained via Pontryagin Maximum Principle, and identifying the average as the geodesic flow of a Riemannian metric. Our method applied in the same case, yields a Finsler metric for the minimum time (instead of minimum energy). We have not yet been able to compute the average system well enough to be sure that this Finsler metric is not, for these precise systems (controlled Kepler problem), a Riemannian metric —this amounts to decide whether or not $\mathcal{E}$ is an ellipsoid— but it is certainly very exciting to compare the two limit objects (if one started with a system with n controls,they would not differ because it is well known that minimizing length or energy is the same in Riemannian geometry).

III. Example

Consider a very academic example of (7), with $(m,n) = (1,2)$ and φ some constant:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = u \begin{pmatrix} \cos\theta \\ \cos(\theta+\varphi) \end{pmatrix} , \quad \dot{\theta} = \frac{1}{\varepsilon} , \quad |u| \le 1 .$$

A computation shows that the set $\mathcal{E}(x_1, x_2)$, that obviously does not depend on (x_1, x_2), is given by $\dot{x}_1^2 + (\dot{x}_2 - \cos\varphi\,\dot{x}_1)^2 / \sin^2\varphi \le 4/\pi^2$. Here, we can write the differential inclusion (11) as a control system. If φ is a multiple of π, assumption (16) fails, and one gets as an average control system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \frac{2\,v}{\pi} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} , \quad |v| \le 1 .$$

If not, one gets a system with two controls:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \frac{2\,v_1}{\pi} \begin{pmatrix} 1 \\ \cos\varphi \end{pmatrix} + \frac{2\,v_2}{\pi} \begin{pmatrix} 0 \\ \sin\varphi \end{pmatrix}$$

with the constraint ${v_1}^2 + {v_2}^2 \le 1$.

Note that constraint is quadratic, so that we do not only get a Finsler metric, but even a Riemannian metric (even a trivial one, i.e. it makes M a Euclidean space!). This example, chosen only to illustrate how the average system can be computed, is too simple to have any generality, and we do not claim that the obtained metric is in general Riemannian (or in other words that the set $\mathcal{E}(I)$ in (10) is always an ellipsoid).

IV. Application

In this section we apply the above described averaging method to the controlled 2-body system, in order to study the asymptotic behavior of time optimal trajectories for low thrust elliptic orbital transfers. For compactness of the present note we will consider the planar case.

The controlled Keplerian system in two dimensions can be written $\ddot{r} = -r/\|r\|^3 + u$ with Γ and r 2-dimensional and $\mu > 0$ a gravitational constant, or equivalently, with $x = (r, v) \in \mathbb{R}^2_* \times \mathbb{R}^2$ (position and speed), and $\varepsilon > 0$ an upperbound on the magnitude of the thrust (control),

$$K_\varepsilon : \quad \dot{x} = f_0 + \Gamma_1 f_1 + \Gamma_2 f_2, \quad {\Gamma_1}^2 + {\Gamma_2}^2 \le \varepsilon^2, \quad (17)$$

with $f_0 = (v, -\mu\, r/\|r\|^3)$, $f_1 = (0, e_1)$ and $f_2 = (0, e_2)$, (e_1, e_2) being an ortho normal frame of $\mathbb{R}^2$.

The free motion ($\dot{x} = f_0(x)$) is well known. All the trajectories project to quadratic curves (ellipses, parabolas or hyperbolas) in the position plane $\mathbb{R}^2_*$. Recall that eccentricity of such a curve is a non negative number e; $e < 1$ for an ellipse (zero for circles), $e = 1$ for a parabola, $e > 1$ for an hyperbola; e can here be given as a smooth function of r, v, constant on solutions of $\dot{x} = f_0(x)$, $e(r, v)$ being, for all $(r, v)\mathbb{R}^2_* \times \mathbb{R}^2$, the excentricity of the projection on $\mathbb{R}^2_*$ of the solution of $\dot{x} = f_0(x)$ starting from this point. We call *elliptic domain* the subset of $\mathbb{R}^2_* \times \mathbb{R}^2$ where $e < 1$ (this is equivalent to negative mechanical energy). IN the elliptic domain all solutions are periodic and project onto ellipses in the plane $\mathbb{R}^2_*$. We are only interested in elliptic orbits. Let O_0 and O_1 be the trajectories in $\mathbb{R}^2_* \times \mathbb{R}^2$ of two such orbits (their projections on the first factor are ellipses). The controlled 2-body system is completely controllable. Consider the time optimal orbital transfer problem (studied for instance in [9], [6] for low thrust):

$$K_\varepsilon\,, \;\; x(0) \in O_0, \;\; x(T) \in O_1, \;\; T \to \min\,. \quad (18)$$

This problem admits a solution for all positive ε (see e.g. these references). Denote the minimum time by T_ε.

Chemical engines have a rather high thrust; transfer problems may then be treated via approximations with impulsive controls (in a sense $\varepsilon \to \infty$, but with durations of the thrust that go to zero). On the contrary, the particularity of low thrust engines is that ε is small (what makes their industrial interest is of course not small thrust but their better efficiency). Hence the idealization of low thrust transfer is captured by the limit behaviour when $\varepsilon \to 0$.

"Open question Q3", stated in [7] :
Does the product $\varepsilon\, T_\varepsilon$ have a limit when ε tends to zero?
We give a partial answer to this question, i.e. only for planar transfers between two elliptic orbits O_0 and O_1 with positive angular momentum and such that the time-optimal trajectories (solutions of K_ε above) stay strictly inside the elliptic domain for all ε (see Theorem 5).

If $c = \det(r, v) > 0$, the Gauss coordinates are $I = (c, e_x, e_y)^T$ (three independent first integrals of the Kepler motion; the excenticity mentioned above is defined by $e^2 = {e_x}^2 + {e_y}^2$) and the cumulated longitude L (in the planar case, it is simply the polar angle between r and a fixed direction). Orbits O_0 and O_1 are defined by the corresponding value of I, say I^0 and I^1. The transfer in (18) means, for some L_0, going from $x(0) = (I^0, L^0)$ to some $x(T) = (I^1, L^1)$ with arbitrary L^1.

With the control Γ expressed in the ortho-radial frame QS associated to the satellite, equation (17) reads, in these coordinates:

$$K_\varepsilon : \begin{cases} \dot{I} \;=\; \Gamma_s\, g_s(I, L) + \Gamma_q\, g_q(I, L), \\ \dot{L} \;=\; \omega(I, L), \\ \|\Gamma\|^2 = {\Gamma_s}^2 + {\Gamma_q}^2 \le \varepsilon^2 \end{cases} \quad (19)$$

with

$$g_s = \begin{pmatrix} \frac{c^2}{\mu Z} \\ \frac{cA}{\mu Z} \\ \frac{cB}{\mu Z} \end{pmatrix}, \; g_q = \begin{pmatrix} 0 \\ \frac{c}{\mu} \sin L \\ -\frac{c}{\mu} \cos L \end{pmatrix}, \; \omega = \frac{\mu^2}{c^3} Z^2, \quad (20)$$

and

$$\begin{cases} Z \;=\; 1 + e_x \cos L + e_y \sin L\,, \\ A \;=\; e_x + (1 + Z) \cos L\,, \\ B \;=\; e_y + (1 + Z) \sin L\,. \end{cases}$$

If one choses t/ε as time instead of t, the above system is of the form (8) with $b = 0$ and $c = 0$. Although we do not have a lowerbound like (9), let us rescale time t to some λ with $d\lambda/dt = \varepsilon\omega$ and take $u = \Gamma/\varepsilon$ as new control; one obtains:

$$K'_\varepsilon : \left\{ \begin{array}{rcl} \frac{dI}{d\lambda} & = & u_s g_s/\omega + u_q g_q/\omega, \\ \frac{dL}{d\lambda} & = & 1/\varepsilon, \end{array} \right. \quad \|u\| \leq 1. \quad (21)$$

Note that systems (17) and (19) are the same, written in different coordinates, while (21) leads to the same trajectories, parametrized by λ instead of time.

Equation (21) belongs to the class of controlled systems (7) that admits (at least on a domain where g_s/ω and g_q/ω are bounded), an averaged controlled system according to section II, namely

$$K : \ \dot{I} \ \in \ \mathcal{E}(I) \ , \quad (22)$$

now independent of ε, where $\mathcal{E}(I)$ is defined according to (10), the expression of $G(I,\theta)u(\theta)\mathrm{d}\theta$ being

$$\left(u_s(L)\, \frac{g_s(I,L)}{\omega(I,L)} + u_q(L)\, \frac{g_q(I,L)}{\omega(I,L)} \right) \mathrm{d}L \ .$$

We have not yet performed an explicit computation of this set $\mathcal{E}(I)$, but its exact descritpion is not needed for the following. We now define the functional J:

$$J(I(.))(\lambda) = \int_0^\lambda \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\omega(I(\ell),s)} ds \right) d\ell, \quad (23)$$

and the following optimal control problem, where the evolution variable is $\lambda \in [0,\Lambda]$ instead of $t \in [0,T]$, and the final time (multiplied by ε) is replaced by J (J stands for $J(I(.),\Lambda)$:

$$K\,, \ I(0) = I_0, \ I(\Lambda) = I_1, \ J \to \min. \quad (24)$$

As announced, the following theorem gives a partial answer to the above open question.

Theorem 5: For some I^0 and I^1, assume that there are positive constants ε_0, $e_M < 1$ and $0 < c_m < c_M$ such that the minimum time solutions of problem K_ε, for $\varepsilon \leq \varepsilon_0$, satisfy, for all time t, $e_x(t)^2 + e_y(t)^2 \leq {e_m}^2$ and $c_m \leq c(t) \leq c_M$. Then $\varepsilon\, T_\varepsilon$ has a limit J^*, which is the value of the average optimal problem (24).

Let us make a few comments on our assumptions. Assuming the initial and final angular momenta are positive, it is easy to show that one must have that inequality on $c(t)$ on time-minimal trajectories. The other assumption, that not only the initial and final orbits are elliptic, i.e. ${e_x^1}^2 + {e_y^1}^2 \leq {e_m}^2$ and ${e_x^0}^2 + {e_y^0}^2 \leq {e_m}^2$ for some $e_m < 1$, but also it remains smaller than such a number for all time and all ε on minimum time trajectories is not explicit. It is possible to show that this is true when I^0 and I^1 are elliptic and "close enough". For two general elliptic orbits, we have not proved this, although it does not seem very likely that, for small ε, minimum time trajectories would go closer and closer to the hyperbolic domain.

Sketch of proof of Theorem 5

With e_M, c_m, c_M given by the theorem, pick e'_M, e''_M, c'_m, c''_m, c'_M, c''_M such that

$$\begin{array}{l} e_M < e'_M < e''_M < 1\,, \\ 0 < c''_m < c'_m < c_m < c_M < c'_M < c''_M\,, \end{array} \quad (25)$$

and define the domain

$$\mathcal{D} = \{(e_x, e_y, c),\ c''_m < c < c''_M \text{ and } {e_x}^2 + {e_y}^2 < {e''_M}^2\}\,.$$

Obviously, with $\omega_m = \mu^2(1-e''_m)^2/{c''_M}^3$ and $\omega_M = \mu^2(1-e''_M)^2/{c''_m}^3$, one has

$$(I,L) \in \mathcal{D}\times S^1 \ \Rightarrow\ \omega_m \leq \omega(I,L) \leq \omega_M\,. \quad (26)$$

This implies in particular that the rescaled system K'_ε (21) and the average system K (22) are well defined on $\mathcal{D}$.

The following lemma states that J given by (23) is the limit of the following J_ε when $\varepsilon \to 0$, and gives a Lipschitz constant for these functionals:

$$J_\varepsilon(L_0, I(.))(\lambda) = \int_0^\lambda \frac{\mathrm{d}\ell}{\omega(I(\ell), L_0 + \ell/\varepsilon)}\,. \quad (27)$$

Lemma 6: There exists $k' > 0$ and $\varepsilon_0 > 0$ such that, for all $\lambda \in [0,1]$, $L_0 \in S^1$, $\varepsilon \in [0,\varepsilon_0]$ and all continuous I and I', $[0,\lambda] \to \mathcal{D}$, one has (convention: $J_0 = J$)

$$\|J_\varepsilon(L_0,I)(\lambda) - J(I)(\lambda)\| \ \leq\ k'\varepsilon\,, \quad (28)$$

$$\|J_\varepsilon(L_0,I)(\lambda) - J_\varepsilon(L_0,I')(\lambda)\| \ \leq\ k' \sup_{[0,\lambda]} \|I - I'\|. \quad (29)$$

Proof: It is exactly the same sort of computations than in the proof of Theorem 3. ■

Let $\mathcal{D}' = \{c'_m \leq c \leq c'_M \text{ and } {e_x}^2 + {e_y}^2 \leq {e'_M}^2\}$; I^0, I^1 are in $\mathcal{D}'$ by construction. Let $\gamma^\sharp$: $[0,\Lambda^\sharp] \to \mathcal{D}'$ be a minimizer for problem (24) on $\mathcal{D}'$ (exists for $\mathcal{D}'$ is compact, depends a priori on c'_m, c'_M, e'_M). Let $J^\sharp = J(\gamma^\sharp)(\Lambda^\sharp)$.

Fix L_0. From Theorem 3, there exists, for all positive ε, a solution $\lambda \mapsto (\gamma^\sharp_\varepsilon(\lambda), \widetilde{L}_\varepsilon(\lambda))$ of (21) with $\gamma^\sharp_\varepsilon(0) = I^0$, $L(0) = L_0$ and $\|\gamma^\sharp_\varepsilon - \gamma^\sharp\|_{[0,\Lambda^\sharp]} < k'\varepsilon$. Each of them rescales to a solution $t \mapsto (I_\varepsilon(t), L_\varepsilon(t))$ of (19), defined on $[0, T^\sharp_\varepsilon]$ with $T^\sharp_\varepsilon = J_\varepsilon(L_0,\gamma_\varepsilon)(\Lambda^\sharp)/\varepsilon$. Since $I_\varepsilon(T^\sharp_\varepsilon) = \gamma_\varepsilon(\Lambda^\sharp)$ and $\gamma(\Lambda^\sharp) = I^1$, one has $\|I_\varepsilon(T^\sharp_\varepsilon) - I^1\| \leq k'\varepsilon$. Then Lemma 7 below, with $M = k'$, implies that there is a $\overline{\tau}$, independent of ε, and, for all ε, a solution of (21) that goes from $I_\varepsilon(T^\sharp_\varepsilon)$ to I^1 in time no more than $\overline{\tau}$. Since, from Lemma 6, $|\varepsilon T^\sharp_\varepsilon - J^\sharp| < 2k'\varepsilon$, the above proves

$$\varepsilon\, T_\varepsilon \ \leq\ J^\sharp + (2k' + \overline{\tau})\,\varepsilon\,, \quad 0 < \varepsilon \leq \varepsilon_0\,. \quad (30)$$

Since $\varepsilon\, T_\varepsilon$ is bounded, we only need to prove that any convergent sequence $\varepsilon_n T_{\varepsilon_n}$ ($\varepsilon_n \to 0$) has the same limit. Consider such a sequence and the corresponding minimizer, $[0, T_{\varepsilon_n}] \to \mathcal{D}'$, $t \mapsto (I_n(t), L_n(t))$; we may rescale it to a solution $\lambda \mapsto (\gamma_n(\lambda), \widetilde{L}_n(\lambda))$ of (21), defined on $[0,\Lambda_n]$ with $\gamma_n(0) = I^0$, $\gamma_n(\Lambda_n) = I^1$ and

$$T_{\varepsilon_n} = \frac{1}{\varepsilon_n}\, J_{\varepsilon_n}(\widetilde{L}_n(0), \gamma_n)(\Lambda_n)\,. \quad (31)$$

From (27), this implies $\Lambda_n \leq \varepsilon_n T_{\varepsilon_n} \omega_M$ and, from (30),

$$\Lambda_n \leq \overline{\Lambda} \ \text{ with } \ \overline{\Lambda} = \left(J^\sharp + (2k' + \overline{\tau})\,\varepsilon_0\right)\omega_M\,.$$

Prolonging γ_n to the interval $[0, \overline{\Lambda}]$ by taking it constant (zero control is allowed) on $[\Lambda_n, \overline{\Lambda}]$, Ascoli-Arzela's theorem implies that a subsequence, that we still denote γ_n, converges uniformly on $[0, \overline{\Lambda}]$ to some γ, which is a solution of the average system (22) according to theorem 3 and remains in $\mathcal{D}'$ by the assumption of the theorem; hence, with Λ the smallest $\lambda \leq \overline{\Lambda}$ such that $\gamma(\lambda) = I^1$, which is also the limit of Λ_n, one must have $J(\gamma)(\Lambda) \geq J^\sharp$. Since, according to Lemma 6, $J_{\varepsilon_n}(\widetilde{L}_n(0), \gamma_n)(\Lambda_n)$ tends to $J(\gamma)(\Lambda)$, with a difference less than $c_1\varepsilon_n$ (c_1 a positive constant), one has, according to (31), $\varepsilon_n T_{\varepsilon_n} \geq J^\sharp + c_1\varepsilon_n$. This and (30) proves that any converging sequence $(\varepsilon_n T_{\varepsilon_n})$, with $\varepsilon_n \to 0$, tends to $J^\sharp$, i.e. that $\varepsilon T_\varepsilon$ tends to $J^\sharp$ when ε tends to 0. This also proves that $J^\sharp$ does not depend on the numbers $e'_M, e''_M, c''_m, c'_m, c'_M, c''_M$ chosen in (25) and hence that $J^\sharp = J^*$, the minimum value without restrictions on c, e_x, e_y of problem (24).

This ends the proof of Theorem 5. The following lemma was needed.

Lemma 7: Consider the balls $\mathcal{B}_\varepsilon$ of radius $M\varepsilon$ around the target orbit O_1, $\forall q \in \mathcal{B}$ let T_q^ε be the optimal time to reach the target orbit then the quantity

$$\Delta = \sup_{0<\varepsilon\leq\delta,\, q\in\mathcal{B}_\varepsilon} T^q \tag{32}$$

is bounded.

Proof: It is a direct consequence of the property that the linearized system is controllable along a Keplerian orbit. Indeed let $x_0(t)$ be the target orbit, which is obtained with the control null. Let x be a trajectory obtain with a control $u(t)$, $x(t) = x_0(t) + \delta x(t)$.

$$\dot{\delta x} = f_0(x_0(t) + \delta x) - f_0(x_0(t)) + \sum_{i=1}^{2} f_i(x_0(t) + \delta x)u_i, \tag{33}$$

The linearized system of state δx along the trajectory $x_0(t)$ is given by

$$\dot{\delta x} = A(t)\delta x + \sum_{i=1}^{2} f_i u_i, \tag{34}$$

with

$$A(t) = \frac{\partial f_0}{\partial x}(x_0(t)) = \begin{bmatrix} 0 & I \\ * & 0 \end{bmatrix} \text{ and } [f_1, f_2, f_3] = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Hence the end point mapping, $F : u \longrightarrow q(1)$ is continuously differentiable and full-rank at 0 (see Theorem 1 p. 57 in [12]). Then the Rank Theorem (see Theorem 52 p 464 in [12]) implies the existence of an inverse map of class $\mathcal{C}^1$, from $\mathcal{O}_{x_0(1)}$ to the space of admissible controls such that $G(x_0(1)) = 0$ and $F(G(y)) = y$ for each $y \in \mathcal{O}_{x_0(1)}$. ∎

V. Conclusion and remarks

This notion of average control system captures a lot of the limiting behavior of the slow variables, and has the advantage that it does not depend at all on the type of controls to be used, but it may fail to capture some phenomenon. For instance, it allows one to treat optimal control problems where the cost depends only on the slow variables, like minimum time: the trajectory that joints two points in minimum time for (11) is ε-close to a trajectory of the original system and hence gives almost minimum time for small ε (using Theorem 3). However, for an optimal control problem where the cost depends, for instance, on the control u, one may not expect to recover from the above results alone the results from [8], or these from [4], [5] on minimum energy.

We cannot yet comute explicitly the average system in general. This is the topic of ongoing research, in particular for orbital transfer with low thrust. Even without this computation, the average system allows to give, in some cases, an estimation of the asymptotics of minimum time low thrust transfer when this thrust goes to zero.

Acknoledgements. It is pleasure to thank Mario Sigalotti from INRIA Lorraine for many discussions on this topic while he was a post-doc at INRIA Sophia Antipolis, and Thierry Dargent, from Alcatel Alenia Space, for introducing us to low-thrust orbit transfer problems, that led to the present developments.

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