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DESIGN OF HOMOGENEOUS TIME-VARYING STABILIZING CONTROL LAWS FOR DRIFTLESS CONTROLLABLE SYSTEMS VIA OSCILLATORY APPROXIMATION OF LIE BRACKETS IN CLOSED LOOP*

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Abstract. A constructive method for time-varying stabilization of smooth driftless controllable systems is developed. It provides time-varying homogeneous feedback laws that are continuous and smooth away from the origin. These feedbacks make the closed-loop system globally exponentially asymptotically stable if the control system is homogeneous with respect to a family of dilations and, using local homogeneous approximation of control systems, locally exponentially asymptotically stable otherwise.

The method uses some known algorithms that construct oscillatory control inputs to approximate motion in the direction of iterated Lie brackets that we adapt to the closed-loop context.

Key words. nonlinear control, stabilization, time-varying stabilization, controllability, Lie brackets

AMS subject classifications. 93D15, 34C29, 93B52

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1. Introduction.

1.1. Related work and contribution.

Stabilization by continuous time-varying feedback laws of nonlinear systems that cannot be stabilized by time-invariant continuous feedback laws has been an ongoing subject of research in the past few years.

The fact that for many controllable systems no continuous stabilizing feedback exists was first pointed out by Sussmann [23]. A simple necessary condition was given by Brockett [1], since known as "Brockett's condition." It allows us to identify a wide class of controllable systems for which no continuous stabilizing feedback exists; these include most controllable driftless systems. More recently, Coron gave a stronger necessary condition [2].

A possible way of stabilizing systems for which these necessary conditions are violated is to use discontinuous (time-invariant) control laws. This has been explored in the literature, but the present work does not go in this direction at all.

The possibility of stabilizing nonlinear controllable systems via *continuous time-varying feedback control laws* was first noticed in the very detailed study of stabilization of one-dimensional systems by Sontag and Sussmann [21]. More recently, smooth stabilizing control laws for some nonholonomic mechanical systems were given by Samson [18]; this was the starting point of a systematic study of time-varying stabilization. Coron [3] proved that all controllable driftless systems may be stabilized by continuous (and even smooth) time-varying feedback and that "most" controllable systems (even with drift) can also be stabilized by continuous time-varying feedback [4]. Pomet deals with a less general class of controllable driftless systems [15].

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From here on, only driftless systems are considered in this paper. After the general existence result given in [3], studies on the subject have focused on methods to *construct* continuous time-varying stabilizing feedback laws and on obtaining feedback laws that provide sufficiently fast convergence.

As far as the constructiveness aspect is concerned, for simplicity let us divide the construction methods into two kinds. The first kind of method applies to rather large classes of controllable driftless systems, such as the work of Coron [3] (general controllable driftless systems; the paper is not oriented toward construction of the control, but a method can be extracted from the proofs), Pomet [15] (controllable driftless systems for which the control Lie algebra is generated by a specific set of vector fields), or of M'Closkey and Murray [12] (same conditions as in [15]). These studies all share the following feature: they use the solution of a linear PDE, or the expression of the flow of a vector field, to construct the control law. This solution, or this flow, has to be calculated beforehand, either analytically or numerically, and this introduces, especially when no analytical solution is available, a degree of complication which may not be necessary. The second kind of method found in the literature provides explicit expressions. Its drawback is that it applies only to specific subclasses of driftless systems, such as models of mobile robots or systems in the "chain form" or "power-form," like the work of Samson [18], Teel, Murray, and Walsh [27], and Sépulchre, Campion, and Vertz [20], among others.

Alternatively, a need to improve the speed of convergence came out of the slow convergence associated with the *smooth* control laws that were first proposed. This concern motivated several studies, starting with the work by M'Closkey and Murray [11], yielding continuous control laws which are not smooth, or even Lipschitz everywhere, but are homogeneous with respect to some dilation, and thus exponentially stabilizing, not in the standard sense but with respect to some homogeneous norm (this notion was introduced by Kawski [7]). See, for instance, further work by the authors of this paper [16, 14] or by M'Closkey and Murray [12], who have also proposed recently a procedure that transforms a given smooth stabilizing control law into a homogeneous one [13]. Except for this last reference, which requires that a smooth stabilizing control law has been designed beforehand, the construction of homogeneous exponentially stabilizing control laws in the literature is restricted to specific subclasses of driftless systems.

The design method described in the present paper has the advantage of being totally explicit, in the sense that it requires only ordinary differentiation and linear algebraic operations, while it applies to general controllable systems and provides exponential stability. This method gives homogeneous feedbacks, which ensure global stability if the control vector fields are homogeneous and local stability otherwise. The fact that it relates controllability with the construction of a stabilizing control law in a more direct way than previous designs also makes it conceptually appealing, all the more so as it may be viewed as converting the open-loop control techniques reported by Liu and Sussmann in [25] and Liu [9] into closed-loop techniques.

However, the generality of the method also has a price. When applied to particular systems for which explicit solutions have long been available, the present method often yields solutions which are significantly more complicated. This comes partly from the complexity of the approximation algorithm proposed in [25, 9], which we use. This is also a consequence of the modifications that we have made to adapt this algorithm to our feedback control objective.

1.2. Outline of the method. Nonlinear controllability results were first derived for driftless systems; see, for instance, the work by Lobry [10], where it is shown that such systems are controllable if and only if any direction in the state space can be obtained as a linear combination of iterated Lie brackets of the control vector fields, at least in real-analytic cases. It was also shown very early on by Haynes and Hermes [5] that, under this same condition, *any* curve in the state-space can be approached by open-loop solutions of the controlled system. (Note that this property is not shared by all controllable systems, but rather is specific to driftless systems.) In these studies the key element is that, in addition to the directions of motion corresponding to the control vector fields, motion along other directions corresponding to iterated Lie brackets is also possible by quickly switching motions along the original control vector fields. Take, for example, a system with two controls

$$\dot{x} \;=\; u_1\, b_1(x) \;+\; u_2\, b_2(x) \tag{1.1}$$

with state x in $\mathbb{R}^5$, and assume that at each point x the vectors

$$b_1(x)\,,\; b_2(x)\,,\; [b_1, b_2](x)\,,\; [b_1, [b_1, b_2]](x)\,,\; [b_2, [b_1, b_2]](x) \tag{1.2}$$

are linearly independent, and thus span $\mathbb{R}^5$. The idea in [5] is the following: first, it is clear that *any* (e.g., differentiable) parameterized curve $t \mapsto \gamma(t)$ is a possible solution of the "extended" system with five controls:

$$\begin{aligned} \dot{x} = {} & v_1\, b_1(x) \;+\; v_2\, b_2(x) \;+\; v_3\, [b_1, b_2](x) \\ & + v_4\, [b_1, [b_1, b_2]](x) \;+\; v_5\, [b_2, [b_1, b_2]](x) \end{aligned} \tag{1.3}$$

(simply decompose $\dot{\gamma}(t)$ on the basis (1.2) to obtain the controls). Then it is proved in [5] that there exists a sequence of (oscillatory) controls $u_1(\varepsilon, t, v_1, v_2, v_3, v_4, v_5)$ and $u_2(\varepsilon, t, v_1, v_2, v_3, v_4, v_5)$ such that the system (1.1) "converges to" the system (1.3) when $\varepsilon \to 0$ in the sense that the solutions of (1.1) with these controls u_k converge uniformly on finite time intervals to the solutions of (1.3). The proof in [5] does not give a process to build these sequences of approximating sequence of oscillatory control, and although the case of a simple bracket (approximating $[b_1, b_2]$ by switching between b_1 and b_2) is elementary and well known, the case above of brackets of order 3 is more complex. The more recent work by Liu [9] and Sussmann and Liu [25] gives an explicit construction of the approximating sequence. The process of building this sequence is amazingly intricate compared to the simplicity of the existence proof in [5]. Of course, the controls u_k are not defined for $\varepsilon = 0$, and both their frequency and their amplitude tend to infinity when ε goes to zero.

Being aware of these results, and faced with the problem of proving that any controllable driftless system may be stabilized by means of a periodic feedback, the most natural idea is probably the following, which we illustrate for (1.1) (5 states, 2 controls):

(a) Stabilize the extended system (1.3) by a control law $v_i(x)$. This is very easy, and $\dot{x}$ may even be assigned to be any desired function, for instance, $-x$.

(b) Use the approximation results and build the controls $u_k(\varepsilon, t, v_1(x), v_2(x), v_3(x), v_4(x), v_5(x))$, according to the process given in [25, 9] so that when ε tends to zero, the system (1.1) controlled with these controls "tends to" the extended system (1.3) controlled with the controls $v_i(x)$.

(c) Since the limit system is asymptotically stable (for instance, $\dot{x} = -x$), and asymptotic stability is somehow robust, the constructed control laws are, it is to be hoped, stabilizing for ε nonzero but small enough. For instance, one may take $\|x\|^2$ as a Lyapunov function for the limit system, its time-derivative along the limit system is $-2\|x\|^2$, and it is tempting to believe that its time-derivative along the original system controlled by $u_k(\varepsilon, t, v_1(x), v_2(x), v_3(x), v_4(x), v_5(x))$ is no larger than $-\|x\|^2$ for ε small enough.

Unfortunately, these arguments, which would have been somewhat simpler than those in [3], are not rigorous as they stand. The meaning of "tends to" in point (b) is very imprecise. In [5], and in [25, 9], only uniform convergence of the trajectories on finite-time intervals are considered. This is not adequate for asymptotic stabilization. The Lyapunov function-based argument in point (c) does not work because, in general, when ε tends to zero, the time derivative of a given function along the system (1.1) in feedback with the controls u_k from point (b) does not tend to the time-derivative of this function along the "limit" system (1.3). In addition, the fact that feedback controls are considered instead of open-loop controls complicates the proofs because the controls depend on the state and therefore may have a very high derivative with respect to time not only through the high frequencies and amplitudes built into the approximation process but also through their dependence on the state, whose speed is proportional to these high amplitudes.

However, we show in the present paper that the above sketch is basically correct, provided that homogeneous controls associated with a homogeneous Lyapunov function are used and that the construction of the approximating sequence is modified to take into account the closed-loop nature of the controls. An argument of the type of point (c) is possible based on a notion of approximation that is not in terms of uniform convergence of trajectories, but in terms of the differential operator defined by derivation along the system.

This paper is organized as follows. After a brief recall of technical material in section 2, we state in section 3 the control objective, make homogeneity assumptions, and explain how they will yield local results for general controllable systems. The design method is developed in section 4 through four steps: choice of the "useful" Lie brackets, construction of the stabilizing controls for the extended system (system (1.3) in the above example), construction of the "state dependent" amplitudes for the feedback law, and construction of the oscillatory controls by the method exposed in [9]; the material from these steps is then gathered to give the control law, and the stabilization result is stated. We present in this section all that is needed for the construction of the control law, but the proofs of some properties needed at each steps, and of the theorem, are given separately in section 7. Section 6 is devoted to a convergence result needed in the proof of the stability theorem; it is a translation in terms of differential operators (instead of trajectories) of the averaging results presented in [25, 9, 26] and in [8]. An illustrative example is given in section 5.

2. Background on homogeneous vector fields. For any $\lambda > 0$, the "dilation operator" δ_λ associated with a "weight vector" $r = (r_1, \ldots, r_n)$ $(r_i > 0)$ is defined on $\mathbb{R}^n$ by

$$\delta_\lambda(x_1, \ldots, x_n) = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n).$$

A function $f \in C^o(\mathbb{R}^n; \mathbb{R})$ is said to be homogeneous of degree τ with respect to the family of dilations (δ_λ) if

$$\forall \lambda > 0, \quad f(\delta_\lambda(x)) = \lambda^\tau f(x)\,.$$

A *homogeneous norm* is any proper continuous positive function that is homogeneous of degree 1.

A continuous vector field X on $\mathbb{R}^n$ is said to be homogeneous of degree σ with respect to the family of dilations (δ_λ) if one of the following equivalent properties is satisfied:

(1) For any $i = 1, \dots, n$, its ith component, i.e., the function $x \mapsto X_i(x)$, is homogeneous of degree $r_i + \sigma$.
(2) For any function h homogeneous of degree $\tau > 0$ with respect to the same dilation, the function $L_X h$ (its Lie derivative along X) is homogeneous of degree $\sigma + \tau$.
(3) For all positive constant λ, the vector field $((\delta_\lambda)_* X)$, conjugate of X by the diffeomorphism δ_λ —away from the origin— satisfies $((\delta_\lambda)_* X)(x) = \lambda^{-\sigma} X(x)$ for $x \neq 0$.

The previous definitions of homogeneity can be extended to time-varying functions and vector fields by considering an "extended dilation":

$$\delta_\lambda(x_1, \dots, x_n, t) = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n, t).$$

Finally, let $f \in C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$, with $f(x, .)$ T-periodic, defining a homogeneous vector field of degree zero with respect to a family of dilations (δ_λ). Then, the two following properties are equivalent (see [7] for the autonomous case):

(i) the origin $x = 0$ of the system $\dot{x} = f(x, t)$ is locally asymptotically stable.
(ii) $x = 0$ is globally *ρ-exponentially asymptotically stable*, i.e., for any homogeneous norm ρ, there exist $K, \gamma > 0$ such that, for any solution $x(.)$ of the system,

$$\rho(x(t)) \leq K \rho(x(0)) e^{-\gamma t}.$$

In what follows, when using the expression *exponentially asymptotically stable*, we will refer to the ρ-exponential asymptotic stability defined above.

3. Problem statement. Consider a smooth driftless controllable system

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x). \tag{3.1}$$

In general, there does not exist a dilation with respect to which the control vector fields are homogeneous. However, controllability implies that after some adequate change of coordinates, there exist a dilation and a controllable *homogeneous approximation* [6, 7]—with respect to this dilation—of the system (3.1) around the origin. Different methods exist to find such a change of coordinates and dilation. For instance, a constructive method (i.e., requiring only algebraic computations and derivations) is given in [22]. Using this method, one obtains a driftless control system with control vector fields homogeneous of degree -1. Moreover, any homogeneous feedback law that asymptotically stabilizes this system also locally asymptotically stabilizes the original system.

The present work constructs a homogeneous feedback that ensures global exponential stabilization for homogeneous systems. Applied to the homogeneous approximation of a general system (3.1), it provides *local* exponential stabilization of (3.1).

Throughout this paper, we always consider a system

$$\dot{x} = \sum_{i=1}^{m} u_i b_i(x), \tag{3.2}$$

where the b_i's are smooth vector fields and the system of coordinates is such that there exist some integers $(r_1, \ldots, r_n)$ such that

(1) each vector field b_i is homogeneous of degree -1 with respect to the family of dilations δ_λ with weights $(r_1, \ldots, r_n)$;

(2) the rank at the origin of the Lie algebra generated by the b_i's is n:

$$\text{Rank}(\text{Lie}\{b_1, \ldots, b_m\}(0)\,) \;=\; n. \tag{3.3}$$

The integer valued weights $r_1, \ldots, r_n$ are now fixed, and we denote

$$P \;=\; \text{Max}\,\{r_i;\ i = 1, \ldots, n\}. \tag{3.4}$$

Our objective is to design feedback laws $u = (u_1, \ldots, u_m) \in C^0(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ such that the origin $x = 0$ of the closed-loop system (3.2) is exponentially asymptotically stable.

Remark 3.1. We require only full rank control Lie algebra at the origin, but controllability follows, because homogeneity allows us to deduce the same rank condition everywhere.

Remark 3.2. We assume that the degrees are all equal to -1. These are the degrees given by the construction of a homogeneous approximation in [22]. If a system is naturally homogeneous, but the degrees are not all equal (if they are equal, a simple scaling makes them all equal to -1), it might be better to use this natural homogeneity than to construct a different homogeneous approximation that will have all the degrees equal to -1. The present method can be adapted to the case when the degrees of homogeneity are not all equal; this requires only a modification of the first step (see Remark 4.3).

4. Controller design. The control design consists of four steps described below.

Step 1 *(selection of Lie brackets).*

In this step, we select some vector fields $\tilde{b}_j$ $(j = 1, \ldots, N)$, obtained as Lie brackets of the control vector fields $b_1, \ldots, b_m$. The $\tilde{b}_j$ are chosen recursively as follows. For any $p = 1, \ldots, P$ (with P defined by (3.4)),

(1) compute all brackets of length p made from the control vector fields b_i $(i = 1, \ldots, m)$;

(2) select among the vector fields so obtained a maximal number of vector fields independent[1] over $\mathbb{R}$. These vector fields are the $\tilde{b}_j$ $(m_{p-1} + 1 \leq j \leq m_p)$. (We set $m_0 = 0$ so that all the integers m_p $(p = 0, \ldots, P)$ are defined, with $N = m_P$.)

It follows from this construction that with each vector field $\tilde{b}_j$ we can associate a Lie bracket of some b_i's, i.e.,

$$\tilde{b}_j \;=\; \mathcal{C}_j(b_{\tau_j^1}, \ldots, b_{\tau_j^{\ell(j)}})\,, \tag{4.1}$$

with

- $\mathcal{C}_j$ a formal bracket and $b_{\tau_j^1}, \ldots, b_{\tau_j^{\ell(j)}}$ the elements that are bracketed (listed in the order they appear in the bracket);
- $\ell(j)$ the number of vector fields that are bracketed in (4.1), i.e.,

$$\ell(j) = p \quad \Leftrightarrow \quad m_{p-1} + 1 \leq j \leq m_p.$$

[1] Recall that some vector fields $X_1, \cdots, X_r$ are said to be *linearly independent over* $\mathbb{R}$ if and only if for any $(\lambda_1, \ldots, \lambda_r)$ in $\mathbb{R}^r$, the vector field $\lambda_1 X_1 + \cdots + \lambda_r X_r$ is identically zero on $\mathbb{R}^n$ only if $\lambda_1 = \cdots = \lambda_r = 0$.

For instance, if we choose a vector field $\tilde{b}_6 = [[b_2, b_1], [b_1, [b_1, b_2]]]$, then we encode this as (4.1) with $\ell(6) = 5$, $\tau_6^2 = \tau_6^3 = \tau_6^4 = 1$, $\tau_6^1 = \tau_6^5 = 2$, and the symbol $\mathcal{C}_6$ defined by $\mathcal{C}_6(z_1, z_2, z_3, z_4, z_5) = [[z_1, z_2], [z_3, [z_4, z_5]]]$. This notation is sloppy but avoids using formal Lie brackets and the evaluation operator (see [24]) from a free Lie algebra to vector fields, which would make the exposition uselessly heavy. Of course, the decomposition (4.1) is not unique in general. From now on, we consider that one decomposition has been chosen and that the $\mathcal{C}_j$'s and τ_j^k's have been defined accordingly.

Remark 4.1. (1) In Step 1 above, we do not need to compute all brackets of length p. More precisely, let $\mathcal{F}$ denote the free Lie algebra generated by some indeterminates $s_1, \dots, s_m$. Then, one can select a basis $\mathcal{B}$ of this Lie algebra (for instance a P. Hall basis, as used by Sussmann and Liu [25, 26] and Liu [9]). If $\mathcal{B}_p$ denotes the elements of $\mathcal{B}$ of order p, then it is clearly sufficient to consider Lie brackets of the b_i obtained by *evaluating* (in the sense of [24]) the elements of $\mathcal{B}_p$ at $s_i = b_i$ $(i = 1, \dots, m)$. One usually takes this into account when checking controllability.

(2) Since the vector fields b_i $(i = 1, \dots, m)$ are homogeneous of degree -1, each bracket of length p of these vector fields is homogeneous of degree $-p$. Moreover, the weights of the dilation being integers, any smooth vector field homogeneous of integer degree is, in fact, polynomial. Using a (finite) basis of the polynomials homogeneous of degree k $(k \in \{0, \dots, P-1\})$, selecting Lie brackets of a given length consists only of computing a basis of a finite dimensional vector space.

(3) We do not need to consider brackets of order larger than P because they are identically zero; indeed, all components of these vector fields are homogeneous of negative degree and, therefore, they would tend to infinity at the origin if they were not identically zero.

Example. Let us illustrate this step on the following academic example:

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= x_3^2(u_1 + u_2), \\ \dot{x}_3 &= u_3, \end{aligned}$$

which is of the form (3.2) with $m = 3$ and

$$b_1 = \frac{\partial}{\partial x_1} + x_3^2 \frac{\partial}{\partial x_2}, \quad b_2 = x_3^2 \frac{\partial}{\partial x_2}, \quad b_3 = \frac{\partial}{\partial x_3}.$$

The control vector fields are homogeneous of degree -1 with respect to the dilation with weights $r_1 = 1$, $r_2 = 3$, and $r_3 = 1$.

For the brackets of length 1, i.e., the control vector fields, b_1 and b_3 are independent at the origin while b_2 is zero at the origin but independent from b_1 and b_3 away from $x_3 = 0$. Hence $m_1 = 3$, and one can take $\tilde{b}_1 = b_1 = \mathcal{C}_1(b_1)$, $\tilde{b}_2 = b_2 = \mathcal{C}_2(b_2)$, and $\tilde{b}_3 = b_3 = \mathcal{C}_3(b_3)$.

At length 2 all the brackets vanish at the origin, but they are not identically zero: $[b_2, b_3] = -2x_3 \frac{\partial}{\partial x_2}$, and $[b_3, b_1] = -[b_2, b_3]$. Since $[b_1, b_2] = 0$, we have $m_2 = 4$. We define, for instance, $\tilde{b}_4 = [b_2, b_3] = \mathcal{C}_4(b_2, b_3)$.

Finally, since $[b_3, [b_2, b_3]] = -2\frac{\partial}{\partial x_2}$, $m_3 = 5$ with, for instance, $\tilde{b}_5 = [b_3, [b_2, b_3]] = \mathcal{C}_5(b_3, b_2, b_3)$. Note that here, due to the origin being a singular point for the distributions spanned by the control vector fields and by the brackets of order at most 2, N is strictly larger than n.

With this general construction, we have the following proposition.

PROPOSITION 4.2. *For any family $(\tilde{b}_j)_{j=1,\dots,N}$ defined as above, we have the following:*

(a) *Let $j_1,\dots,j_n$ be such that* $\mathrm{Span}\{\tilde{b}_{j_1}(0),\dots,\tilde{b}_{j_n}(0)\} = \mathbb{R}^n$. *Then*

$$\forall x \in \mathbb{R}^n, \quad \mathrm{Span}\{\tilde{b}_{j_1}(x),\dots,\tilde{b}_{j_n}(x)\} = \mathbb{R}^n .$$

(b) *Any vector field b that can be written as a Lie bracket of order p of some b_i's is a linear combination of the $\tilde{b}_j$'s with $\ell(j) = p$, i.e.,*

$$b = \sum_{j=m_{p-1}+1}^{m_p} \lambda_j \tilde{b}_j = \sum_{\ell(j)=p} \lambda_j \tilde{b}_j$$

for some real numbers $\lambda_j \in \mathbb{R}$.

(c) *The vector fields $\{\tilde{b}_j\}_{j=1,\dots,N}$ are linearly independent over $\mathbb{R}$.*

(The proof is in section 7.1.)

Remark 4.3. If the degrees of the vector fields b_i are not all equal, the above construction has to be modified. More precisely, in the recursive construction of the family $(\tilde{b}_j)_{j=1,\dots,N}$, we have to consider an induction on the degree of homogeneity instead of an induction on the length of the Lie brackets. (Note that this is just a generalization of the above construction, since for vector fields of the same degree -1 the set of Lie brackets of length p is the same as the set of Lie brackets of degree $-p$.) This means that at each step, we have to compute the set of Lie brackets of a certain degree and select from among them a finite number of vector fields that form a basis of this set.

Step 2 *(stabilization of the extended system).*

Let a be a smooth vector field, homogeneous of degree zero with respect to the family of dilations (δ_λ), and such that the origin $x = 0$ of the system $\dot{x} = a(x)$ is asymptotically stable. One may take, for instance, $a(x) = -x$. In view of Proposition 4.2(a), the $n \times n$ matrix whose columns are $\tilde{b}_{j_1}(x),\dots,\tilde{b}_{j_n}(x)$ is invertible for all x. Define the functions $\tilde{u}_j$ ($j = 1,\dots,N$) by

$$\text{(4.2)} \qquad \bullet \begin{pmatrix} \tilde{u}_{j_1}(x) \\ \vdots \\ \tilde{u}_{j_n}(x) \end{pmatrix} = \Big(\tilde{b}_{j_1}(x),\dots,\tilde{b}_{j_n}(x)\Big)^{-1} a(x),$$

$$\bullet\ \tilde{u}_j = 0 \quad \forall j \notin \{j_1,\dots,j_n\}.$$

These functions are obviously such that

$$\text{(4.3)} \qquad a = \sum_{j=1}^{N} \tilde{u}_j \, \tilde{b}_j,$$

and furthermore, we may state this proposition.

PROPOSITION 4.4. *For any $j = 1,\dots,N$, the above-constructed function $\tilde{u}_j$ is in $C^\infty(\mathbb{R}^n - \{0\};\mathbb{R}) \cap C^0(\mathbb{R}^n;\mathbb{R})$ and is homogeneous of degree $\ell(j)$.*

Proof. Continuity and smoothness away from the origin are inherited from the vector fields $\tilde{b}_j$ and the vector field a. Each $\tilde{u}_{j_k}$ is homogeneous of degree $\ell(j_k)$ because the lth component of the vector field a is homogeneous of degree r_l and the element (k,l) of the matrix $(\tilde{b}_{j_1}(x),\dots,\tilde{b}_{j_n}(x))^{-1}$ is homogeneous of degree $\ell(j_k) - r_l$. This last statement is true because the element (k,l) of the matrix $(\tilde{b}_{j_1}(x),\ldots,\tilde{b}_{j_n}(x))$ is homogeneous of degree $r_l-\ell(j_k)$ for the vector field; $\tilde{b}_{j_k}$ is an iterated Lie bracket of $\ell(j_k)$ homogeneous vector fields of degree -1 and hence is homogeneous of degree $-\ell(j_k)$. $\square$

Step 3 *(construction of the state-dependent amplitudes).*

This step consists of finding some functions $v_j^k \in C^\infty(\mathbb{R}^n-\{0\};\mathbb{R})\cap C^0(\mathbb{R}^n;\mathbb{R})$ $(j=1,\ldots,N,\ k=1,\ldots\ell(j))$ homogeneous of degree one and such that

$$\sum_{j=1}^{N}\tilde{u}_j\,\mathcal{C}_j(b_{\tau_j^1},\ldots,b_{\tau_j^{\ell(j)}})=\sum_{j=1}^{N}\mathcal{C}_j(b_{\tau_j^1}v_j^1,\ldots,b_{\tau_j^{\ell(j)}}v_j^{\ell(j)}). \tag{4.4}$$

Recall that the $\mathcal{C}_j$'s, defined in Step 1, are the brackets associated with the $\tilde{b}_j$'s, i.e.,

$$\tilde{b}_j=\mathcal{C}_j(b_{\tau_j^1},\ldots,b_{\tau_j^{\ell(j)}})\,. \tag{4.5}$$

The construction of the functions v_j^k is based on the following lemma.

LEMMA 4.5. *Let* $\mathcal{C}(b_{i_1},\ldots,b_{i_p})$ $(i_k\in\{1,\ldots,m\})$ *be any Lie bracket of some vector fields* b_{i_k} $(i_k\in\{1,\ldots,m\})$*, and* $v_k\in C^\infty(\mathbb{R}^n-\{0\};\mathbb{R})\cap C^0(\mathbb{R}^n;\mathbb{R})$ $(k=1,\ldots,p)$ *some functions homogeneous of degree* 1*. Then,*

(i) $\mathcal{C}(b_{i_1}v_1,\ldots,b_{i_p}v_p)=v_1\ldots v_p\,\mathcal{C}(b_{i_1},\ldots,b_{i_p})-\sum_{j=1}^{m_{p-1}}h_j\ \tilde{b}_j$;

(ii) *for any* $j=1,\ldots,m_{p-1}$, $h_j\in C^\infty(\mathbb{R}^n-\{0\};\mathbb{R})\cap C^0(\mathbb{R}^n;\mathbb{R})$ *is homogeneous of degree* $\ell(j)$.

The proof of this lemma, left to the reader, follows from Proposition 4.2(b) by a direct induction on the length p of the bracket $\mathcal{C}(b_{i_1}v_1,\ldots,b_{i_p}v_p)$. It is a generalization of the fact that for two functions v_1 and v_2, and vector fields b_{i_1} and b_{i_2}, $[v_1b_{i_1},v_2b_{i_2}]=v_1v_2[b_{i_1},b_{i_2}]-v_2(L_{b_{i_2}}v_1)b_{i_1}+v_1(L_{b_{i_1}}v_2)b_{i_2}$.

Note that the functions h_j in Lemma 4.5 can be explicitly computed by expressing brackets of order not larger than $p-1$ as linear combinations of $\tilde{b}_1,\ldots,\tilde{b}_{m_{p-1}}$.

Based on Lemma 4.5, the functions v_j^k can be constructed recursively as follows.

Step $p=P$: For any $j\in\{m_{P-1}+1,\ldots,m_P\}$, we define

$$v_j^P=\frac{\tilde{u}_j}{\rho^{P-1}}\ \text{ and } v_j^k=\rho\ (k=1,\ldots,P-1)\,, \tag{4.6}$$

with ρ any homogeneous norm in $C^\infty(\mathbb{R}^n-\{0\};\mathbb{R})\cap C^0(\mathbb{R}^n;\mathbb{R})$ (for instance, one may take $\rho(x)=(\sum|x_i|^{\frac{q}{r_i}})^{\frac{1}{q}}$ with $q=2\prod_{i=1}^n r_i$).

In view of (4.5), (4.6), and Lemma 4.5, we have

$$\sum_{j=m_{P-1}+1}^{m_P}\mathcal{C}_j(b_{\tau_j^1}v_j^1,\ldots,b_{\tau_j^P}v_j^P)\;=\;\sum_{j=m_{P-1}+1}^{m_P}\tilde{u}_j\tilde{b}_j-\sum_{j=1}^{m_{P-1}}h_j^P\tilde{b}_j \tag{4.7}$$

with h_j^P $(j=1,\ldots,m_{P-1})$ obtained by expanding the brackets in the left-hand side of (4.7) with respect to the variables v_j^k and their derivatives.

Step $1\le p<P$: Assume that the functions v_j^k $(j=m_p+1,\ldots,m_P,\ k=1,\ldots,\ell(j))$ and h_j^k $(j=m_p+1,\ldots,m_P,\ k=p+1,\ldots,P)$ have been computed in Steps P to $p+1$ and satisfy the induction assumption

$$\sum_{j=m_p+1}^{N}\mathcal{C}_j(b_{\tau_j^1}v_j^1,\ldots,b_{\tau_j^{\ell(j)}}v_j^{\ell(j)})\;=\;\sum_{j=m_p+1}^{N}\tilde{u}_j\tilde{b}_j-\sum_{j=1}^{m_p}h_j^{p+1}\tilde{b}_j\,. \tag{4.8}$$

We define, for any $j \in \{m_{p-1}+1, \ldots, m_p\}$,

$$v_j^p = \frac{1}{\rho^{p-1}}(\tilde{u}_j + h_j^{p+1}) \text{ and } v_j^k = \rho \ (k = 1, \ldots, p-1). \tag{4.9}$$

In view of (4.5), (4.9), and Lemma 4.5, we have

$$\sum_{j=m_{p-1}+1}^{m_p} \mathcal{C}_j(b_{\tau_j^1} v_j^1, \ldots, b_{\tau_j^p} v_j^p) = \sum_{j=m_{p-1}+1}^{m_p} (\tilde{u}_j + h_j^{p+1})\tilde{b}_j + \sum_{j=1}^{m_{p-1}} (h_j^{p+1} - h_j^p)\tilde{b}_j \tag{4.10}$$

for an adequate choice of the h_j^p $(j = m_{p-1}+1, \ldots, m_p)$ obtained again by expanding the brackets in the left-hand side of (4.7) with respect to the variables v_j^k and their derivatives. In view of (4.8) and (4.10), we have

$$\sum_{j=m_{p-1}+1}^{N} \mathcal{C}_j(b_{\tau_j^1} v_j^1, \ldots, b_{\tau_j^{\ell(j)}} v_j^{\ell(j)}) = \sum_{j=m_{p-1}+1}^{N} \tilde{u}_j \tilde{b}_j - \sum_{j=1}^{m_{p-1}} h_j^p \tilde{b}_j \tag{4.11}$$

so that the induction assumption (4.8) on Steps P to $p+1$ is also true for Steps P to p.

The computation of the functions v_j^k and h_j^k ends after Step $p = 1$ has been performed. Let us remark that in the last step ($p = 1$), there is no function h_j^p to compute. With this construction, we have the next proposition.

Proposition 4.6. *Consider the functions v_j^k defined above. Then*

(a) *each v_j^k $(j = 1, \ldots, N,\ k = 1, \ldots, \ell(j))$ belongs to $C^\infty(\mathbb{R}^n - \{0\}; \mathbb{R}) \cap C^0(\mathbb{R}^n; \mathbb{R})$ and is homogeneous of degree 1;*

(b) (4.4) *is satisfied.*

Proof. Point (b) is a direct consequence of (4.11) with $p = 1$. Point (a) is an easy consequence of Proposition 4.4, equations (4.6) and (4.9), and Lemma 4.5. □

Step 4 (oscillatory approximation of Lie brackets).

The last step of our construction relies on the work of Liu [9] and Sussmann and Liu [25, 26]. More precisely, consider a control system

$$\dot{x} = \sum_{\alpha=1}^{A} u_\alpha X_\alpha(x) \tag{4.12}$$

with $X_1, \ldots, X_A$ some smooth vector fields on a smooth n-dimensional manifold, a "Lie bracket extended" system

$$\dot{x} = \sum_{\beta=1}^{B} w_\beta X_\beta(x) \qquad (B \geq A), \tag{4.13}$$

where the A first vector fields are the same as in (4.12), and the other vector fields are Lie brackets of $X_1, \ldots, X_A$. In [9], an algorithm is given that builds, for any set of integrable functions of time w_β $(\beta = 1, \ldots, B)$, some "highly oscillatory" functions of time u_α^ε such that the trajectories of (4.12), with $u_\alpha = u_\alpha^\varepsilon$, approximate those of (4.13).

We do not describe this algorithm here, though we use the notation

$$u_\alpha^\varepsilon = \mathcal{F}(\alpha, \varepsilon, (w_\beta)_{1 \leq \beta \leq B}), \tag{4.14}$$

where $\mathcal{F}$ is a function described algorithmically in [9]. It depends only on which Lie brackets have to be performed to obtain the vector fields $X_{A+1}, \ldots, X_B$ from the vector fields $X_1, \ldots, X_A$. It is of the form

$$(4.15)\quad u_\alpha^\varepsilon(t) \;=\; \eta_{\alpha,0}(t) + \varepsilon^{-\frac{1}{2}} \sum_{\omega \in \Omega(2,\alpha)} \eta_{\omega,\alpha}(t) e^{i\omega t/\varepsilon} + \sum_{n=3}^{N} \varepsilon^{\frac{n-1}{n}} \sum_{\omega \in \Omega(n,\alpha)} \eta_\omega(t) e^{i\omega t/\varepsilon},$$

with N the length of the higher order bracket X_β in (4.13), $\eta_{\alpha,0}, \eta_{\omega,\alpha}$, and η_ω some functions, and $\Omega_{2,\alpha}$, $\Omega_{n,\alpha}$ some finite subsets of $\mathbb{R}$, that are all built precisely in [9]. In particular, the construction of the "approximating inputs" u_α^ε given in [9] implies the following.

THEOREM 4.7 (see [9]). *For any T $(0 < T < +\infty)$ and any family w_β $(\beta = 1, \ldots, B)$ of integrable functions on $[0,T]$, the functions u_α^ε $(\alpha = 1, \ldots, A)$ given by* (4.14)*, where $\mathcal{F}$ symbolizes the algorithm described in* [9]*, are integrable and are such that the trajectories of* (4.12)–(4.15) *converge to the trajectories of* (4.13) *in the following sense: For any $p \in \mathbb{R}^n$, if the system* (4.13) *with $x(0) = p$ has a unique solution x^∞ defined on $[0,T]$ and if x_ε is a maximal solution of system* (4.12)–(4.15) *with $x(0) = p$, then x_ε is defined on $[0,T]$ for ε small enough and converges uniformly to x^∞ on $[0,T]$ as $\varepsilon \to 0$.*

Remark 4.8. (1) The functions u_α^ε in (4.15) are real-valued because each $\Omega_{n,\alpha}$ $(n = 2, \ldots, N)$ is symmetric $(\omega \in \Omega_{n,\alpha} \Rightarrow -\omega \in \Omega_{n,\alpha})$, $\eta_{-\omega} = \overline{\eta_\omega}$, and $\eta_{-\omega,\alpha} = \overline{\eta_{\omega,\alpha}}$.

(2) If the functions w_β in (4.13) are constant, the functions $\eta_{\alpha,0}, \eta_{\omega,\alpha}$, and η_ω are also constant.

Consider now the following two systems:

$$(4.16)\qquad \dot{x} = \sum_{j=1}^{N} \sum_{s=1}^{\ell(j)} u_{j,s} b_{\tau_j^s} v_j^s\,,$$

$$(4.17)\qquad \dot{x} = \sum_{j=1}^{N} \mathcal{C}_j(b_{\tau_j^1} v_j^1, \ldots, b_{\tau_j^{\ell(j)}} v_j^{\ell(j)})\,.$$

Systems (4.16) and (4.17) are of the same form as (4.12) and (4.13), respectively, with the vector fields X_α being the $b_{\tau_j^s} v_j^s$'s (with α a double index (j,s)), the vector fields X_β being these plus the brackets in (4.17), i.e., $\mathcal{C}_j(X_{j,1}, \ldots, X_{j,\ell(j)})$, $1 \le j \le N$, and each w_β in (4.13) being constant: 0 in front of the X_β's that are also X_α's and 1 in front of the added brackets. Note that since each original vector field from (3.2) appears many times in the brackets selected in Step 1, we consider here as independent control vector fields in (4.12) some vector fields that are in fact "multiples" of each other: for instance if the vector field b_1 appears more than one time, we have $\tau_j^s = \tau_{j'}^{s\prime} = 1$ for some $(j,x) \neq (j', s')$, and $v_j^s b_1$ and $v_{j'}^{s\prime} b_1$ are distinct control vector fields X_α in (4.12).

Following Liu's algorithm, we construct some functions

$$u_{j,s}^\varepsilon \;=\; \mathcal{F}\left((j,s), \varepsilon, (0, \ldots, 0, 1, \ldots, 1)\right),$$

where $\mathcal{F}$ is the notation introduced in (4.14), such that the trajectories of (4.16)–(4.18) (which exist on any time interval because the system is degree zero homogeneous) converge uniformly on any time interval $[0,T]$ to those of (4.17), as ε tends to zero.

Recall (see (4.15)) that they are of the form

$$u^{\varepsilon}_{j,s}(t) = \eta_{j,s,0} + \varepsilon^{-\frac{1}{2}} \sum_{\omega\in\Omega(2,j,s)} \eta_{\omega,j,s} e^{i\omega t/\varepsilon} + \sum_{n=3}^{P} \varepsilon^{\frac{n-1}{n}} \sum_{\omega\in\Omega(n,j,s)} \eta_{\omega} e^{i\omega t/\varepsilon}\,. \tag{4.18}$$

Note that the functions η in (4.18) are constant in view of Remark 4.8 above. We rewrite system (4.16)–(4.18) as

$$\dot{x} = \sum_{i=1}^{m} \left(\sum_{(j,s):\tau^s_j=i} u^{\varepsilon}_{j,s}(t) v^s_j(x) \right) b_i(x)\,. \tag{4.19}$$

Our final control laws are defined by

$$u^{\varepsilon}_i(x,t) = \sum_{(j,s):\tau^s_j=i} u^{\varepsilon}_{j,s}(t) v^s_j(x)\,. \tag{4.20}$$

As stated in the following theorem, they ensure asymptotic stability of system (3.2) for "sufficiently large" frequencies.

THEOREM 4.9. *Let the controls u^{ε}_i be these described above. Then, the vector field in the right-hand side of the time-varying closed-loop system*

$$\dot{x} = \sum_{i=1}^{m} u^{\varepsilon}_i(x,t)\, b_i(x) \tag{4.21}$$

is homogeneous of degree zero and, for $\varepsilon > 0$ sufficiently small, the origin is exponentially uniformly asymptotically stable. (See the proof in section 7.3.)

Remark 4.10. Our construction a priori implies uniform convergence of the trajectories of (4.21) to those of (4.17), the origin of which is asymptotically stable from (4.3) to (4.5). However, this is not enough to infer asymptotic stability of (4.21). In the proofs, and in section 6, we introduce a stronger kind of convergence (DO-convergence), sufficient to infer asymptotic stability of (4.21). However, we quote uniform convergence here (instead of the DO-convergence, which we really need) because we base our construction on [9]. It makes the present construction clearer. (To construct the controls, one needs only to follow the algorithm in [9]; the kind of convergence does not matter.) Also, using the convergence result from [9] (Theorem 4.7) provides a shortcut in the proof on DO-convergence. This may make the paper less self-contained, but it avoids reproducing some difficult calculations made in [9].

5. An illustrative example. We now illustrate the control design method shown in section 4. Let us consider the following system in $\mathbb{R}^4$:

$$\dot{x} = b_1 u_1 + b_2 u_2\,, \tag{5.1}$$

with $b_1 = \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3}$ and $b_2 = \frac{\partial}{\partial x_4}$, which can be used to model the kinematic equations of a car-like mobile robot. One easily verifies that the vector fields b_1 and b_2 are homogeneous of degree -1 with respect to the family of dilations of weight $r = (1,3,2,1)$, and that this system is controllable. We follow this example with the four steps of our control design procedure.

Step 1. Since $[b_1,b_2] = -\frac{\partial}{\partial x_3}$, $[b_1,[b_1,b_2]] = \frac{\partial}{\partial x_2}$, and $[b_2,[b_2,b_1]] = 0$, the family $(\tilde{b}_j)$ is directly given by

$$\begin{aligned} (\tilde{b}_j) = (\tilde{b}_1,\tilde{b}_2,\tilde{b}_3,\tilde{b}_4) &= (b_1, b_2, [b_1,b_2], [b_1,[b_1,b_2]]) \\ &= (\mathcal{C}_1(b_{\tau^1_1}), \mathcal{C}_2(b_{\tau^1_2}), \mathcal{C}_3(b_{\tau^1_3}, b_{\tau^2_3}), \mathcal{C}_4(b_{\tau^1_4}, b_{\tau^2_4}, b_{\tau^3_4})). \end{aligned} \tag{5.2}$$

This implies that $\tau_1^1 = 1, \tau_2^1 = 2, \tau_3^1 = 1, \tau_3^2 = 2, \tau_4^1 = \tau_4^2 = 1, \tau_4^3 = 2$, and that $m_1 = 2$, $m_2 = 3$, and $m_3 = N = 4$.

Step 2. Let us, for instance, define the vector field a by $a(x) = -x$. (The origin $x = 0$ of $\dot{x} = a(x)$ is obviously asymptotically stable.) Then the integers j_k are simply defined by $j_k = k$ $(k = 1, \dots, 4)$. By a direct computation, one obtains the following expression for the functions $\tilde{u}_j$:

$$\begin{aligned} (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4)^T(x) &= (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4)^{-1}(x)\, a(x) \\ &= (-x_1, -x_4, -x_1 x_4 + x_3, x_1 x_3 - x_2)^T . \end{aligned} \tag{5.3}$$

Step 3. From Step 1, the brackets $\mathcal{C}_k$ are defined by

$$\mathcal{C}_1(x_1) = x_1, \mathcal{C}_2(x_2) = x_2, \mathcal{C}_3(x_1, x_2) = [x_1, x_2], \mathcal{C}_4(x_1, x_1, x_2) = [x_1, [x_1, x_2]] .$$

We now follow the procedure exposed in section 4.

Step $p = P = 3$: The functions v_4^1, v_4^2, and v_4^3 are given, in view of (4.6), by

$$v_4^1 = v_4^2 = \rho, \quad v_4^3 = \frac{\tilde{u}_4}{\rho^2} , \tag{5.4}$$

with $\rho \in C^\infty(\mathbb{R}^4 - \{0\}; \mathbb{R}) \cap C^0(\mathbb{R}^4; \mathbb{R})$ a homogeneous norm. (For instance, one may take $\rho(x) = (x_1^{12} + x_2^4 + x_3^6 + x_4^{12})^{\frac{1}{12}}$.)

We also compute the functions h_j^P involved in (4.7). A tedious but simple calculation gives

$$\begin{aligned} h_1^3 &= v_4^2 v_4^3 L_{[b_1, b_2]} v_4^1 + v_4^2 L_{b_1} v_4^3 L_{b_2} v_4^1 + v_4^1 L_{b_1}(v_4^3 L_{b_2} v_4^2) - v_4^3 L_{b_1} v_4^1 L_{b_2} v_4^2 , \\ h_2^3 &= -v_4^1 L_{b_1}(v_4^2 L_{b_1} v_4^3) , \\ h_3^3 &= -v_4^1 L_{b_1} v_4^2 v_4^3 - v_4^1 v_4^2 L_{b_1} v_4^3 . \end{aligned} \tag{5.5}$$

Step $p = 2$: The functions v_3^1 and v_3^2 are given, in view of (4.9), by

$$v_3^1 = \rho, \quad v_3^2 = \frac{(\tilde{u}_3 + h_3^3)}{\rho} . \tag{5.6}$$

The functions h_1^2 and h_2^2 defined by (4.10) can be computed using (5.6):

$$\begin{aligned} h_1^2 &= h_1^3 + v_3^2 L_{b_2} v_3^1 , \\ h_2^2 &= h_2^3 - v_3^1 L_{b_1} v_3^2 . \end{aligned} \tag{5.7}$$

Step $p = 1$: Finally, the functions v_1^1 and v_2^1 are defined, from (4.9) again, by

$$v_1^1 = \tilde{u}_1 + h_1^2, \quad v_2^1 = \tilde{u}_2 + h_2^2 . \tag{5.8}$$

Step 4. First, we need to find functions $u_{j,s}$ $(j = 1, \dots, 4\,, s = 1, \dots, \ell(j))$ such that the trajectories of the system

$$\dot{x} = \sum_{j=1}^{4} \sum_{s=1}^{\ell(j)} u_{j,s} b_{\tau_j^s} v_j^s \tag{5.9}$$

converge uniformly to those of the system

$$\dot{x} = \sum_{j=1}^{4} \mathcal{C}_j(b_{\tau_j^1} v_j^1, \dots, b_{\tau_j^{\ell(j)}} v_j^{\ell(j)}) . \tag{5.10}$$

We remark that, in view of (5.2), (5.4), and (5.6), the vector fields $b_{\tau_3^1}v_3^1, b_{\tau_4^1}v_4^1$, and $b_{\tau_4^2}v_4^2$ are in fact identical. As a consequence, there are only 5—not 7 (the number of terms in the sum (5.9))—different vector fields in (5.9) or (5.10). Therefore, the system (5.9) can be rewritten as

$$\dot{x} = \sum_{i=1}^{5} u_i X_i \tag{5.11}$$

with $X_1 = b_1 v_1^1, X_2 = b_2 v_2^1, X_3 = b_1 v_3^1 = b_1 v_4^1 = b_1 v_4^2, X_4 = b_2 v_3^2$, and $X_5 = b_2 v_4^3$, and u_1, u_2, u_3, u_4, u_5 standing, respectively, for $u_{1,1}$, $u_{2,1}$, $u_{3,1} + u_{4,1} + u_{4,2}$, $u_{3,2}$, and $u_{4,3}$, and the system (5.10) can then be rewritten as

$$\dot{x} = X_1 + X_2 + [X_3, X_4] + [X_3, [X_3, X_5]]\,. \tag{5.12}$$

We choose some candidate functions u_i, for the approximation of trajectories of (5.12) by solutions of (5.11), of the following form:

$$\left\{\begin{array}{lcl} u_1(t) &=& \eta_{1,0}\,,\\ u_2(t) &=& \eta_{2,0}\,,\\ u_3(t) &=& \varepsilon^{-\frac{1}{2}}\eta_{\omega_{1,1}}\cos\omega_{1,1}t/\varepsilon + \varepsilon^{-\frac{2}{3}}\left(\eta_{\omega_{2,1}}\cos\omega_{2,1}t/\varepsilon + \eta_{\omega_{2,2}}\cos\omega_{2,2}t/\varepsilon\right),\\ u_4(t) &=& \varepsilon^{-\frac{1}{2}}\eta_{\omega_{1,2}}\sin\omega_{1,2}t/\varepsilon,\\ u_5(t) &=& \varepsilon^{-\frac{2}{3}}\eta_{\omega_{2,3}}\cos\omega_{2,3}t/\varepsilon \end{array}\right. \tag{5.13}$$

with $\omega_{k,j}$ defined for instance by

$$\begin{array}{rcl} \Omega_1 &=& \{\omega_{1,1}, \omega_{1,2}\} = \{\frac{7}{2}, -\frac{7}{2}\}\\ \text{and}\quad \Omega_2 &=& \{\omega_{2,1}, \omega_{2,2}, \omega_{2,3}\} = \{2, 3, -5\}. \end{array}$$

Note in particular that each set Ω_k is "minimally cancelling" (MC) in the sense of [9, 25, 26]. Using [9, Theorem 5.1] (see also [9, section 8]), where a very similar example is treated), one can show that the trajectories of system (5.11)–(5.13) converge to those of the system

$$\dot{x} = \eta_{1,0}X_1 + \eta_{2,0}X_2 - \frac{\eta_{\omega_{1,1}}\eta_{\omega_{1,2}}}{2\omega_{1,1}}[X_3, X_4] - \frac{\eta_{\omega_{2,1}}\eta_{\omega_{2,2}}\eta_{\omega_{2,3}}}{4\omega_{2,1}\omega_{2,2}}[X_3, [X_3, X_5]]\,. \tag{5.14}$$

In order to identify system (5.12) with system (5.14), one can, for instance, define

$$\eta_{1,0} = \eta_{2,0} = \eta_{\omega_{1,1}} = \eta_{\omega_{2,1}} = \eta_{\omega_{2,2}} = 1$$

and

$$\eta_{\omega_{1,2}} = -2\omega_{1,1},\ \eta_{\omega_{2,3}} = -4\omega_{2,1}\omega_{2,2}\,.$$

Expressing the right-hand term of (5.11) as a function of the control vector fields b_1 and b_2, we finally obtain the expression of our stabilizing feedbacks:

$$\left\{\begin{array}{lcl} u_1^\varepsilon(x,t) &=& u_1(t/\varepsilon)v_1^1(x) + u_3(t/\varepsilon)v_3^1(x),\\ u_2^\varepsilon(x,t) &=& u_2(t/\varepsilon)v_2^1(x) + u_4(t/\varepsilon)v_3^2(x) + u_5(t/\varepsilon)v_4^3(x) \end{array}\right. \tag{5.15}$$

with the u_i's defined by (5.13) and the v_j^s's defined by (5.4), (5.6), and (5.8).

Although the above expression of the control laws appears quite simple, it is, in fact, quite involved due to the terms contained in the v_j^s's, and in particular due to the functions h_j^p defined by (5.5) and (5.7). This is a negative aspect of our construction: solving the equation (4.4) in the v_j^s's leads to heavy computations.

6. Convergence of highly oscillatory vector fields as differential operators. As explained in the introduction (section 1.2), the convergence results which are implicitly contained in [5], and explicitly in [25, 9] or [8], in terms of uniform convergence of solutions on finite time intervals, are not sufficient here. In this section, we state separately the convergence result that is used to prove Theorem 4.9. The word convergence is perhaps a bit farfetched since there is no notion of limit in the topological sense; the convergence is more of an algebraic nature: we simply decompose the operator as the sum of a nonoscillating term (the "limit") and a term which is a differential operator—of order higher than 1—whose coefficients are, when ε goes to zero and x remains in a compact set, $\mathcal{O}(\varepsilon^\gamma)$, with $\gamma > 0$. However, this result will prove to be sufficient for our needs. It is also sufficient to recover the uniform convergence stated in [5, 8, 25, 9]. In what follows, $\mathcal{T}$ denotes any time interval (possibly infinite).

DEFINITION 6.1. *Let F^ε ($\varepsilon \in (0, \varepsilon_0]\, \varepsilon_0 > 0$) and F^0 be vector fields on $\mathbb{R}^{1+n}$, defined by $F^\varepsilon(t,x) = \frac{\partial}{\partial t} + f(\varepsilon,t,x)$ and $F^0(t,x) = \frac{\partial}{\partial t} + f^0(t,x)$ with $f \in \mathcal{C}^0((0,\varepsilon_0] \times \mathcal{T} \times \mathbb{R}^n; \mathbb{R}^n) \cap \mathcal{C}^\infty((0,\varepsilon_0] \times \mathcal{T} \times (\mathbb{R}^n - \{0\}); \mathbb{R}^n)$, and $f^0 \in \mathcal{C}^0(\mathcal{T} \times \mathbb{R}^n; \mathbb{R}^n) \cap \mathcal{C}^\infty(\mathcal{T} \times (\mathbb{R}^n - \{0\}); \mathbb{R}^n)$.*

We say that F^ε converges as a differential operator of order one on functions of t and x, in brief "DO-converges," to F^0, as $\varepsilon \longrightarrow 0$, if

$$F^\varepsilon = F^0 + \varepsilon^{\gamma_1} \left(F^\varepsilon D_1^\varepsilon - D_1^\varepsilon \frac{\partial}{\partial t} \right) + \varepsilon^{\gamma_2} D_2^\varepsilon . \tag{6.1}$$

The above equality is understood as an equality of differential operators. γ_1 and γ_2 are strictly positive reals, and D_1^ε and D_2^ε are differential operators whose coefficients are continuous, smooth outside the origin, and locally uniformly bounded when $\varepsilon \to 0$, i.e., there exists $\varepsilon_0 > 0$ such that for all compact subset K of $\mathbb{R}^n$, each component of these differential operators is bounded for $(\varepsilon, t, x) \in (0, \varepsilon_0] \times \mathcal{T} \times K$.

This kind of convergence carries with it two important properties.

PROPOSITION 6.2. *Suppose that a vector field F^ε DO-converges, as $\varepsilon \longrightarrow 0$, to a vector field F^0 on a time interval $\mathcal{T}$. Then we have the following.*

(1) *The trajectories of $\dot{x} = f(\varepsilon,t,x)$ converge uniformly to those of $\dot{x} = f^0(t,x)$ on finite time intervals. More precisely, let $[0,T] \subset \mathcal{T}$, and let x^0 be the (unique) solution of*

$$\begin{aligned} \dot{x} &= f^0(t,x), \\ x(0) &= x_0 . \end{aligned} \tag{6.2}$$

Then, for ε small enough, the unique solution x^ε of

$$\begin{aligned} \dot{x} &= f(\varepsilon,t,x), \\ x(0) &= x_0 \end{aligned} \tag{6.3}$$

is defined on $[0,T]$, and $x^\varepsilon(t)$ converges to $x^0(t)$ uniformly on $[0,T]$.

(2) *If $\mathcal{T} = [0,+\infty)$, and if all vector fields in (6.1) are homogeneous of degree zero and f^0 is autonomous, then, if the origin of*

$$\dot{x} = f^0(x) \tag{6.4}$$

is asymptotically stable, the origin of

$$\dot{x} = f(\varepsilon,t,x) \tag{6.5}$$

is (exponentially) asymptotically stable too for $\varepsilon > 0$ sufficiently small.

Proof. We prove (1). First, we rewrite (6.1) as

$$F^\varepsilon(I - \varepsilon^{\gamma_1} D_1^\varepsilon) = F^0 - \varepsilon^{\gamma_1} D_1^\varepsilon \frac{\partial}{\partial t} + \varepsilon^{\gamma_2} D_2^\varepsilon .$$

This is an equality between differential operators. We apply each side to the coordinate functions x_i. $D_1^\varepsilon x_i$ and $D_2^\varepsilon x_i$ are simply the coefficients in front of $\frac{\partial}{\partial x_i}$ in the expression of the differential operator D_1^ε or D_2^ε. This implies (coordinate by coordinate) that the differential equation (6.3) can be rewritten

$$\frac{\mathrm{d}}{\mathrm{d}t}(x - \varepsilon^{\gamma_1} d_1(\varepsilon, t, x)) = f^0(t, x) + \varepsilon^{\gamma_2} d_2(\varepsilon, t, x), \tag{6.6}$$

where $d_i(\varepsilon, t, x)$ ($i \in \{1, 2\}$) is the vector whose jth component is the coefficient of $\frac{\partial}{\partial x_j}$ in D_i^ε. This implies that the difference between $x^0(t) - x^\varepsilon(t)$ satisfies

$$\begin{aligned}\|x^\varepsilon(t) - x^0(t)\| \le{}& \varepsilon^{\gamma_1}\|d_1(\varepsilon, t, x^0(t))\| + \varepsilon^{\gamma_1}\|d_1(\varepsilon, t, x_0)\| \\ &+ \int_0^t \|f^0(\tau, x^\varepsilon(\tau)) - f^0(\tau, x^0(\tau))\|\mathrm{d}\tau + \varepsilon^{\gamma_2}\int_0^t \|d_2(\varepsilon, \tau, x^\varepsilon(\tau))\|\mathrm{d}\tau.\end{aligned}$$

The standard Gronwall lemma then yields, $\forall\ \varepsilon \in (0, \varepsilon_0]$ and $\forall\ t \in [0, T]$ such that x^ε remains in the interior of a certain compact neighborhood K of the trajectory x^0, the estimate $\|x^\varepsilon(t) - x^0(t)\| \le (2\varepsilon^{\gamma_1} + T\varepsilon^{\gamma_2})Me^{\lambda t}$, where λ is a Lipschitz constant (with respect to x) of F on $[0, T] \times K$ and M is an upperbound on $(0, \varepsilon_0] \times [0, T] \times K$ for both $\|d_1\|$ and $\|d_2\|$. This proves (1).

Let us prove (2). Since the right-hand side of (6.4) is homogeneous of degree zero, there exists [17] a *homogeneous* and autonomous Lyapunov function V, positive definite, whose derivative along (6.4) is given by

$$\dot{V}_{(6.4)} = F^0 V = -W. \tag{6.7}$$

Here $X\,V$, for X a vector field, denotes the Lie derivative of V along X with W homogeneous positive definite of the same degree as V, i.e.,

$$W(x) \ge c\,V(x). \tag{6.8}$$

Let us now compute the derivative of V along system (6.5). From (6.1) and (6.7),

$$\dot{V}_{(6.5)} = F^\varepsilon V = -W + \varepsilon^{\gamma_1} F^\varepsilon D_1^\varepsilon V - \varepsilon^{\gamma_1} D_1^\varepsilon \frac{\partial V}{\partial t} + \varepsilon^{\gamma_2} D_2^\varepsilon V,$$

which can be rewritten, since V is autonomous, as

$$F^\varepsilon V_\varepsilon = -W + \varepsilon^{\gamma_2} D_2^\varepsilon V, \tag{6.9}$$

with

$$V_\varepsilon = V - \varepsilon^{\gamma_1} D_1^\varepsilon V. \tag{6.10}$$

Since, by assumption, the operators D_1^ε and D_2^ε are homogeneous of degree zero, and locally uniformly bounded with respect to $\varepsilon > 0$, one has, since V is positive definite,

$$|D_1^\varepsilon V| \le k\,V, \quad |D_2^\varepsilon V| \le k\,V$$

$\forall\, \varepsilon > 0$. Hence, for ε sufficiently small, V_ε is arbitrarily close to V and hence positive definite, and also

$$\dot{V}_\varepsilon \;=\; F^\varepsilon V_\varepsilon \;\le\; -\frac{c}{2}V. \tag{6.11}$$

Therefore, for ε small enough, V_ε is a strict Lyapunov function for system (6.5). This ends the proof of 2 via Lyapunov's first method. □

Before stating our convergence result, we recall two definitions introduced in [25, 9].

Definition 6.3 (see [25, 9]). *Let Ω be a finite subset of $\mathbb{R}$ and $|\Omega|$ denote the number of elements of Ω. The set Ω is said to be MC if and only if*

(i) $\sum_{\omega\in\Omega} \omega = 0$;

(ii) *this is the only zero sum with at most $|\Omega|$ terms taken in Ω with possible repetitions:*

$$\left.\begin{array}{c} \displaystyle\sum_{\omega\in\Omega} \lambda_\omega \omega = 0 \\ (\lambda_\omega)_{\omega\in\Omega} \in \mathbb{Z}^{|\Omega|} \\ \displaystyle\sum_{\omega\in\Omega} |\lambda_\omega| \le |\Omega| \end{array}\right\} \Longrightarrow \left\{ \begin{array}{rcl} (\lambda_\omega)_{\omega\in\Omega} & = & (0,\dots,0), \\ & \text{or} & (1,\dots,1), \\ & \text{or} & (-1,\dots,-1). \end{array}\right. \tag{6.12}$$

For example, a set $\{\omega_1, \omega_2\}$ is MC if and only if $\omega_2 = -\omega_1$ with $\omega_1 \neq 0$, a set $\{\omega_1, \omega_2, \omega_3\}$ is MC if and only if $\omega_3 = -\omega_1 - \omega_2$ with $\omega_1 \neq 0$, $\omega_2 \neq 0$, $\omega_1 + \omega_2 \neq 0$, $\omega_1 - \omega_2 \neq 0$, $\omega_1 + 2\omega_2 \neq 0$, $2\omega_1 + \omega_2 \neq 0$, $\omega_1 - 2\omega_2 \neq 0$, $2\omega_1 - \omega_2 \neq 0\ldots$.

Definition 6.4 (see [25, 9]). *Let $(\Omega_\alpha)_{\alpha\in I}$ be a finite family of finite subsets Ω_α of $\mathbb{R}$. The family $(\Omega_\alpha)_{\alpha\in I}$ is said to be "independent with respect to p" if and only if*

$$\left.\begin{array}{l} \bullet\ \displaystyle\sum_{\alpha\in I}\sum_{\omega\in\Omega_\alpha} \lambda_\omega \omega = 0 \\ \bullet\ (\lambda_\omega)_{\omega\in\Omega_\alpha,\alpha\in I} \in \mathbb{Z}^{\Sigma|\Omega_\alpha|} \\ \bullet\ \displaystyle\sum_{\alpha\in I}\sum_{\omega\in\Omega} |\lambda_\omega| \le p \end{array}\right\} \Longrightarrow \sum_{\omega\in\Omega_\alpha} \lambda_\omega\omega = 0 \quad \forall \alpha \in I. \tag{6.13}$$

For example, the sets $(\{\omega_1,\omega_2,\omega_3\}, \{\omega_4,\omega_5\})$ are both MC and independent with respect to 2 if and only if $\omega_3 = -\omega_1 - \omega_2$ and $\omega_5 = -\omega_4$ with $\omega_1 \neq 0$, $\omega_2 \neq 0$, $\omega_1 + \omega_2 \neq 0$, $\omega_1 - \omega_2 \neq 0$, $\omega_1 + 2\omega_2 \neq 0$, $2\omega_1 + \omega_2 \neq 0$, $\omega_1 - 2\omega_2 \neq 0$, $2\omega_1 - \omega_2 \neq 0$, $\omega_4 \neq 0$ (this is MC), and $\omega_1 + \omega_4 \neq 0$, $\omega_1 - \omega_4 \neq 0$, $\omega_2 + \omega_4 \neq 0$, $\omega_2 - \omega_4 \neq 0$, $\omega_1 + \omega_2 + \omega_4 \neq 0$, $\omega_1 + \omega_2 - \omega_4 \neq 0$ (this is independence).

We are now ready to state our convergence result.

Theorem 6.5. *Let N be a positive integer and consider, for $j = 1, \ldots, N$,*

- *some vector fields $X_j^s \in C^\infty(\mathbb{R}^n - \{0\}; \mathbb{R}^n) \cap C^0(\mathbb{R}^n; \mathbb{R}^n)$ $(s = 1, \ldots, \ell(j))$,*
- *some smooth complex valued functions of time η_j^s $(s = 1, \ldots, \ell(j))$ such that, for some M,*

$$\left|\eta_j^s(t)\right| \le M \ \text{ and } \ \left|\dot{\eta}_j^s(t)\right| \le M \quad \forall t \in \mathcal{T}, \tag{6.14}$$

- *some sets $\Omega_j = \{\omega_j^1, \ldots, \omega_j^{\ell(j)}\}$ of real numbers such that $\omega_j^s = 0$ if $\ell(j) = 1$, Ω_j is MC if $\ell(j) \ge 2$, and the family $(\Omega_j)\,(\ell(j) \ge 2)$ is independent with respect to $P \triangleq \mathrm{Max}_j \ell(j)$.*

Then the vector field

$$F^\varepsilon \;=\; \frac{\partial}{\partial t} + \sum_{j=1}^{N}\sum_{s=1}^{\ell(j)} \alpha_{j,\varepsilon}^s X_j^s\,, \tag{6.15}$$

with

$$\alpha_{j,\varepsilon}^s(t) = 2\varepsilon^{-\frac{\ell(j)-1}{\ell(j)}}\,\Re\left(\eta_j^s(t)\,e^{i\omega_j^s t/\varepsilon}\right), \tag{6.16}$$

DO-converges, as $\varepsilon \to 0$*, to the vector field*

$$F^0 = \frac{\partial}{\partial t} + \sum_{j=1}^{N} \frac{2}{\ell(j)}\Re\left(\frac{\eta_j^1\cdots\eta_j^{\ell(j)}}{i^{\ell(j)-1}}\right)\; B_j \tag{6.17}$$

$$\textit{with}\quad B_j \;=\; \sum_{\sigma\in\mathfrak{S}(\ell(j))} \frac{[X_j^{\sigma(1)},[X_j^{\sigma(2)},[\ldots,X_j^{\sigma(\ell(j))}]\ldots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)}+\omega_j^{\sigma(2)})\ldots(\omega_j^{\sigma(1)}+\ldots+\omega_j^{\sigma(\ell(j)-1)})}\,.$$

Furthermore, if all the vector fields X_j^s *are homogeneous of degree zero, then all the differential operators in* (6.1) *are homogeneous of degree zero also.*

Remark 6.6. This result is very much related to the theory of "normal forms" for time-varying differential equations, as shown, for instance, in [19, Chapter 6]. Let us recall (see [19] for details) that a vector field $\frac{\partial}{\partial t} + \varepsilon f^0(t,x)\frac{\partial}{\partial x}$ is said to be in normal form if and only if $[\frac{\partial}{\partial t}, f^0\frac{\partial}{\partial x}] = 0$, i.e., if f^0 does not depend on t. For a system

$$(\Sigma_\varepsilon)\qquad \dot{x} \;=\; f(\varepsilon,t,x),$$

finding a normal form means finding a change of coordinates $x \mapsto y = x+\alpha(\varepsilon,x)$ that transforms (Σ_ε) into

$$(\Sigma_0)\qquad \dot{y} \;=\; \varepsilon f_0(y).$$

In general, deciding whether a normal form exists for a system, and then possibly finding this normal form, is a difficult problem and there are no systematic tools available.

Let us, however, rephrase Theorem 6.5 in the terms of [19]. By a time-scaling $t \mapsto \varepsilon t$, the system $\dot{x} = f(\varepsilon,x,t)$, where f is defined by $F^\varepsilon = \frac{\partial}{\partial t} + f\frac{\partial}{\partial x}$, with F^ε given by (6.15), is rewritten as

$$(\Sigma'_\varepsilon)\qquad \dot{x} \;=\; \varepsilon f_1(t,x) \,+\, \varepsilon^{1/2} f_2(t,x) \,+\, \cdots \,+\, \varepsilon^{1/P} f_P(t,x).$$

In the context of "normal forms," Theorem 6.5 states that (Σ_0), with f^0 defined by $F^0 = \frac{\partial}{\partial t} + f^0\frac{\partial}{\partial x}$ and F^0 given by (6.17), is a normal form for (Σ'), up to terms of higher order in ε.

7. Proofs.

7.1. Proof of Proposition 4.2 (section 4). Point (b) is strictly a consequence of the construction. Point (c) follows from the fact that if a linear combination of all the vector fields $\tilde{b}_j$ with constant real coefficients is identically zero, then homogeneity implies that each linear combination where only the terms corresponding to the brackets of same length must also be zero, and since by construction all the brackets

$\tilde{b}_j$ of same length are linearly independent over $\mathbb{R}$, this implies that all the coefficients are zero.

Let us prove point (a). Recall that any Lie bracket of length $p > P$ made with the vector fields b_i is identically zero (see Remark 4.1). From this fact, the controllability assumption (3.3), and the construction itself, there clearly exist integers $j_1, \ldots, j_n \in \{1, \ldots, N\}$ such that $\{\tilde{b}_{j_1}(0), \ldots, \tilde{b}_{j_n}(0)\}$ is a basis of $\mathbb{R}^n$. Hence $\{\tilde{b}_{j_1}(x), \ldots, \tilde{b}_{j_n}(x)\}$ is a basis of $\mathbb{R}^n$ for x in some neighborhood W of the origin. Let us show that this is true for any x in $\mathbb{R}^n$. Let x be outside W. There exist $\lambda > 0$ such that $\bar{x} = \delta_\lambda(x)$ is in W and hence $\{\tilde{b}_{j_1}(\bar{x}), \ldots, \tilde{b}_{j_n}(\bar{x})\}$ is a basis of $\mathbb{R}^n$. This implies, since δ_λ is a local diffeomorphism from a neighborhood of x to a neighborhood of $\bar{x}$, that

$$\left\{ \left((\delta_\lambda^{-1})_* \tilde{b}_{j_1} \right)(x), \ldots, \left((\delta_\lambda^{-1})_* \tilde{b}_{j_n} \right)(x) \right\}$$

is also a basis of $\mathbb{R}^n$. Now, from the homogeneity, $(\delta_\lambda^{-1})_* \tilde{b}_{j_k} = \lambda^{-\ell(j_k)} \tilde{b}_{j_k}$. This proves point (a). ∎

7.2. Proof of Theorem 6.5. In [5, 8, 25, 9], the main ingredient of the proof was iterated integrations by parts. Here we mimic these integrations by parts but at the level of products of differential operators instead of integrals along the solutions. The closed-loop vector field F^ε can be rewritten as

$$F^\varepsilon = \frac{\partial}{\partial t} + \sum_{\substack{1 \le j \le N \\ \ell(j)=1}} 2\eta_j^1 X_j^1 + \sum_{\substack{1 \le j \le N \\ \ell(j)\ge 2}} \sum_{s=1}^{\ell(j)} \varepsilon^{-\frac{\ell(j)-1}{\ell(j)}} \left(\eta_j^s e^{i\omega_j^s t/\varepsilon} + \overline{\eta_j^s} e^{-i\omega_j^s t/\varepsilon} \right) X_j^s. \tag{7.1}$$

Let us make some conventions and definitions, used only in the present proof. We define the sets of indices

$$J = \{j \in \{1, \ldots, N\}, \ell(j) \ge 2\} = \{m_1 + 1, \ldots, N\}, \tag{7.2}$$

$$J_l = \{j \in \{1, \ldots, N\}, \ell(j) = l\} = \{m_{l-1} + 1, \ldots, m_l\}, \tag{7.3}$$

$$K_j = \{-\ell(j), -\ell(j) - 1, \ldots, -1, 1, 2, \ldots \ell(j)\} \tag{7.4}$$

and the sets of pairs of indices

$$I = \{(j, s), j \in J, s \in K_j\} = \bigcup_{j \in J} \{j\} \times K_j, \tag{7.5}$$

$$I_l = \{(j, s) \in I, \ell(j) = l\} = \bigcup_{j \in J_l} \{j\} \times K_j. \tag{7.6}$$

We call F_1 the vector field

$$F_1 = \sum_{\substack{1 \le j \le N \\ \ell(j)=1}} 2\eta_j^1 X_j^1 = \sum_{(j,s) \in I_1} 2\eta_j^s X_j^s. \tag{7.7}$$

Clearly, if we define, for $s < 0$, the real numbers ω_j^s, the complex numbers η_j^s, and the vector fields X_j^s by

$$\left.\begin{array}{rcl} \omega_j^{-s} &=& -\omega_j^s \\ \eta_j^{-s} &=& \overline{\eta_j^s} \\ X_j^{-s} &=& X_j^s \end{array}\right\} \text{ for } j \in J,\ s \in K_j,\ s > 0, \tag{7.8}$$

the vector field F^ε from (7.1) may be rewritten as

$$F^\varepsilon = \frac{\partial}{\partial t} + F_1 + \sum_{(j,s)\in I} \varepsilon^{-\frac{\ell(j)-1}{\ell(j)}} \eta_j^s e^{i\omega_j^s t/\varepsilon} X_j^s \tag{7.9}$$

$$= \frac{\partial}{\partial t} + F_1 + \varepsilon^{-\frac{1}{2}} F_2^\varepsilon + \varepsilon^{-\frac{2}{3}} F_3^\varepsilon + \cdots + \varepsilon^{-\frac{P-1}{P}} F_P^\varepsilon, \tag{7.10}$$

where

$$F_l^\varepsilon = \sum_{(j,s)\in I_l} \eta_j^s e^{i\omega_j^s t/\varepsilon} X_j^s. \tag{7.11}$$

Note that the interest of (7.10) is that the negative powers of ε are written apart, and the vector fields F_j^ε have the "boundedness" property that their coefficients are continuous functions of x and t, smooth outside $x=0$, indexed by $\varepsilon>0$, and locally uniformly bounded with respect to $\varepsilon>0$. (It is not the case of F^ε itself because of the negative powers of ε.) In the remainder of the proof, we shall always write the negative powers of ε apart so that all the differential operators written as capital letters never contain coefficients that are unbounded when ε goes to zero.

We now define a certain number of differential operators $F^\varepsilon_{p_1,p_2,\dots,p_d}$ of order d for d between 1 and P, and for all d-tuple $(p_1,p_2,\dots,p_d)$ of integers such that

$$\left\{\begin{array}{rcl} & 1\le p_k\le P \quad \text{for } 1\le k\le d, & \\ & \frac{1}{p_1}+\cdots+\frac{1}{p_{d-1}}\le 1, & \\ (p_1,p_2) & \ne & (2,2), \\ (p_1,p_2,p_3) & \ne & (3,3,3), \\ & \vdots & \\ (p_1,\dots,p_{d-1}) & \ne & (d-1,\dots,d-1). \end{array}\right. \tag{7.12}$$

We define $F^\varepsilon_{p_1,p_2,\dots,p_d}$ to be equal to

$$\sum_{(\,(j_1,s_1)\,,\dots,\,(j_d,s_d)\,)\,\in\, I^d(p_1,\dots,p_d)} \frac{\eta_{j_1}^{s_1}\eta_{j_2}^{s_2}\cdots\eta_{j_d}^{s_d} e^{i(\omega_{j_1}^{s_1}+\cdots+\omega_{j_d}^{s_d})\frac{t}{\varepsilon}}\, X_{j_d}^{s_d}\, X_{j_{d-1}}^{s_{d-1}}\cdots X_{j_1}^{s_1}}{i^{(d-1)}\omega_{j_1}^{s_1}(\omega_{j_1}^{s_1}+\omega_{j_2}^{s_2})\cdots(\omega_{j_1}^{s_1}+\cdots+\omega_{j_{d-1}}^{s_{d-1}})}, \tag{7.13}$$

where $I^d(p_1,\dots,p_d)$ is the set of d-tuples of indices $((j_1,s_1),\dots,(j_d,s_d))$ such that $\ell(j_k)=p_k$, and which are neither a collection of $\frac{d}{2}$ pairs of the form (j,s), $(j,-s)$ nor such that, for some (even) k, $2\le k\le d$, $((j_1,s_1),\dots,(j_k,s_k))$ would be a collection of $\frac{k}{2}$ pairs of the form (j,s), $(j,-s)$. More precisely, $I^d(p_1,\dots,p_d)$ may be defined recursively by $I^1(p)=I_1$ and

$$((j_1,s_1),\dots,(j_d,s_d)) \in I^d(p_1,\dots,p_d)$$

$$\Leftrightarrow \left\{\begin{array}{l} \bullet\ (j_k,s_k)\in I_{p_k}\ \forall\, k, \\ \bullet\ ((j_1,s_1),\dots,(j_{d-1},s_{d-1}))\in I^{d-1}(p_1,\dots,p_{d-1}), \\ \bullet\ \text{there exists no permutation } \tau\in\mathfrak{S}(d) \\ \quad\ \text{such that } \left(j_{\tau(k)},s_{\tau(k)}\right) = (j_k,-s_k). \end{array}\right. \tag{7.14}$$

With the above definition of the sets of indices $I^d(p_1,\ldots,p_d)$, the denominators in (7.13) cannot be zero because of the following lemma.

LEMMA 7.1. *Let* $((j_1,s_1),\ldots,(j_d,s_d)) \in I^d$ *(see the definition of* I *in (7.5)) be such that* $\omega_{j_1}^{s_1}+\cdots+\omega_{j_d}^{s_d} = 0$. *Then*

- *either* $(\ell(j_1),\ldots,\ell(j_d)) = (d,\ldots,d)$ *and there exists a permutation* $\sigma \in \mathfrak{S}(d)$ *such that the p-tuple* $((j_1,s_1),\ldots,(j_d,s_d))$ *is exactly equal to* $((j,\sigma(1)),\ldots,(j,\sigma(p)))$ *or* $((j,-\sigma(1)),\ldots,(j,-\sigma(p)))$ *(with* $j_1=\cdots=j_d=j$*),*
- *or* $\frac{1}{\ell(j_1)}+\cdots+\frac{1}{\ell(j_d)} > 1$,
- *or there exists a permutation* $\tau \in \mathfrak{S}(d)$ *such that* $(j_{\tau(k)},s_{\tau(k)}) = (j_k,-s_k)\ \forall\ k$.

Proof. The equality $\omega_{j_1}^{s_1}+\cdots+\omega_{j_d}^{s_d}=0$ may be rewritten as

$$\sum_{\substack{j\in\{1,\ldots,N\}\\ \ell(j)\geq 2}} \sum_{s=1}^{\ell(j)} \lambda_j^s \omega_j^s = 0, \tag{7.15}$$

where the integer λ_j^s is equal to the number of times that (j,s) appears in $((j_1,s_1),\ldots,(j_d,s_d))$ minus the number of times $(j,-s)$ appears. Of course, (7.15) may be rewritten as

$$\sum_{j\in J}\sum_{\omega\in\Omega_j} \lambda_\omega \omega = 0$$

with $\lambda_{\omega_j^s} = \lambda_j^s$. Note that

$$\sum_{\omega} |\lambda_w| = \sum_{j,s} |\lambda_j^s| \leq d \leq P.$$

Hence, from the assumption that the sequences of frequencies are mutually "independent with respect to P" and are all MC (see (6.12)–(6.13)), each $(\lambda_j^1,\ldots,\lambda_j^{\ell(j)})$ is equal to either $(0,\ldots,0)$, $(1,\ldots,1)$, or $(-1,\ldots,-1)$. If it is different from $(0,\ldots,0)$ for at least one j, then all the couples $(j,1),\ldots,(j,\ell(j))$ or all the couples $(j,-1),\ldots,(j,-\ell(j))$ appear in $((j_1,s_1),\ldots,(j_d,s_d))$. If $d=\ell(j)$ for this j, i.e., if $((j_1,s_1),\ldots,(j_d,s_d))$ is a reordering of $((j,1),\ldots,(j,\ell(j)))$, or of $((j,-1),\ldots,(j,-\ell(j)))$, then we are in the first case of the lemma; if $d>\ell(j)$, then there is at least another couple (j',s') in $((j_1,s_1),\ldots,(j_d,s_d))$ and hence the sum $\frac{1}{\ell(j_1)}+\cdots+\frac{1}{\ell(j_d)}$ can be no less than $1+\frac{1}{\ell(j')}$ and we are in the second case of the lemma. Let us now examine the case where all the $(\lambda_j^1,\ldots,\lambda_j^{\ell(s)})$'s are equal to $(0,\ldots,0)$. This means that for all j,s, the couple (j,s) and the couple $(j,-s)$ appear the same number of times in $((j_1,s_1),\ldots,(j_d,s_d))$. This allows one to build the permutation having the property required in the third point of the lemma: it is the one that exchanges 1 with the first k_1 such that $(j_{k_1},s_{k_1})=(j_1,-s_1)$, 2 (3 if $k_1=2$) with the first $k_2\neq k_1$ such that $(j_{k_2},s_{k_2})=(j_2,-s_2)$, and so on. ∎

We shall now prove the following two facts.

Fact 1. For all q, $1\leq q\leq P$, there exist $\gamma_{1,q}$ and $\gamma_{2,q}$ strictly positive such that

$$\begin{aligned} F^\varepsilon = \frac{\partial}{\partial t} + F_1 + \sum_{p=2}^{q}(-1)^{p-1}F^\varepsilon_{\underbrace{p,p,\ldots,p}_{p\text{ times}}} \\ + \varepsilon^{\gamma_{1,q}}\left(F^\varepsilon D^\varepsilon_{1,q} - D^\varepsilon_{1,q}\frac{\partial}{\partial t}\right) + \varepsilon^{\gamma_{2,q}} D^\varepsilon_{2,q} \end{aligned} \tag{7.16}$$

$$+ \sum_{\substack{(p_1,\ldots,p_q)\in\{2,\ldots,P\}^q,\\ \frac{1}{p_1}+\cdots+\frac{1}{p_q}\le 1,\\ (p_1,\ldots,p_q)\neq(q,\ldots,q)}} (-1)^{q-1}\varepsilon^{-\left(1-\frac{1}{p_1}-\cdots-\frac{1}{p_q}\right)} F^\varepsilon_{p_1,\ldots,p_q}$$

Fact 2. For all p, $1\le p\le P$, there exist $\gamma'_{1,p}$ and $\gamma'_{2,p}$ strictly positive such that

$$F^\varepsilon_{\underbrace{p,\,p,\,\ldots,\,p}_{p\text{ times}}} = 2\,\frac{(-1)^{p-1}}{p}\sum_{j\in J_p}\Re\left(\frac{\eta_j^1\cdots\eta_j^p}{i^{(p-1)}}\right)B_j + \varepsilon^{\gamma'_{1,p}}\left(F^\varepsilon D'^\varepsilon_{1,p} - D'^\varepsilon_{1,p}\frac{\partial}{\partial t}\right) + \varepsilon^{\gamma'_{2,p}}D'^\varepsilon_{2,p} \tag{7.17}$$

with

$$B_j = \sum_{\sigma\in\mathfrak{S}(\ell(j))}\frac{[X_j^{\sigma(1)},[X_j^{\sigma(2)},[\ldots,X_j^{\sigma(\ell(j))}]\ldots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)}+\omega_j^{\sigma(2)})\cdots(\omega_j^{\sigma(1)}+\cdots+\omega_j^{\sigma(\ell(j)-1)})}. \tag{7.18}$$

These two facts imply Theorem 6.5. Indeed, for $q=P$, the last sum in (7.16) is empty since $\frac{1}{p_1}+\cdots+\frac{1}{p_P}\le 1$ with all the integers p_j no larger that P implies $(p_1,\ldots,p_P)=(P,\ldots,P)$. Hence for $q=P$, (7.16) reads

$$F^\varepsilon = \frac{\partial}{\partial t} + F_1 + \sum_{p=2}^{P}(-1)^{p-1}F^\varepsilon_{\underbrace{p,\,p,\,\ldots,\,p}_{p\text{ times}}} + \varepsilon^{\gamma_{1,P}}\left(F^\varepsilon D^\varepsilon_{1,P} - D^\varepsilon_1\frac{\partial}{\partial t}\right) + \varepsilon^{\gamma_{2,P}}D^\varepsilon_{2,P}. \tag{7.19}$$

Substituting in the above the expression of $F^\varepsilon_{p,\ldots,p}$ given by (7.17), one clearly gets (6.1) with the appropriate differential operators D_1^ε and D_2^ε and the appropriate positive real numbers γ_1 and γ_2.

Proof of Fact 1. We prove (7.16) by induction on q, from $q=1$ to $q=P$.

For $q=1$, the sum on the first line of (7.16) is empty, one may take $D^\varepsilon_{1,1}, D^\varepsilon_1$, and $D^\varepsilon_{2,1}$ to be zero, and (7.16) is simply (7.10).

Let us now suppose that (7.16) holds for a certain $q\ge 1$ and let us prove it for $q+1$. This is done through a manipulation on differential operators that more or less mimics an integration by parts. Since we shall use it elsewhere, let us explain it on a "general" differential operator Y before applying it.

Consider a differential operator of order d on functions of t and x that does not contain derivations with respect to t:

$$Y = \sum_{\text{multi-indices } I \text{ of length } d}\eta_I(t)a_I(t,x)\frac{\partial^{|I|}}{\partial x_I}. \tag{7.20}$$

Define $Y^{[-1]}$ and $Y^{[1]}$ to be

$$Y^{[-1]} = \sum_{\text{multi-indices } I \text{ of length } d}\eta_I(t)\left(\int_*^t a_I(\tau,x)\mathrm{d}\tau\right)\frac{\partial^{|I|}}{\partial x_I}, \tag{7.21}$$

$$Y^{[1]} = \sum_{\text{multi-indices } I \text{ of length } d}\frac{\mathrm{d}\eta_I}{\mathrm{d}t}(t)\left(\int_*^t a_I(\tau,x)\mathrm{d}\tau\right)\frac{\partial^{|I|}}{\partial x_I}. \tag{7.22}$$

Note that these are defined up to a function of x (through the initial time in the integrals) and that $Y^{[1]}$ is zero if the η's are constants. The derivative with respect to t of $Y^{[-1]}$ is $Y + Y^{[1]}$ in the following sense:

$$Y + Y^{[1]} \;=\; \left[\frac{\partial}{\partial t}\,,\,Y^{[-1]}\right] \;=\; \frac{\partial}{\partial t}\,Y^{[-1]} \;-\; Y^{[-1]}\,\frac{\partial}{\partial t}. \tag{7.23}$$

Indeed it is obvious that for any smooth function h of x and t, one has

$$Y.h\,(t,x) \;+\; Y^{[1]}.h\,(t,x) \;=\; \frac{\partial}{\partial t}\left(Y^{[-1]}.h\,(t,x)\right) \;-\; Y^{[-1]}.\frac{\partial h}{\partial t}\,(t,x), \tag{7.24}$$

simply because $\frac{\partial}{\partial t}$ commutes with $\frac{\partial}{\partial^{|I|}x_I}$. Then we rewrite (7.23) in the following way:

$$\begin{aligned} Y &= \left[\frac{\partial}{\partial t}\,,\,Y^{[-1]}\right] \;-\; Y^{[1]} \\ &= F^\varepsilon\,Y^{[-1]} \;-\; \left(\sum_{r=1}^{P}\varepsilon^{-\frac{r-1}{r}}\,F_r^\varepsilon\right)Y^{[-1]} \;-\; Y^{[-1]}\,\frac{\partial}{\partial t} \;-\; Y^{[1]}. \end{aligned} \tag{7.25}$$

In order to prove that if (7.16) holds for q, it also holds for $q+1$, we apply the identity (7.25) with

$$\begin{aligned} Y &= F^\varepsilon_{p_1,\ldots,p_q}, \\ Y^{[-1]} &= \varepsilon\,G^\varepsilon_{p_1,\ldots,p_q}, \\ Y^{[1]} &= \varepsilon\,H^\varepsilon_{p_1,\ldots,p_q}, \end{aligned}$$

for

$$(p_1,\ldots,p_q)\neq(q,\ldots,q) \;\text{ and }\; \frac{1}{p_1}+\cdots+\frac{1}{p_q}\leq 1, \tag{7.26}$$

where $G^\varepsilon_{p_1,\ldots,p_q}$ and $H^\varepsilon_{p_1,\ldots,p_q}$ are given by

$$G^\varepsilon_{p_1,\ldots,p_q} = \sum_{((j_1,s_1),\ldots,(j_q,s_q))\,\in\,I^q(p_1,\ldots,p_q)} \frac{\eta_{j_1}^{s_1}\cdots\eta_{j_q}^{s_q}\,e^{i(\omega_{j_1}^{s_1}+\cdots+\omega_{j_q}^{s_q})t/\varepsilon}\,X_{j_q}^{s_q}X_{j_{q-1}}^{s_{q-1}}\ldots X_{j_1}^{s_1}}{i^q\omega_{j_1}^{s_1}(\omega_{j_1}^{s_1}+\omega_{j_2}^{s_2})\ldots(\omega_{j_1}^{s_1}+\cdots+\omega_{j_q}^{s_q})} \tag{7.27}$$

$$H^\varepsilon_{p_1,\ldots,p_q} = \sum_{((j_1,s_1),\ldots,(j_q,s_q))\in I^q(p_1,\ldots,p_q)} \frac{\left(\frac{d}{dt}\left(\eta_{j_1}^{s_1}\cdots\eta_{j_q}^{s_q}\right)\right)e^{i(\omega_{j_1}^{s_1}+\cdots+\omega_{j_q}^{s_q})t/\varepsilon}\,X_{j_q}^{s_q}X_{j_{q-1}}^{s_{q-1}}\ldots X_{j_1}^{s_1}}{i^q\omega_{j_1}^{s_1}(\omega_{j_1}^{s_1}+\omega_{j_2}^{s_2})\ldots(\omega_{j_1}^{s_1}+\cdots+\omega_{j_q}^{s_q})}. \tag{7.28}$$

Note that the denominators are nonzero because, from Lemma 7.1, the definition (7.14) of the set of indices $I^q(p_1,\ldots,p_q)$ precisely removes the terms where the denominators would be zero.

Then (7.25) with the above expressions for Y, $Y^{[1]}$, and $Y^{[-1]}$ yields

$$F^\varepsilon_{p_1,\ldots,p_q} \;=\; -\sum_{r=1}^{P}\varepsilon^{\frac{1}{r}}F_r^\varepsilon G^\varepsilon_{p_1,\ldots,p_q} + \;\varepsilon\,F^\varepsilon G^\varepsilon_{p_1,\ldots,p_q} - \;\varepsilon\,G^\varepsilon_{p_1,\ldots,p_q}\frac{\partial}{\partial t} \;-\; \varepsilon\,H^\varepsilon_{p_1,\ldots,p_q}. \tag{7.29}$$

From (7.27) and (7.11) we have

$$\begin{aligned} &F_r^\varepsilon G^\varepsilon_{p_1,\ldots,p_q} \\ &\quad= \sum_{\substack{((j_1,s_1),\ldots,(j_q,s_q))\,\in\,I^q(p_1,\ldots,p_q)\\ (j_{q+1},s_{q+1})\,\in\,I_r}} \frac{\eta_{j_1}^{s_1}\cdots\eta_{j_{q+1}}^{s_{q+1}}\,e^{i(\omega_{j_1}^{s_1}+\cdots+\omega_{j_{q+1}}^{s_{q+1}})t/\varepsilon}\,X_{j_{q+1}}^{s_{q+1}}X_{j_q}^{s_q}\ldots X_{j_1}^{s_1}}{i^q\omega_{j_1}^{s_1}(\omega_{j_1}^{s_1}+\omega_{j_2}^{s_2})\ldots(\omega_{j_1}^{s_1}+\cdots+\omega_{j_q}^{s_q})}. \end{aligned}$$

From (7.14), the $(q+1)$-tuples $((j_1,s_1),\dots,(j_{q+1},s_{q+1}))$, which are in $I^q(p_1,\dots,p_q)\times I_r$ but not in $I^{q+1}(p_1,\dots,p_q,r)$, are such that there exists a permutation τ of the set of integers $\{1,\dots,q+1\}$ for which

$$\begin{aligned}((j_{\tau(1)},s_{\tau(1)}),(j_{\tau(2)},s_{\tau(2)}),\dots,(j_{\tau(q+1)},s_{\tau(q+1)}))\\ =((j_1,-s_1),(j_2,-s_2),\dots,(j_{q+1},-s_{q+1})),\end{aligned}\tag{7.30}$$

and this is possible only if q is odd. Equations (7.8) (for X and ω, not for η) and (7.30) imply that the term corresponding to $((j_1,-s_1),(j_2,-s_2),\dots,(j_{q+1},-s_{q+1}))$ is equal to

$$\frac{\eta_{j_{\tau(1)}}^{s_{\tau(1)}}\cdots\eta_{j_{\tau(q+1)}}^{s_{\tau(q+1)}}e^{i(\omega_{j_{\tau(1)}}^{s_{\tau(1)}}+\cdots+\omega_{j_{\tau(q+1)}}^{s_{\tau(q+1)}})t/\varepsilon}X_{j_{q+1}}^{s_{q+1}}X_{j_q}^{s_q}\cdots X_{j_1}^{s_1}}{i^q(-\omega_{j_1}^{s_1})(-\omega_{j_1}^{s_1}-\omega_{j_2}^{s_2})\dots(-\omega_{j_1}^{s_1}-\cdots-\omega_{j_q}^{s_q})}$$

which, since q must be odd (if not, there is no such term), is equal to

$$-\frac{\left(\prod_{k=1}^{q+1}\eta_{j_{\tau(k)}}^{s_{\tau(k)}}\right)e^{i(\omega_{j_1}^{s_1}+\cdots+\omega_{j_{q+1}}^{s_{q+1}})t/\varepsilon}X_{j_{q+1}}^{s_{q+1}}X_{j_q}^{s_q}\cdots X_{j_1}^{s_1}}{i^q\omega_{j_1}^{s_1}(\omega_{j_1}^{s_1}+\omega_{j_2}^{s_2})\cdots(\omega_{j_1}^{s_1}+\cdots+\omega_{j_q}^{s_q})}$$

and τ gives the change of index in the product allowing us to say that this is the opposite of the term corresponding to $((j_1,s_1),(j_2,s_2),\dots,(j_{q+1},s_{q+1}))$. Hence these terms sum to zero in the above sum. From (7.14), this implies $F_r^\varepsilon G_{p_1,\dots,p_q}^\varepsilon = F_{p_1,\dots,p_q,r}^\varepsilon$. Substituting this in (7.29) yields (we rename r as p_{q+1})

$$F_{p_1,\dots,p_q}^\varepsilon = -\sum_{p_{q+1}=1}^{P}\varepsilon^{\frac{1}{p_{q+1}}}F_{p_1,\dots,p_q,p_{q+1}}^\varepsilon+\varepsilon F^\varepsilon G_{p_1,\dots,p_q}^\varepsilon-\varepsilon G_{p_1,\dots,p_q}^\varepsilon\frac{\partial}{\partial t}-\varepsilon H_{p_1,\dots,p_q}^\varepsilon.$$

Hence (7.16) yields

$$\begin{aligned}F^\varepsilon=&\frac{\partial}{\partial t}+F_1+\sum_{p=2}^{q}(-1)^{p-1}F^\varepsilon_{\underbrace{p,p,\dots,p}_{p\text{ times}}}\\
&+\varepsilon^{\gamma_{1,q}}\left(F^\varepsilon D_{1,q}^\varepsilon-D_1^\varepsilon\frac{\partial}{\partial t}\right)+\varepsilon^{\gamma_{2,q}}D_{2,q}^\varepsilon\\
&+(-1)^q\sum_{\substack{(p_1,\dots,p_q)\in\{2,\dots,P\}^q\\ \frac{1}{p_1}+\cdots+\frac{1}{p_q}\le 1\\ (p_1,\dots,p_q)\neq(q,\dots,q)\\ p_{q+1}\in\{1,\dots,P\}}}\varepsilon^{-\left(1-\frac{1}{p_1}-\cdots-\frac{1}{p_{q+1}}\right)}F_{p_1,\dots,p_q,p_{q+1}}^\varepsilon\\
&+(-1)^{q-1}\sum_{\substack{(p_1,\dots,p_q)\in\{2,\dots,P\}^q,\\ \frac{1}{p_1}+\cdots+\frac{1}{p_q}\le 1\\ (p_1,\dots,p_q)\neq(q,\dots,q)}}\varepsilon^{\frac{1}{p_1}+\cdots+\frac{1}{p_q}}\left(F^\varepsilon G_{p_1,\dots,p_q}^\varepsilon-G_{p_1,\dots,p_q}^\varepsilon\frac{\partial}{\partial t}-H_{p_1,\dots,p_q}^\varepsilon\right).\end{aligned}\tag{7.31}$$

The term corresponding to $(p_1,\dots,p_{q+1})=(q+1,\dots,q+1)$ in the sum on the third line is $(-1)^qF_{q+1,\dots,q+1}^\varepsilon$, it adds to the sum on the first line and this yields the first line of (7.16) for $q+1$. The other terms in this sum such that $\frac{1}{p_1}+\cdots+\frac{1}{p_{q+1}}\le 1$ yield exactly the third line of (7.16) for $q+1$, and the terms in this sum such that

$\frac{1}{p_1} + \cdots + \frac{1}{p_{q+1}} > 1$, as well as all the last sum, add up with the second line to give the second line (the "small" terms) of (7.16) for $q+1$. This proves (7.16) for $q+1$ and ends the proof by induction of Fact 1.

Proof of Fact 2. From the definition (7.13) of $F^{\varepsilon}_{p_1,p_2,\ldots,p_d}$, we have

(7.32)

$$F^{\varepsilon}_{\underbrace{p,\ldots,p}_{p\text{ times}}} = \sum_{\substack{((j_1,s_1),\ldots,(j_p,s_p)) \in I^p(p,\ldots,p) \\ \omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p}=0}} \frac{\eta^{s_1}_{j_1}\eta^{s_2}_{j_2}\cdots\eta^{s_p}_{j_p}\, X^{s_p}_{j_p} X^{s_{p-1}}_{j_{p-1}} \ldots X^{s_1}_{j_1}}{i^{(p-1)}\omega^{s_1}_{j_1}(\omega^{s_1}_{j_1}+\omega^{s_2}_{j_2})\ldots(\omega^{s_1}_{j_1}+\cdots+\omega^{s_{p-1}}_{j_{p-1}})}$$
$$+ \sum_{\substack{((j_1,s_1),\ldots,(j_p,s_p)) \in I^p(p,\ldots,p) \\ \omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p}\neq 0}} \frac{\eta^{s_1}_{j_1}\eta^{s_2}_{j_2}\cdots\eta^{s_p}_{j_p} e^{i(\omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p})t/\varepsilon}\, X^{s_p}_{j_p} X^{s_{p-1}}_{j_{p-1}} \ldots X^{s_1}_{j_1}}{i^{(p-1)}\omega^{s_1}_{j_1}(\omega^{s_1}_{j_1}+\omega^{s_2}_{j_2})\ldots(\omega^{s_1}_{j_1}+\cdots+\omega^{s_{p-1}}_{j_{p-1}})}.$$

Now, apply (7.20), (7.21), (7.22), and (7.25) with Y equal to the second sum, and therefore

$$Y^{[-1]} \;=\; \varepsilon\, G^{\varepsilon}_{p,\ldots,p}, \quad Y^{[1]} \;=\; \varepsilon\, H^{\varepsilon}_{p,\ldots,p},$$

with

(7.33)

$$G^{\varepsilon}_{\underbrace{p,\ldots,p}_{p\text{ times}}} = \sum_{\substack{((j_1,s_1),\ldots,(j_p,s_p)) \in I^p(p,\ldots,p) \\ \omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p}\neq 0}} \frac{\eta^{s_1}_{j_1}\cdots\eta^{s_p}_{j_p} e^{i(\omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p})t/\varepsilon}\, X^{s_p}_{j_p} X^{s_{p-1}}_{j_{p-1}} \cdots X^{s_1}_{j_1}}{i^p\omega^{s_1}_{j_1}(\omega^{s_1}_{j_1}+\omega^{s_2}_{j_2})\ldots(\omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p})};$$

(7.34)

$$H^{\varepsilon}_{\underbrace{p,\ldots,p}_{p\text{ times}}} = \sum_{\substack{((j_1,s_1),\ldots,(j_p,s_p)) \in I^p(p\ldots p) \\ \omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p}\neq 0}} \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\eta^{s_1}_{j_1}\cdots\eta^{s_p}_{j_p}\right)\right) e^{i(\omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p})t/\varepsilon}\, X^{s_p}_{j_p} X^{s_{p-1}}_{j_{p-1}} \cdots X^{s_1}_{j_1}}{i^p\omega^{s_1}_{j_1}(\omega^{s_1}_{j_1}+\omega^{s_2}_{j_2})\ldots(\omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p})}.$$

This allows us to rewrite the second sum in (7.32) as

$$\varepsilon\, F^{\varepsilon}\, G^{\varepsilon}_{p,\ldots,p} \;-\; \varepsilon\, G^{\varepsilon}_{p,\ldots,p}\,\frac{\partial}{\partial t} \;-\; \varepsilon\, H^{\varepsilon}_{p,\ldots,p} \;-\; \sum_{r=1}^{P} \varepsilon^{\frac{1}{r}} F^{\varepsilon}_{p,\ldots,p,r}$$

with

(7.35)

$$F^{\varepsilon}_{\underbrace{p,\ldots,p}_{p\text{ times}},r} = \sum_{\substack{((j_1,s_1),\ldots,(j_p,s_p))\ \in\ I^p(p,\ldots,p)\\ \omega_{j_1}^{s_1}+\cdots+\omega_{j_p}^{s_p}\neq 0\\ (j_{p+1},s_{p+1})\in I_r}} \frac{\eta_{j_1}^{s_1}\cdots\eta_{j_{p+1}}^{s_{p+1}} e^{i(\omega_{j_1}^{s_1}+\cdots+\omega_{j_{p+1}}^{s_{p+1}})t/\varepsilon} X_{j_{p+1}}^{s_{p+1}} X_{j_p}^{s_p}\ldots X_{j_1}^{s_1}}{i^p\omega_{j_1}^{s_1}(\omega_{j_1}^{s_1}+\omega_{j_2}^{s_2})\ldots(\omega_{j_1}^{s_1}+\cdots+\omega_{j_p}^{s_p})}.$$

Let us now consider the *first* sum in (7.32). From Lemma 7.1 and the fact that, from (7.14), a p-tuple that is in $I^p(p,\ldots,p)$ cannot be of the type described in the third item of this lemma, all the p-tuples $((j_1,s_1),\ldots,(j_p,s_p))$ in $I^p(p,\ldots,p)$ such that $\omega_{j_1}^{s_1}+\cdots+\omega_{j_p}^{s_p}=0$ are exactly of the form $((j,\sigma(1)),\ldots,(j,\sigma(p)))$ or $((j,-\sigma(1)),\ldots,(j,-\sigma(p)))$ with $\ell(j)=p$ and $\sigma\in\mathfrak{S}(p)$. Hence the first sum may be rewritten (recall that $X_j^{-s}=X_j^s$) as

$$2\sum_{j\in J_p}\Re\left(\frac{\eta_j^1\cdots\eta_j^p}{i^{p-1}}\right)C_j$$

with

$$C_j = \sum_{\sigma\in\mathfrak{S}(p)}\frac{X_j^{\sigma(p)}X_j^{\sigma(p-1)}\cdots X_j^{\sigma(1)}}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)}+\omega_j^{\sigma(2)})\cdots(\omega_j^{\sigma(1)}+\cdots+\omega_j^{\sigma(p-1)})}. \tag{7.36}$$

If one replaces in the above sum σ by $\sigma\circ\tau$, where τ is the permutation that sends $(1,2,\ldots,p)$ on $(p,p-1,\ldots,1)$ (change of indices in the summation), one gets

$$C_j = \sum_{\sigma\in\mathfrak{S}(p)}\frac{X_j^{\sigma(1)}X_j^{\sigma(2)}\cdots X_j^{\sigma(p)}}{(\omega_j^{\sigma(p)}+\cdots+\omega_j^{\sigma(2)})(\omega_j^{\sigma(p)}+\cdots+\omega_j^{\sigma(3)})\cdots(\omega_j^{\sigma(p)}+\omega_j^{\sigma(p-1)})\omega_j^{\sigma(p)}}.$$

Since $\omega_j^1+\cdots+\omega_j^p=0$, the denominator may be transformed:

$$C_j = (-1)^{p-1}\sum_{\sigma\in\mathfrak{S}(p)}\frac{X_j^{\sigma(1)}X_j^{\sigma(2)}\ldots X_j^{\sigma(p)}}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)}+\omega_j^{\sigma(2)})\cdots(\omega_j^{\sigma(1)}+\cdots+\omega_j^{\sigma(p-1)})}.$$

Finally, a combinatorial computation in the free Lie algebra (see [8], or [9] in which this identity is also obtained but in a less computational way) gives

$$\sum_{\sigma\in\mathfrak{S}(p)}\frac{X_j^{\sigma(1)}X_j^{\sigma(2)}\ldots X_j^{\sigma(p)}}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)}+\omega_j^{\sigma(2)})\cdots(\omega_j^{\sigma(1)}+\cdots+\omega_j^{\sigma(p-1)})} = \frac{1}{p}\sum_{\sigma\in\mathfrak{S}(p)}\frac{[X_j^{\sigma(1)},[X_j^{\sigma(2)},[\cdots,X_j^{\sigma(p)}]\cdots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)}+\omega_j^{\sigma(2)})\cdots(\omega_j^{\sigma(1)}+\cdots+\omega_j^{\sigma(p-1)})}.$$

Hence $C_j = \frac{(-1)^{p-1}}{p} B_j$ with B_j given by (7.18). Substituting the above in (7.32) yields

$$
\begin{aligned}
(7.37) \quad F^{\varepsilon}_{\underbrace{p,\ldots,p}_{p \text{ times}}} &= \frac{2\,(-1)^{p-1}}{p} \sum_{j \in J_p} \Re\left(\frac{\eta_j^1 \cdots \eta_j^p}{i^{p-1}}\right) B_j \\
&+ \varepsilon \left(F^{\varepsilon}\, G^{\varepsilon}_{\underbrace{p,\ldots,p}_{p \text{ times}}} - G^{\varepsilon}_{\underbrace{p,\ldots,p}_{p \text{ times}}} \frac{\partial}{\partial t} \right) - \varepsilon\, H^{\varepsilon}_{p,\ldots,p} - \sum_{r=1}^{P} \varepsilon^{\frac{1}{r}} F^{\varepsilon}_{\underbrace{p,\ldots,p,r}_{p \text{ times}}}.
\end{aligned}
$$

This clearly yields (7.17), ends the proof of Fact 2, and hence ends the proof of Theorem 6.5.

7.3. Proof of Theorem 4.9. Let $F^{\varepsilon} = \frac{\partial}{\partial t} + f^{\varepsilon}$ with f^{ε} the vector field associated with the right-hand side of (4.21), and let $G = \frac{\partial}{\partial t} + g$ with g the vector field associated with the right-hand side of (4.17). First, we show that F^{ε} DO-converges (see Definition 6.1) to G as ε tends to zero.

Since (4.21) is the same as (4.16) with $u_{j,s} = u^{\varepsilon}_{j,s}$ given by (4.18), F^{ε} can be expressed in the form (6.15), with all X_j^s's homogeneous of degree zero because each X_j^s corresponds to one of the $b_{\tau_j^s} v_j^s$'s and, from Proposition 4.6, all vector fields $b_{\tau_j^s} v_j^s$ are homogeneous of degree zero. We can apply Theorem 6.5 because the sets $\Omega_{n,\alpha}$ ($n = 2, \ldots, N$) in the construction of Theorem 4.7 are MC and linearly independent with respect to P (see [9, section 5]). It implies that F^{ε} DO-converges, as ε tends to zero, to a vector field $F^0 = \frac{\partial}{\partial t} + f^0$ of the form (6.17), and in the definition (6.1) of DO-convergence, all differential operators are homogeneous of degree zero. We claim that $G = F^0$. Indeed, from Proposition 6.2, the property of DO-convergence implies the uniform convergence of the trajectories on finite time intervals. Therefore, the trajectories of (4.21) converge to those of $\dot{x} = f^0(t,x)$; however, from Theorem 4.7 (recall that (4.21) is the same as (4.16)–(4.18)), they converge to the trajectories of (4.17). This implies that the systems $\dot{x} = g(t,x)$ and $\dot{x} = f^0(t,x)$ are the same because they have the same trajectories. Hence, $F^0 = G$.

Finally, since F^{ε} DO-converges to $G = \frac{\partial}{\partial t} + g$ (with g autonomous) and since all differential operators in the definition of DO-convergence are homogeneous of degree zero, the asymptotic stability of the origin of (4.21), for $\varepsilon > 0$ small enough, will follow from Proposition 6.2 if we can show that the origin of (4.17) is asymptotically stable. This is a direct consequence of (4.3) to (4.5). □

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