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STRICT CONTROL LYAPUNOV FUNCTIONS FOR
HOMOGENEOUS JURDJEVIC-QUINN TYPE
SYSTEMS

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Abstract: This paper presents a method to design an explicit control Lyapunov function for affine and homogeneous systems that satisfy the so-called “Jurdjevic-Quinn conditions”. For these systems a positive definite function is known that can only be made non increasing by feedback ; a control Lyapunov function is obtained via a deformation of this function. As an example of its applications, this method allows one to construct an explicit control Lyapunov function for the stabilization of the angular velocity of an underactuated rigid body.

Keywords: Feedback stabilization, control Lyapunov function, Lyapunov design.

1. INTRODUCTION
Let us consider a control affine system:
\[
\dot{x} = f_0(x) + \sum_{k=1}^{m} u_k f_k(x) \tag{1}
\]
with state \(x \in \mathbb{R}^n\) and control \((u_1, \ldots, u_m) = u \in \mathbb{R}^m\). The \(f_k\)’s are smooth vector fields in \(\mathbb{R}^n\). The “Lyapunov design” of a stabilizing control law (see (Bacciotti, 1992)), based on Lyapunov direct method for stability of ordinary differential equations (see (Hahn, 1967)) consists in finding a continuous feedback law and a positive definite and radially unbounded function \(V\), such that \(V\) is (strictly) decreasing along the trajectories of the closed-loop system. Artstein’s theorem (Artstein, 1983; Sontag, 1990; Bacciotti, 1992) points out exactly which relations (control Lyapunov function, small control property, see below) a function has to satisfy in order to allow existence of a feedback law that makes it decrease. Construction of such a feedback is then explicit.

Definition 1. (Control Lyapunov function). A differentiable function \(V\) is a control Lyapunov function (clf) for the system (1) if and only if it is definite positive and radially unbounded and it satisfies for all \(x \in \mathbb{R}^n \setminus \{0\}\):
\[
\begin{align*}
L_{f_1} V(x) &= 0 \\
& \vdots \\
L_{f_m} V(x) &= 0 \implies L_{f_0} V(x) < 0 \tag{2}
\end{align*}
\]

Definition 2. (Small control property). A positive definite radially unbounded function \(V\) satisfies the small control property for system (1) if and only if, for all \(\epsilon > 0\), there is a \(\delta > 0\) such that for all \(x \in \mathbb{R}^n \setminus \{0\}\),
\[
\|x\| < \delta \implies \exists u \left\{ \|u\| < \epsilon \right\} \left\{ L_{f_0} V(x) < 0 \right\} \tag{3}
\]

Proposition 3. (Artstein’s Theorem). There exists a continuous state feedback control law \(u = \phi(x)\) that assigns \(V\) to be a strict Lyapunov function for the system (1), if and only if \(V\) is a control Lyapunov function which satisfies the small control property.

We do not mention the construction of \(\phi\) because we will only need it in the case of a homogeneous system, see Proposition 6. Unfortunatly
there is no systematic method of obtaining such a function $V$. Nevertheless, sometimes (for example in mechanical systems) one can easily obtain a function $V_0$ non increasing along the solutions of the closed-loop system, but not strictly decreasing. We may call such a function $V_0$ a “non strict Lyapunov function”. LaSalle’s Invariance principle (see (LaSalle, 1968; Hahn, 1967)) implies that, under some additional conditions, the origin is indeed asymptotically stable.

For the purpose of proving asymptotic stability, this $V_0$ is as good as a usual Lyapunov function, but there are many advantages in having a “strict” control Lyapunov function $V$ at hand. The negative definite function $V$ allows one to quantify robustness to model errors or perturbations in the control, see (Coron et al., 1995, section 3.3). Also, it can be used to perform Lyapunov redesign to enhance robustness properties, see (Khalil, 1992, section 5.5). Finally, some simple backstepping-like techniques for adding vector fields are homogeneous.

Note that the “converse Lyapunov theorems” (see for instance (Kurzweil, 1956; Hahn, 1967)) imply the existence of a function that is strictly decreasing along the solutions of this closed loop systems and hence a control Lyapunov function that is assigned to decrease strictly by this same control law. However these theorems are far from giving an explicit construction of these functions. This paper is a preliminary step in the direction of understanding how to design a control Lyapunov function as precisely as possible. We restrict our attention to systems of Jurdjevic-Quinn type, with the additional conditions that all the vector fields are homogeneous.

The paper is organized as follows. Section 2 is devoted to making precise the class of system we consider. In section 3, the idea of re-shaping a Lyapunov function in the direction of a vector field is presented, and the main result is given. Section 4 presents an application to the control of an underactuated rigid body.

2. PROBLEM STATEMENT

We consider only the class of systems (1) that meets the following conditions, often called “Jurdjevic-Quinn conditions” (see (Jurdjevic and Quinn, 1978; Gauthier, 1984; Bacciotti, 1992; Outbib and Sallet, 1992)). Let us give the assumption we need here.

For $V_0 : \mathbb{R}^n \to \mathbb{R}$ a function, we call $W_L(V_0)$ the following set:

$$W_L(V_0) = \left\{ x, L^{k+1}_{i}V_0(x) = L^k_{i}L^k_{j}V_0(x) = 0 \quad \forall k = 1, \ldots, m; \quad i = 1, \ldots, L \right\}$$

**Assumption 4.** (Jurdjevic-Quinn conditions). We assume that:

$$f_0(x) = 0 \iff x = 0 ,$$

and that a function $V_0 : \mathbb{R}^n \to \mathbb{R}$ is known and has the following three properties: it is positive definite and radially unbounded; it satisfies:

$$\forall x \in \mathbb{R}^n, L_kV_0(x) = 0 ;$$

it is such that there is an $L$ such that

$$W_L(V_0) \text{ is reduced to } \{0\} .$$

Under these conditions, an asymptotically stabilizing feedback control is easily obtained (see (Jurdjevic and Quinn, 1978; Gauthier, 1984; Outbib and Sallet, 1992; Bacciotti, 1992)), but only a “non strict Lyapunov function” is known.

**Proposition 5.** If system (1) satisfies Assumption 4, then a continuous asymptotically stabilizing feedback control law is given by:

$$u_k(x) = -L_{f_k}V_0(x) \quad (k = 1, \ldots, m) .$$

On top of Jurdjevic-Quinn assumptions, we will restrict our attention to homogeneous vector fields. Instead of recalling homogeneity of functions and vector fields with respect to a dilation, we refer to (Kawski, 1990). In the sequel, a family of dilations $(\delta_n)$ is fixed, and “homogeneous of degree $d$” means homogeneous of degree $d$ with respect to this family of dilations. Besides classical properties inherent to homogeneity, we use the following result, proved in (Bacciotti, 1992), that gives a simple formula for the feedback $\phi$ of Artstein’s theorem in the homogeneous case.

**Proposition 6.** If each vector field $f_k (k = 0, \ldots, m)$ is homogeneous of degree $c_k$ and $V$ is a homogeneous control Lyapunov function of degree $d > 0$ that satisfies the small control property, then

$$u_k(x) = -\alpha L_{f_k}V(x) \parallel x \parallel ^{c_0 - 2c_k - d}$$

is homogeneous of degree $c_0 - c_k$ (this makes the right-hand side of the closed-loop system $\dot{x} = f_0(x) + \sum_{k=1}^{m} u_k(x)f_k(x)$ homogeneous of degree $c_0$), and defines a control law that assigns the control Lyapunov function $V$ to be strictly decreasing if, $T$ being the set $\{x, \parallel x \parallel = 1, L_{f_k}V(x) \geq 0\}$,

$$\alpha > \sup_T \left( \frac{L_{f_k}V(x)}{(\sum_{k=1}^{m}(L_{f_k}V(x))^2)\parallel x \parallel ^{c_0 - 2c_k - d}} \right) .$$
3. RESHAPING LYAPUNOV FUNCTIONS VIA THE FLOW OF A VECTOR FIELD

Given a function $V_0$, a complete vector field $G$ (complete means that $\phi^G_\lambda$ — see below — is defined from $\mathbb{R}^n$ to $\mathbb{R}^n$ for all $\lambda \in \mathbb{R}$) and a positive real number $\lambda$, let us define

$$V^G_\lambda = V_0 \circ \phi^G_\lambda$$

(10)

with $\phi^G_\lambda$ the flow of $G$ at time $\lambda$ defined by:

$$\frac{\partial}{\partial \lambda} (\phi^G_\lambda (x)) = G(\phi^G_\lambda (x)) ,$$

$$\phi^G_0 (x) = x .$$

(11)

We shall use functions of this type and show that for a suitable vector field $G$ and suitable values of $\lambda$, $V^G_\lambda$ is a control Lyapunov function. The drawback is that, in general, it is impossible to give an explicit expression of the flow $\phi^G_\lambda$ with some usual functions. For this reason, we define another function $W^G_\lambda$ which will satisfy some of the properties of $V^G_\lambda$:

$$W^G_\lambda (x) = V_0 (x + \lambda G(x)) .$$

(12)

**Theorem 7.** Suppose that each vector field $f_k$ ($k = 0, \ldots, m$) is homogeneous of degree $c_k$. Suppose also that the affine control system (1) satisfies Jurdjevic-Quinn conditions (assumption 4) with a function $V_0$ that is homogeneous of degree $d$. Let $G$ be a homogeneous vector field of degree 0 which satisfies for all $x$ in $\mathbb{R}^n$,

$${L}_f_i V_0 (x) = 0$$

$$\vdots$$

$${L}_f_m V_0 (x) = 0$$

$$\implies {L}_f G V_0 (x) < 0$$

(13)

Then there exists a real $\lambda_0$ such that for all $\lambda$ that satisfies $0 < \lambda < \lambda_0$, $V^G_\lambda$ and $W^G_\lambda$ are homogeneous control Lyapunov functions of degree $d$.

Of course, to construct a control Lyapunov function, there remains to construct a vector field $G$ satisfying (13).

**Theorem 8.** Let the affine control system (1) be of Jurdjevic-Quinn type with respect to $V_0$, supposed to be homogeneous of degree $d$. Assume that the $f_k$ ($k = 0, \ldots, m$) are homogeneous of degree $c_k$. Then the vector field $G$ defined by (14) and (15) is homogeneous of degree 0 and satisfies the condition (13).

$$G = \sum_{i=0}^{L-1} \sum_{k=1}^{m} \lambda_{i,k} a_{i,k} f_k$$

(14)

with $\lambda_{i,k}$ ($i = 0, \ldots, L - 1; k = 1, \ldots, m$) are some functions defined by

$$\lambda_{i,k} = \sum_{j=1}^{L-1} (-1)^{j+1} \frac{L_{ad}^m (2j-1+j) f_k V_0}{(2V_0)^{m,e}}$$

(15)

$$\alpha_{j,k} = \frac{(2j+1)c_0 + 2c_k + d}{2}$$

The readers can find the proof of these theorems in the last section.

These two theorems give a method to design a strict Lyapunov function for homogeneous systems that satisfy Jurdjevic-Quinn conditions (assumption 4) with a homogeneous function $V_0$:

- First design a vector field $G$ that satisfies the condition (13). Theorem 8 gives a “universal” method to obtain it. However, on some examples, other solutions for $G$ are simpler.
- Then compute $W^G_\lambda$ according to the explicit formula (12). This function is a control Lyapunov function for $\lambda$ positive and small enough. If it is possible to compute explicitly the flow of $G$, then one may also take $V^G_\lambda$ given by (10) as a control Lyapunov function.
- Finally, design a feedback law that makes $V^G_\lambda$ or $W^G_\lambda$ strictly decreasing. Equations (8) and (9) give a possibility to obtain it.

4. EXAMPLE: THE RIGID BODY

4.1 Problem statement

The problem of stabilization of the velocity of a rigid body with two torques has already been studied in the literature (see (Brockett, 1983; Aeyels, 1985; Morin, 1996)). We study the following model:

$$\begin{cases}
\dot{\omega}_1 = c_1 \omega_2 \omega_3 + \frac{\tau_1}{j_1} \\
\dot{\omega}_2 = c_2 \omega_1 \omega_3 + \frac{\tau_2}{j_2} \\
\dot{\omega}_3 = c_3 \omega_1 \omega_2
\end{cases}$$

(16)

with $c_1 = \frac{j_2 - j_3}{j_1}$, $c_2 = \frac{j_3 - j_1}{j_2}$, $c_3 = \frac{j_1 - j_2}{j_3}$

A preliminary feedback is proposed in (Morin, 1996) to make the system satisfy the Jurdjevic-Quinn hypotheses.

$$\begin{cases}
\tau_1 = j_1 (-c_1 \omega_2 \omega_3 + \beta c_3 \omega_1 \omega_2) + k_1 \\
\tau_2 = j_2 (-c_2 + \mu c_3) \omega_1 \omega_3) + k_2
\end{cases}$$

We obtain the new system:

$$\dot{\omega} = f_0 (\omega) + k_1 f_1 + k_2 f_2$$

(17)

$$f_0 = \begin{bmatrix}
\beta c_3 \omega_1 \omega_2 \\
-\mu c_3 \omega_1 \omega_3 \\
c_3 \omega_1 \omega_2
\end{bmatrix},

f_1 = \begin{bmatrix}
\frac{1}{j_1} \\
0 \\
0
\end{bmatrix},

f_2 = \begin{bmatrix}
0 \\
\frac{1}{j_2} \\
0
\end{bmatrix} .$$

This system is of Jurdjevic-Quinn type with the
positive definite and radially unbounded function $V_0$ defined by
\[
V_0 = \frac{1}{2}(\omega_1 - \beta \omega_3)^2 + \frac{1}{2} \omega_2^2 + \frac{\mu}{2} \omega_3^2
\] (18)

In fact it is easy to verify that the set $W_0(V_0)$ is reduced to $\{0\}$. Using Theorem 5, one obtains, as in (Morin, 1996), a stabilizing control law. We are going to apply our method to design a control Lyapunov function as a deformation of $V_0$.

4.2 Our method

The system (17) is a Jurdjevic-Quinn type system with an homogeneous non strict Lyapunov function $V_0$. In addition the vector fields $f_0$, $f_1$ et $f_2$ are homogeneous of degree 1, $-1$ et $-1$. As a consequence we are able to design a strict Lyapunov function (this is new to our knowledge).

Design of $G$.

One can use theorem 8 to conceive the vector field $G$. But we have find a more simple $G$ that satisfies the condition (13). The verification is left to the reader.

\[
\forall \omega \in \mathbb{R}^3, G(\omega) = \begin{bmatrix} 0 \\
\frac{\mu \omega_3 \omega_2^2}{2V_0}\omega_3^2 \\
0 \end{bmatrix} \] (19)

Design of the control Lyapunov function.

The flow of $G$ is not easy to compute explicitly. Formula (12) yields the following control Lyapunov function for $\lambda$ that satisfies $0 < \lambda < \frac{1}{\sqrt{3}}$:
\[
W_\lambda^G = \frac{1}{2}(\omega_1 - \beta \omega_3)^2 + \frac{1}{2} (\omega_2 + \lambda \frac{\mu \omega_3 \omega_2^2}{2V_0})^2
\]
\[
+ \frac{\mu}{2} \omega_3^2
\]

Design of the new control law.

The equation (8) and (9) give a stabilizing feedback law for the system:
\[
\begin{align*}
\{ k_1 & = -\alpha \|\omega\| L_f \lambda W_\lambda^G \\
\{ k_2 & = -\alpha \|\omega\| L_f \lambda W_\lambda^G \}
\end{align*}
\] (20)

with $\|\cdot\|$ an homogeneous norm and $\alpha$ sufficiently large:
\[
\alpha > \sup_T \frac{L_f \lambda W_\lambda^G(\omega)}{(\|L_f \lambda W_\lambda^G(\omega)\|^2 + (L_f \lambda W_\lambda^G(\omega))^2)\|\omega\|}
\]

where $T = \{\omega, \|\omega\| = 1, L_f \lambda W_\lambda^G(\omega) \geq 0\}$.

4.3 Robustness to misalignment of actuators

Our method does not a priori improve the robustness of the control law given in (Morin, 1996) but it gives a tool (the negative definite function $W_\lambda^G$) to analyse it. Let us quantify, for instance the tolerable errors on actuators misalignment with the principal axes. We consider the initial system (16):
\[
\dot{\omega} = X_0(\omega) + \tau_1 f_1 + \tau_2 f_2
\] (21)

with $(\tau_1, \tau_2)$ defined by the addition of the preliminary feedback and the control law we have made $(k_1, k_2)$.

Let $\varphi_1$ and $\varphi_2$ be angle between the position of the torques and the principal axes. That means the vector fields $f_1$ and $f_2$ of the system (21) are given by:
\[
f_1 = \frac{1}{j_1} \begin{bmatrix} \cos(\varphi_1) \\
\sin(\varphi_1)\cos(\varphi_1) \\
\sin(\varphi_1)\sin(\varphi_1) \end{bmatrix}
f_2 = \frac{1}{j_2} \begin{bmatrix} \cos(\varphi_2) \\
\sin(\varphi_2)\cos(\varphi_2) \\
\sin(\varphi_2)\sin(\varphi_2) \end{bmatrix}
\]

Then we have:
\[
W_\lambda^G = L_f \lambda W_\lambda^G + k_1 L_f \lambda W_\lambda^G + k_2 L_f \lambda W_\lambda^G + \tau_1 L_f \lambda W_\lambda^G + \tau_2 L_f \lambda W_\lambda^G
\]

We defined three constants $\beta_1$, $\beta_2$ and $\beta_3$ :
\[
\begin{align*}
\beta_1 & = \sup_{|\omega| = 1} L_f \lambda W_\lambda^G + k_1 L_f \lambda W_\lambda^G + k_2 L_f \lambda W_\lambda^G < 0 \\
\beta_2 & = \sup_{|\omega| = 1} \tau_1 \|\frac{\partial W_\lambda^G}{\partial \omega}\| > 0 \\
\beta_3 & = \sup_{|\omega| = 1} \tau_2 \|\frac{\partial W_\lambda^G}{\partial \omega}\| > 0
\end{align*}
\]

Then we deduce :
\[
W_\lambda^G \leq -\beta_1 |\omega|^3 + \beta_2 |\omega|^3 \sin^2(\varphi) + \beta_3 |\omega|^3 \sin^2(\varphi)
\]

Consequently, if $\varphi_1$ and $\varphi_2$ are small enough to satisfy the following equation,
\[
\beta_2 \sin^2(\varphi/2) + \beta_3 \sin^2(\varphi/2) < \beta_1
\]

then the origin remains an asymptotically stable equilibrium.

5. CONCLUSION

Theorems 7 and 8 give a method to design a control Lyapunov function when non strict Lyapunov function has been used to prove asymptotic stability of an equilibrium point. This method is applied to Jurdjevic-Quinn system which satisfy some hypotheses of homogeneity. For non homogeneous system some preliminary results has been obtained in (Faubourg, 1997), but we are not able to design a control Lyapunov function. In order to extend the method to non homogeneous system, it would be necessary to find conditions on the vector field $G$ that express more precisely the fact that $V_\lambda^G$ or $W_\lambda^G$ are control Lyapunov functions.

Moreover, we have established that the functions $V_\lambda^G$ or $W_\lambda^G$ are control Lyapunov functions for
enough small λ. This is not surprising because the condition (13) imposed on the vector field G express a good condition at λ = 0. An extension would be to find other conditions on G to make these functions be control Lyapunov function for large λ.

Appendix A. APPENDIX : PROOFS.

A.1 Proof of theorem 7

Let us first establish two lemmas. $W^G_λ$ and $V^G_λ$ are, of course, defined by (10) and (12).

Lemma 9. For all $x$ in $\mathbb{R}^n$ and all $λ > 0$, we have:

\[
\frac{d}{dλ}_{λ=0} (L_{f_0}V^G_λ)(x) = L_{f_0}L_GV_0(x) \quad (A.1)
\]

\[
\frac{d}{dλ}_{λ=0} (L_{f_0}W^G_λ)(x) = L_{f_0}L_GW_0(x) \quad (A.2)
\]

Proof : Let the function $χ$ be defined by $χ(τ, λ, x) = V_0(φ^G_0(τφ^G_0(x)))$. From the definition (10) of $V_λ$ and the definition of the Lie derivative $(L_{f_0})_0ψ(ξ) = \frac{d}{dτ}|_t=0 ψ(φ^G_0(ξ))$ for any function $ψ$ and point $ξ$, the left-hand side of (A.1) is equal to $\frac{d^2}{dλ^2}$. On the other hand, applying twice the same definition of the Lie derivative (along $G$ and after $f_0$), the right-hand side is equal to $\frac{d^2}{dλ^2}$. This proves (A.1) because the partial derivatives commute. Relation (A.2) is true for the same reasons because, for any function $ψ$ and any point $ξ$, $ψ(φ^G_0(ξ))$ and $ψ(ξ + λG)(ξ))$ have the same partial derivative with respect to $λ$ at $λ = 0$.

Lemma 10. If $V_0$ is homogeneous of degree $d$ and the vector field $G$ is homogeneous of degree zero, the functions $W^G_λ$ and $V^G_λ$ are homogeneous of degree $d$ for all $λ ≥ 0$.

Proof : Since $G$ is homogeneous of degree zero, we have $δ_μ ^* G = G$ for all $μ > 0$. On the other hand, the flow of $δ_μ ^* G$ is the conjugate of the flow of $G$:

$φ^{G_δ_μ ^*} = δ_μ ^* φ^G_δ ^* δ_μ ^{-1}$

Clearly these two relations imply that the dilation $δ_μ$ commutes with the flow $φ^G_δ$. This implies that, for all $μ$ and $λ$,

$V^G_λ δ_μ = V_0 δ_μ φ^G_δ = V_0 δ_μ φ^G_δ = μ^d V^G_λ$

This precisely means that $V^G_λ$ is homogeneous of degree $d$. The same property holds for $W^G_λ$ because $G$ homogeneous of degree zero implies that $G(δ_μ (x)) = δ_μ (G(x))$ for all $x$ (this formula only makes sense in the coordinates where the definition (12) was written).

Now, we are going to give the proof that $V^G_λ$ is a control Lyapunov function. The proof that $W^G_λ$ satisfies the same property follows exactly along the same lines because only the properties from the above two lemmas are used.

We introduce some new notations to simplify the proof :

$Γ = \{ x, L_{f_1}V_0(x) = \ldots = L_{f_m}V_0(x) = 0 \}$

$E = \{ x, L_{f_0}L_GV_0(x) ≥ 0 \}$

$S = \{ x, |x| = 1 \}$

the norm in the definition of $S$ is the homogeneous norm. Clearly, condition (13) is equivalent to :

$Γ \cap E = \{ 0 \}$

(A.3)

Hence $Γ \cap S$ and $E \cap S$ are two disjoint compact sets, that do not intersect, so that the distance between them is strictly positive. Let $q'$ be such that $0 < q' < d(Γ \cap S, E \cap S)$, and define the set

$C_2 = \{ x ∈ S, d(x, Γ \cap S) ≤ q' \}$.  

(A.4)

It obviously has an empty intersection with $E$.

Let us now finish the proof in three steps :

First, we prove that that there exists a $λ_1 > 0 \text{ s.t }$

\[
0 < λ < λ_1, \quad x ∈ C_2 \implies L_{f_0}V^G_λ(x) < 0 \quad (A.5)
\]

Define the function $χ$ from $\mathbb{R} \times \mathbb{R}^n$ by $ξ(λ, x) = L_{f_0}V^G_λ(X)$. It is continuously differentiable at least away from $x = 0$, and in particular on $[0, +∞) \times C_2$. Let $−β = \max_{x ∈ C_2}(L_{f_0}L_GV_0(x))$. $β$ is positive because the set $C_2$ is compact and $(L_{f_0}L_GV_0)$ is negative on $C_2$ (because $C_2 \cap E = \emptyset$).

From lemma 9, this implies $\max_{x ∈ C_2} \frac{∂χ}{∂x}(0, x) = −β$, but $λ \mapsto \max_{x ∈ C_2} \frac{∂χ}{∂λ}(λ, x)$ is continuous on $[0, +∞)$, and consequently it remains larger smaller that $−β$ on a certain interval $[0, λ_1]$ with $λ_1 > 0$.

By integrating with respect to $λ$ on the interval $[0, λ_1]$ and taking into account the fact that, from (5), $χ(0, x)$ is identically zero, one obtains that $χ(λ, x) < −\frac{β λ_1}{2}$ for $(λ, x)$ in $[0, λ_1] \times C_2$. This clearly proves (A.5).

Second, we prove that, defining for all $λ ≥ 0$ the set $Γ_λ$ by $Γ_λ = \{ x ∈ S, \ L_{f_k}V^{G_k}(x) = 0; k = 1 \ldots m \}$, there exists a $λ_2 > 0 \text{ s.t }$

\[
0 < λ < λ_2, \implies Γ_λ \subset C_2 \quad (A.6)
\]

Let us prove that by contradiction. If this assertion is not true. Then

$\forall λ_2 > 0, \exists λ_0 < λ_2, \exists x_0 ∈ S \backslash C_2,$

$L_{f_k}V^{G_k}(x) = \ldots = L_{f_m}V^{G_m}(x) = 0$
For example, we choose \( \lambda_2 = \frac{1}{n} \), hence we can design a suite \((\lambda_n)\) and a suite \((x_n)\) such that

\[
\begin{align*}
\lambda_n &< \frac{1}{n} \\
x_n &\in S \setminus C_2 \\
L_{f_n}V^{G}_{\lambda_n}(x_n) &= \ldots = L_{f_m}V^{G}_{\lambda_n}(x_n) = 0
\end{align*}
\]

is compact. Consequently we can define a suite \((y_n)\) extracted from \((x_n)\) such that this suite tends to \( y \) in \( (S \setminus C_2) \). \((x, \lambda) \rightarrow L_{f_n}V^{G}_{\lambda}(x)\) is a continuous function. Consequently, \( L_{f_n}V_0(y) = 0 \), and \( y \) belongs to \( \Gamma \) which is included in \( C_2 \). It is impossible for \( y \) to belong to \( (S \setminus C_2) \) and to the interior of \( C_2 \). Consequently the equation (A.6) is true.

To finish the proof, define \( \lambda_0 \) by \( \lambda_0 = \min(\lambda_1, \lambda_2) \), and let us prove that for all \((\lambda, x)\) such that \( 0 < \lambda \leq \lambda_0 \), \( x \neq 0 \), and

\[
L_{f_n}V^{G}_{\lambda}(x) = \ldots = L_{f_m}V^{G}_{\lambda}(x) = 0 \quad (A.7)
\]

we have \( L_{f_n}V^{G}_{\lambda}(x) < 0 \). This obviously implies that \( V^{G}_{\lambda} \) is a control Lyapunov function if \( 0 < \lambda \leq \lambda_0 \). Since \( x \neq 0 \), there exists \( y \in S \) and a real number \( \mu > 0 \) such that \( x = \delta_{\mu}(y) \). From the homogeneity properties of the vector fields \( f_k \) (assumptions of the theorem) and the function \( V^{G}_{\lambda} \) (lemma 10), each function \( L_{f_k}V^{G}_{\lambda} \) is homogeneous of degree \( c_k + d \), and hence (A.7) implies:

\[
L_{f_n}V^{G}_{\lambda}(y) = \ldots = L_{f_m}V^{G}_{\lambda}(y) = 0
\]

Hence \( y \) is in \( \Gamma_{\lambda} \), and fact that \( 0 < \lambda < \lambda_2 \) plus property (A.6) imply that \( y \) is in \( C_2 \). Since \( 0 < \lambda < \lambda_1 \), the property (A.5) implies \( L_{f_n}V^{G}_{\lambda}(y) < 0 \). Then the homogeneity of \( L_{f_n}V^{G}_{\lambda} \) implies that \( L_{f_n}V^{G}_{\lambda}(x) < 0 \) as well and gives the result. \( \blacksquare \)

### A.2 Proof of theorem 8

A simple computation proves, using (14) and (15), that for all \( x \) in \( \mathbb{R}^n \),

\[
L_{f_n}L_{G}V_0(x) = -\sum_{i=1}^{L} \sum_{k=1}^{m} (L_{ad_{f_n}(f_k)}V_0(x))^2 + \sum_{k=1}^{m} L_{f_n}\lambda_0(x) L_{f_n}V_0(x)
\]

Hence both equation (6) and previous equality imply that for all \( x \) in \( \mathbb{R}^n \setminus \{0\} \),

\[
\begin{align*}
L_{f_n}V_0(x) &= 0 \\
L_{f_m}V_0(x) &= 0
\end{align*} \quad \implies \quad L_{f_n}L_{G}V_0(x) < 0
\]

It follows that the condition (13) is satisfied by the vector field \( G \). One can easily verify that \( G \) is also homogeneous of degree 0. \( \blacksquare \)

### Appendix B. REFERENCES


