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STRICT CONTROL LYAPUNOV FUNCTIONS FOR HOMOGENEOUS JURDJEVIC-QUINN TYPE SYSTEMS

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Abstract: This paper presents a method to design an explicit control Lyapunov function for affine and homogeneous systems that satisfy the so-called "Jurdjevic-Quinn conditions". For these systems a positive definite function is known that can only be made non increasing by feedback ; a control Lyapunov function is obtained via a deformation of this function. As an example of its applications, this method allows one to construct an explicit control Lyapunov function for the stabilization of the angular velocity of an underactuated rigid body.

Keywords: Feedback stabilization, control Lyapunov function, Lyapunov design.

1. INTRODUCTION

Let us consider a control affine system :

$$\dot{x} = f_0(x) + \sum_{k=1}^{m} u_k f_k(x) \tag{1}$$

with state $x \in I\!R^n$ and control $(u_1, \ldots, u_m) = u \in I\!R^m$. The f_k's are smooth vector fields in $I\!R^n$. The "Lyapunov design" of a stabilizing control law (see (Bacciotti, 1992)), based on Lyapunov direct method for stability of ordinary differential equations (see (Hahn, 1967)) consists in finding a continuous feedback law and a positive definite and radially unbounded function V, such that V is (strictly) decreasing along the trajectories of the closed-loop system. Artstein's theorem (Artstein, 1983; Sontag, 1990; Bacciotti, 1992) points out exactly which relations (control Lyapunov function, small control property, see below) a function has to satisfy in order to allow existence of a feedback law that makes it decrease. Construction of such a feedback is then explicit.

Definition 1. (Control Lyapunov function). A differentiable function V is a control Lyapunov function (clf) for the system (1) if and only if it is definite positive and radially unbounded and it satisfies for all x in $I\!R^n\backslash\{0\}$:

$$\left.\begin{array}{c} L_{f_1}V(x) = 0 \\ \vdots \\ L_{f_m}V(x) = 0 \end{array}\right\} \Longrightarrow L_{f_0}V(x) < 0 \tag{2}$$

Definition 2. (Small control property). A positive definite radially unbounded function V satisfies the *small control property* for system (1) if and only if, for all $\epsilon > 0$, there is a $\delta > 0$ such that for all $x \in I\!R^n\backslash\{0\}$,

$$\|x\| < \delta \Longrightarrow \exists u \left\{ \begin{array}{l} \|u\| < \epsilon \\ L_{f_0 + \sum_{k=1}^{m} u_k f_k} V(x) < 0 \end{array} \right. \tag{3}$$

Proposition 3. (Artstein's Theorem). There exists a continuous state feedback control law $u = \phi(x)$ that assigns V to be a strict Lyapunov function for the system (1), if and only if V is a control Lyapunov function which satisfies the small control property.

We do not mention the construction of ϕ because we will only need it in the case of a homogeneous system, see Proposition 6. Unfortunatly there is no systematic method of obtaining such a function V. Nevertheless, sometimes (for example in mechanical systems) one can easily obtain a positive definite and radially unbounded V_0 and some controls that make V_0 non increasing along the solutions of the closed-loop system, but not strictly decreasing. We may call such a function V_0 a "*non strict Lyapunov function*". LaSalle's Invariance principle (see (LaSalle, 1968; Hahn, 1967)) implies that, under some additional conditions, the origin is indeed asymptotically stable.

For the purpose of proving asymptotic stability, this V_0 is as good as a usual Lyapunov function, but there are many advantages in having a "strict" control Lyapunov function V at hand. The negative definite function $\dot{V}$ allows one to quantify robustness to model errors or perturbations in the control, see (Coron *et al.*, 1995, section 3.3). Also, it can be used to perform Lyapunov redesign to enhance robustness properties, see (Khalil, 1992, section 5.5). Finally, some simple backstepping-like techniques for adding an integrator require a strict Lyapunov function (see (Coron *et al.*, 1995)).

Note that the "converse Lyapunov theorems" (see for instance (Kurzweil, 1956; Hahn, 1967)) imply the *existence* of a function that is strictly decreasing along the solutions of this closed loop systems and hence a control Lyapunov function that is assigned to decrease strictly by this same control law. However these theorems are far from giving an explicit construction of these functions. This paper is a prelimary step in the direction of understanding how to design a control Lyapunov function as precisely as possible. We restrict our attention to systems of Jurdjevic-Quinn type, with the additional conditions that all the vector fields are homogeneous.

The paper is organized a follows. Section 2 is devoted to making precise the class of system we consider. In section 3, the idea of re-shaping a Lyapunov function in the direction of a vector field is presented, and the main result is given. Section 4 presents an application to the control of an underactuated rigid body.

2. PROBLEM STATEMENT

We consider only the class of systems (1) that meets the following conditions, often called "Jurdjevic-Quinn conditions" (see (Jurjevic and Quinn, 1978; Gauthier, 1984; Bacciotti, 1992; Outbib and Sallet, 1992)). Let us give the assumption we need here.

For $V_0 : \mathbb{R}^n \to \mathbb{R}$ a function, we call $\mathcal{W}_L(V_0)$ the following set:

$$\mathcal{W}_L(V_0) = \left\{ \begin{array}{l} x,\ L_{f_0}^{i+1}V_0(x) = L_{f_0}^{i}L_{f_k}V_0(x) = 0 \\ k = 1\dots m;\ i = 1\dots L \end{array} \right\}$$

Assumption 4. (Jurdjevic-Quinn conditions). We assume that:

$$f_0(x) = 0 \iff x = 0\ , \tag{4}$$

and that a function $V_0 : \mathbb{R}^n \to \mathbb{R}$ is known and has the following three properties : it is positive definite and radially unbounded ; it satisties :

$$\forall x \in \mathbb{R}^n,\ L_{f_0}V_0(x) \ = \ 0\ ; \tag{5}$$

it is such that there is an L such that

$$\mathcal{W}_L(V_0) \quad \text{is reduced to} \quad \{0\}\ . \tag{6}$$

Under these conditions, an asymptotically stabilizing feedback control is easily obtained (see (Jurjevic and Quinn, 1978; Gauthier, 1984; Outbib and Sallet, 1992; Bacciotti, 1992)), but only a "non strict Lyapunuv function" is known.

Proposition 5. If system (1) satisfies Assumtion 4, then a continuous asymptotically stabilizing feedback control law is given by :

$$u_k(x) = -L_{f_k}V_0(x) \quad (k = 1, \dots, m)\ . \tag{7}$$

On top of Jurdjevic-Quinn assumptions, we will restrict our attention to *homogeneous* vector fields. Instead of recalling homogeneity of functions and vector fields with respect to a dilation, we refer to (Kawski, 1990). In the sequel, a family of dilations (δ_μ) is fixed, and "homogeneous of degree d" means homogeneous of degree d with respect to this family of dilations. Besides classical properties inherent to homogeneity, we use the following result, proved in (Bacciotti, 1992), that gives a simple formula for the feedback ϕ of Artstein's theorem in the homogeneous case.

Proposition 6. If each vector field f_k $(k = 0, \dots, m)$ is homogeneous of degree c_k and V is a homogeneous control Lyapunov function of degree $d > 0$ that satisfies the small control property, then

$$u_k(x) = -\alpha L_{f_k}V(x)\|x\|^{c_0-2c_k-d} \tag{8}$$

is homogeneous of degree $c_0 - c_k$ (this makes the right-hand side of the closed-loop system $\dot{x} = f_0(x) + \sum_1^m u_k(x)f_k(x)$ homogeneous of degree c_0), and defines a control law that assigns the control Lyapunov function V to be strictly decreasing if, T being the set $\{x,\ \|x\| = 1,\ L_{f_0}V(x) \geq 0\}$,

$$\alpha > \sup_T \frac{L_{f_0}V(x)}{(\sum_{k=1}^m (L_{f_k}V(x))^2)\|x\|^{c_0-2c_k-d}}\ . \tag{9}$$

3. RESHAPING LYAPUNOV FUNCTIONS VIA THE FLOW OF A VECTOR FIELD

Given a function V_0, a complete vector field G (complete means that ϕ_λ^G —see below— is defined from $\mathbb{R}^n$ to $\mathbb{R}^n$ for all $\lambda \in \mathbb{R}$) and a positive real number λ, let us define

$$V_\lambda^G = V_0 \circ \phi_\lambda^G \tag{10}$$

with ϕ_λ^G the flow of G at time λ defined by :

$$\begin{aligned} \frac{\partial}{\partial \lambda}(\phi_\lambda^G(x)) &= G(\phi_\lambda^G(x)) \ , \\ \phi_0^G(x) &= x \ . \end{aligned} \tag{11}$$

We shall use functions of this type and show that for a suitable vector field G and suitable values of λ, V_λ^G is a control Lyapunov function. The drawback is that, in general, it is impossible to give an explicit expression of the flow ϕ_λ^G with some usual functions. For this reason, we define another function W_λ^G which will satisfy some of the properties of V_λ^G :

$$W_\lambda^G(x) = V_0(x + \lambda G(x)) \ . \tag{12}$$

Theorem 7. Suppose that each vector field f_k ($k = 0, \ldots, m$) is homogeneous of degree c_k. Suppose also that the affine control system (1) satisfies Jurdjevic-Quinn conditions (assumption 4) with a function V_0 that is homogeneous of degree d. Let G be a homogeneous vector field of degree 0 which satisfies, for all x in $\mathbb{R}^n$,

$$\left.\begin{array}{c} L_{f_1}V_0(x) = 0 \\ \vdots \\ L_{f_m}V_0(x) = 0 \end{array}\right\} \Longrightarrow L_{f_0}L_G V_0(x) < 0 \tag{13}$$

Then there exists a real λ_0 such that for all λ that satisfies $0 < \lambda < \lambda_0$, V_λ^G and W_λ^G are homogeneous control Lyapunov functions of degree d.

Of course, to construct a control Lyapunov function, there remains to construct a vector field G satisfying (13).

Theorem 8. Let the affine control system (1) be of Jurdjevic-Quinn type with respect to V_0, supposed to be homogeneous of degree d. Assume that the f_k ($k = 0, \ldots, m$) are homogeneous of degree c_k. Then the vector field G defined by (14) and (15) is homogeneous of degree 0 and satisfies the condition (13).

$$G = \sum_{i=0}^{L-1} \sum_{k=1}^{m} \lambda_{i,k} ad_{f_0}^i f_k \tag{14}$$

with $\lambda_{i,k}$ ($i = 0, \ldots, L-1; k = 1, \ldots, m$) are some functions defined by

$$\begin{cases} \lambda_{i,k} = \displaystyle\sum_{j=i}^{L-1} (-1)^{j+1} \frac{L_{ad_{f_0}^{(2j-i+1)}(f_k)} V_0}{(2V_0)^{\alpha_{j,k}}} \\ \alpha_{j,k} = \dfrac{(2j+1)c_0 + 2c_k + d}{2} \end{cases} \tag{15}$$

The readers can find the proof of these theorems in the last section.

These two theorems give a method to design a strict Lyapunov function for homogeneous systems that satisfy Jurdjevic-Quinn conditions (assumption 4) with a homogeneous function V_0 :

- First design a vector field G that satisfies the condition (13). Theorem 8 gives a "universal" method to obtain it. However, on some example, other solutions for G are simpler.
- Then compute W_λ^G according to the explicit formula (12). This function is a control Lyapunov function for λ positive and small enough. If it is possible to compute explicitly the flow of G, then one may also take V_λ^G given by (10) as a control Lyapunov function.
- Finally, design a feedback law that makes V_λ^G or W_λ^G strictly decreasing. Equations (8) and (9) give a possibility to obtain it.

4. EXAMPLE: THE RIGID BODY

4.1 *Problem statement*

The problem of stabilization of the velocity of a rigid body with two torques has already been studied in the literature (see (Brockett, 1983; Aeyels, 1985; Morin, 1996)). We study the following model :

$$\begin{cases} \dot{\omega}_1 = c_1\omega_2\omega_3 + \dfrac{\tau_1}{j_1} \\ \dot{\omega}_2 = c_2\omega_1\omega_3 + \dfrac{\tau_2}{j_2} \\ \dot{\omega}_3 = c_3\omega_1\omega_2 \end{cases} \tag{16}$$

with $c_1 = \frac{j_2 - j_3}{j_1}$, $c_2 = \frac{j_3 - j_1}{j_2}$, $c_3 = \frac{j_1 - j_2}{j_3}$

A preliminary feedback is proposed in (Morin, 1996) to make the system satisfy the Jurdjevic-Quinn hypotheses.

$$\begin{cases} \tau_1 = j_1(-c_1\omega_2\omega_3 + \beta c_3\omega_1\omega_2) + k_1 \\ \tau_2 = j_2(-(c_2 + \mu c_3)\omega_1\omega_3) + k_2 \end{cases}$$

We obtain the new system :

$$\dot{\omega} = f_0(\omega) + k_1 f_1 + k_2 f_2 \tag{17}$$

$$f_0 = \begin{bmatrix} \beta c_3\omega_1\omega_2 \\ -\mu c_3\omega_1\omega_3 \\ c_3\omega_1\omega_2 \end{bmatrix}, \ f_1 = \begin{bmatrix} \frac{1}{j_1} \\ 0 \\ 0 \end{bmatrix}, \ f_2 = \begin{bmatrix} 0 \\ \frac{1}{j_2} \\ 0 \end{bmatrix}.$$

This system is of Jurdjevic-Quinn type with the positive definite and radially unbounded function V_0 defined by

$$V_0 = \frac{1}{2}(\omega_1 - \beta\omega_3)^2 + \frac{1}{2}\omega_2^2 + \frac{\mu}{2}\omega_3^2 \quad (18)$$

In fact it is easy to verify that the set $\mathcal{W}_1(V_0)$ is reduced to $\{0\}$. Using Theorem 5, one obtains, as in (Morin, 1996), a stabilizing control law. We are going to apply our method to design a control Lyapunov function as a deformation of V_0.

4.2 *Our method*

The system (17) is a Jurdjevic-Quinn type system with an homogeneous non strict Lyapunov function V_0. In addition the vector fields f_0, f_1 et f_2 are homogeneous of degree 1, -1 et -1. As aconsequence we are able to design a strict Lyapunov function (this is new to our knowledge).

Design of G.
One can use theorem 8 to conceive the vector field G. But we have find a more simple G that satisfies the condition (13). The verification is left to the reader.

$$\forall \omega \in I\!R^3,\ G(\omega) = \begin{bmatrix} 0 \\ \dfrac{\beta\mu c_3\omega_3^2}{(2V_0)^{\frac{1}{2}}} \\ 0 \end{bmatrix} \quad (19)$$

Design of the control Lyapunov function.
The flow of G is not easy to compute explicitly. Formula (12) yields the following control Lyapunov function for λ that satisfies $0 < \lambda < \frac{1}{\beta c_3}$:

$$W_\lambda^G = \frac{1}{2}(\omega_1 - \beta\omega_3)^2 + \frac{1}{2}\left(\omega_2 + \lambda\frac{\beta\mu c_3\omega_3^2}{\sqrt{2V_0}}\right)^2 + \frac{\mu}{2}\omega_3^2$$

Design of the new control law.
The equation (8) and (9) give a stabilizing feedback law for the system:

$$\begin{cases} k_1 = -\alpha\|\omega\| L_{f_1} W_\lambda^G \\ k_2 = -\alpha\|\omega\| L_{f_2} W_\lambda^G \end{cases} \quad (20)$$

with $\|.\|$ an homogeneous norm and α sufficiently large:

$$\alpha > \sup_T \frac{L_{f_0} W_\lambda^G(\omega)}{((L_{f_1} W_\lambda^G(\omega))^2 + (L_{f_2} W_\lambda^G(\omega))^2)\|\omega\|}$$

where $T = \{\omega,\ \|\omega\| = 1,\ L_{f_0} W_\lambda^G(\omega) \geq 0\}$.

4.3 *Robustness to misalignment of actuators*

Our method does not a priori improve the robustness of the control law given in (Morin, 1996) but it gives a tool (the negative definite function $\dot{W}_L^G$) to analyse it. Let us quantify, for instance the tolerable errors on actuators misalignment with the principal axes. We consider the initial system (16):

$$\dot{\omega} = X_0(\omega) + \tau_1 f_1 + \tau_2 f_2 \quad (21)$$

with (τ_1, τ_2) defined by the addition of the preliminary feedback and the control law we have made (k_1, k_2).

Let φ_1 and φ_2 be angle between the position of the torques and the principal axes. That means the vector fields f_1 and f_2 of the system (21) are given by :

$$f_1 = \frac{1}{j_1}\begin{bmatrix} cos(\varphi_1) \\ sin(\varphi_1)cos(\theta_1) \\ sin(\varphi_1)sin(\theta_1) \end{bmatrix} \quad f_2 = \frac{1}{j_2}\begin{bmatrix} sin(\varphi_2)cos(\theta_2) \\ cos(\varphi_2) \\ sin(\varphi_2)sin(\theta_2) \end{bmatrix}$$

Then we have :

$$\begin{aligned} \dot{W}_\lambda^G &= L_{f_0} W_\lambda^G + k_1 L_{\hat{f}_1} W_\lambda^G + k_2 L_{\hat{f}_2} W_\lambda^G \\ &+ \tau_1 L_{f_1 - \hat{f}_1} W_\lambda^G + \tau_2 L_{f_2 - \hat{f}_2} W_\lambda^G \end{aligned}$$

We defined three constants β_1, β_2 and β_3 :

$$\begin{cases} -\beta_1 = \sup\limits_{\|\omega\|=1} L_{f_0} W_\lambda^G + k_1 L_{\hat{f}_1} W_\lambda^G + k_2 L_{\hat{f}_2} W_\lambda^G < 0 \\ \beta_2 = \sup\limits_{\|\omega\|=1} \tau_1 \|\dfrac{\partial W_\lambda^G}{\partial \omega}\| > 0 \\ \beta_3 = \sup\limits_{\|\omega\|=1} \tau_2 \|\dfrac{\partial W_\lambda^G}{\partial \omega}\| > 0 \end{cases}$$

Then we deduce :
$\dot{W}_\lambda^G \leq -\beta_1\|\omega\|^3 + \beta_2\|\omega\|^3 sin^2(\frac{\varphi_1}{2}) + \beta_3\|\omega\|^3 sin^2(\frac{\varphi_2}{2})$
Consequently, if φ_1 and φ_2 are small enough to satisfy the following equation,

$$\beta_2 sin^2(\frac{\varphi_1}{2}) + \beta_3 sin^2(\frac{\varphi_2}{2}) < \beta_1$$

then the origin remains an asymptotically stable equilibrium.

5. CONCLUSION

Theorems 7 and 8 give a method to design a control Lyapunov function when non strict Lyapunov function has been used to prove asymptotic stability of an equilibrium point. This method is applied to Jurdjevic-Quinn system which satisfy some hypotheses of homogeneity. For non homogeneous system some preliminary results has been obtained in (Faubourg, 1997), but we are not able to design a control Lyapunov function. In order to extend the method to non homogeneous system, it would be necessary to find conditions on the vector field G that express more precisely the fact that V_λ^G or W_λ^G are control Lyapunov functions.

Moreover, we have established that the functions V_λ^G or W_λ^G are control Lyapunov functions for enough small λ. This is not surprising because the condition (13) imposed on the vector field G express a good condition at $\lambda = 0$. An extension would be to find other conditions on G to make these functions be control Lyapunov function for large λ.

Appendix A. APPENDIX : PROOFS.

A.1 *Proof of theorem 7*

Let us first establish two lemmas. W_λ^G and V_λ^G are, of course, defined by (10) and (12).

Lemma 9. For all x in $I\!R^n$ and all $\lambda > 0$, we have :

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} \left(L_{f_0} V_\lambda^G\right)(x) = L_{f_0} L_G V_0(x) \quad \text{(A.1)}$$
$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} \left(L_{f_0} W_\lambda^G\right)(x) = L_{f_0} L_G W_0(x) \quad \text{(A.2)}$$

Proof : Let the function χ be defined by $\chi(t,\lambda,x) = V_0(\phi_G^\lambda(\phi_{f_0}^t(x)))$. From the definition (10) of V_λ and the definition of the Lie derivative ($L_{f_0}\psi(\xi) = \frac{d}{dt}\big|_{t=0}\psi(\phi_{f_0}^t(\xi))$ for any function ψ and point ξ), the left-hand side of (A.1) is equal to $\frac{\partial^2}{\partial\lambda\partial t}$. On the other hand, applying twice the same definition of the Lie derivative (along G and along f_0), the right-hand side is equal to $\frac{\partial^2}{\partial t\partial\lambda}$. This proves (A.1) because the partial derivatives commute. Relation (A.2) is true for the same reasons because, for any function ψ and any point ξ, $\psi(\phi_G^\lambda(\xi))$ and $\psi(\xi+\lambda G(\xi))$ nave the same partial derivative with respect to λ at $\lambda = 0$. ∎

Lemma 10. If V_0 is homogeneous of degree d and the vector field G is homogeneous of degree zero, the functions W_λ^G and V_λ^G are homogeneous of degree d for all $\lambda \geq 0$.

Proof : Since G is homogeneous of degree zero, we have $\delta_\mu * G = G$ for all $\mu > 0$. On the other hand, the flow of $\delta_\mu * G$ is the conjugate of the flow of G :

$$\phi_{\delta_\mu * G}^t \quad = \quad \delta_\mu \circ \phi_G^t \circ (\delta_\mu)^{-1}$$

Clearly these two relations imply that the dilation δ_μ commutes with the flow ϕ_G^t. This implies that, for all μ and λ,
$V_\lambda^G \circ \delta_\mu \;=\; V_0 \circ \phi_G^\lambda \circ \delta_\mu \;=\; V_0 \circ \delta_\mu \circ \phi_G^\lambda \;=\; \mu^d\, V_\lambda^G$
This precisely means that V_λ^G is homogeneous of degree d. The same property holds for W_λ^G because G homogeneous of degree zero implies that $G(\delta_\mu(x)) = \delta_\mu(G(x))$ for all x (this formula only makes sense in the coordinates where the definition (12) was written). ∎

Now, we are going to give the proof that V_λ^G is a control Lyapunov function. The proof that W_λ^G satisfies the same property follows exactly along the same lines because only the properties from the above two lemmas are used.

We introduce some new notations to simplify the proof :

$$\begin{aligned} \Gamma &= \{x,\ L_{f_1}V_0(x) = \ldots = L_{f_m}V_0(x) = 0\} \\ E &= \{x,\ L_{f_0}L_G V_0(x) \geq 0\} \\ S &= \{x, \|x\| = 1\} \end{aligned}$$

the norm in the definition of S is the homogeneous norm . Clearly, condition (13) is equivalent to :

$$\Gamma \cap E \quad = \quad \{0\} \qquad \text{(A.3)}$$

Hence $\Gamma \cap S$ and $E \cap S$ are two disjoint compact sets, that do not intersect, so that the distance between them is strictly positive. Let q' be such that $0 < q' < d\,(E \cap S, \Gamma \cap S)$, and define the set

$$C_2 = \{x \in S,\ d(x, \Gamma \textstyle\bigcap S) \leq q'\}\ . \qquad \text{(A.4)}$$

It obviously has an empty intersection with E.

Let us now finish the proof in three steps :
First, we prove that that there exists a $\lambda_1 > 0$ s.t ,

$$\left.\begin{array}{r} 0 < \lambda < \lambda_1 \\ x \in C_2 \end{array}\right\} \implies \quad L_{f_0}V_\lambda^G(x) < 0 \qquad \text{(A.5)}$$

Define the function χ from $I\!R \times I\!R^n$ by $\xi(\lambda, x) = L_{f_0}V_\lambda^G(X)$. It is continuously differentiable at least away from $x = 0$, and in particular on $[0,+\infty) \times C_2$. Let $-\beta = \max_{x\in C_2}(L_{f_0}L_G V_0(x))$. β is positive because the set C_2 is compact and $(L_{f_0}L_G V_0)$ is negative on C_2 (because $C_2 \cap E = \emptyset$). From lemma 9, this implies $\max_{x\in C_2} \dfrac{\partial\chi}{\partial\lambda}(0,x) = -\beta$, but $\lambda \mapsto \max_{x\in C_2} \dfrac{\partial\chi}{\partial\lambda}(\lambda,x)$ is continuous on $[0,+\infty)$, and consequently it remains larger smaller that $-\frac{\beta}{2}$ on a certain interval $[0,\lambda_1]$ with $\lambda_1 > 0$. By integrating with respect to λ on the interval $[0,\lambda_1]$ and taking into account the fact that, from (5), $\chi(0,x)$ is identically zero, one obtains that $\chi(\lambda,x) < -\frac{\beta\lambda}{2}$ for (λ,x) in $[0,\lambda_1] \times C_2$. This clearly proves (A.5).

Second, we prove that, defining for all $\lambda \geq 0$ the set Γ_λ by $\Gamma_\lambda = \{x \in S\ ,\ L_{f_k}V_\lambda^G(x) = 0; k = 1\ldots m\}$, there exists a $\lambda_2 > 0$ s.t,

$$0 < \lambda < \lambda_2,\ \Rightarrow\ \Gamma_\lambda \subset C_2 \qquad \text{(A.6)}$$

Let us prove that by contradiction. If this assertion is not true. Then

$$\forall\lambda_2 > 0,\ \exists\lambda, 0 < \lambda < \lambda_2\ ,\ \exists x \in S \backslash C_2,$$
$$L_{f_1}V_\lambda^G(x) = \ldots = L_{f_m}V_\lambda^G(x) = 0$$

For example, we choose $\lambda_2 = \frac{1}{n}$, hence we can design a suite (λ_n) and a suite (x_n) such that

$$\begin{cases} \lambda_n < \dfrac{1}{n} \\ x_n \in S\backslash C_2 \\ L_{f_1} V^G_{\lambda_n}(x_n) = \ldots = L_{f_m} V^G_{\lambda_n}(x_n) = 0 \end{cases}$$

$\overline{(S\backslash C_2)}$ is compact. Consequently we can define a suite (y_n) extracted from (x_n) such that this suite tends to y in $\overline{(S\backslash C_2)}$. $(x, \lambda) \longrightarrow L_{f_k} V^G_\lambda(x)$ is a continuous function. Consequently, $L_{f_k} V_0(y) = 0$, and y belongs to Γ which is included in C_2. It is impossible for y to below to $\overline{(S\backslash C_2)}$ and to the interior of C_2. Consequently the equation (A.6) is true.

To finish the proof, define λ_0 by $\lambda_0 = \min(\lambda_1, \lambda_2)$, and let us prove that for all (λ, x) such that $0 < \lambda \leq \lambda_0$, $x \neq 0$, and

$$L_{f_1} V^G_\lambda(x) = \ldots = L_{f_m} V^G_\lambda(x) = 0 \quad \text{(A.7)}$$

we have $L_{f_0} V^G_\lambda(x) < 0$. This obviously implies that V^G_λ is a control Lyapunov function if $0 < \lambda \leq \lambda_0$. Since $x \neq 0$, there exists $y \in S$ and a real number $\mu > 0$ such that $x = \delta_\mu(y)$. From the homogeneity properties of the vector fields f_k (assumptions of the theorem) and the function V^G_λ (lemma 10), each function $L_{f_k} V^G_\lambda$ is homogeneous of degree $c_k + d$, and hence (A.7) implies :

$$L_{f_1} V^G_\lambda(y) = \ldots = L_{f_m} V^G_\lambda(y) = 0$$

Hence y is in Γ_λ, and fact that $0 < \lambda < \lambda_2$ plus property (A.6) imply that y is in C_2. Since $0 < \lambda < \lambda_1$, the property (A.5) implies $L_{f_0} V^G_\lambda(y) < 0$. Then the homogeneity of $L_{f_0} V^G_\lambda$ implies that $L_{f_0} V^G_\lambda(x) < 0$ as well and gives the result. ∎

A.2 *Proof of theorem 8*

A simple computation proves, using (14) and (15), that for all x in $I\!R^n$,

$$\begin{aligned} L_{f_0} L_G V_0(x) &= -\sum_{i=1}^{L} \sum_{k=1}^{m} (L_{ad^i_{f_0}(f_k)} V_0(x))^2 \\ &\quad + \sum_{k=1}^{m} L_{f_0} \lambda_{0,k}(x) L_{f_k} V_0(x) \end{aligned}$$

Hence both equation (6) and previous equality imply that for all x in $I\!R^n \backslash \{0\}$,

$$\left. \begin{aligned} L_{f_1} V_0(x) &= 0 \\ &\vdots \\ L_{f_m} V_0(x) &= 0 \end{aligned} \right\} \Longrightarrow L_{f_0} L_G V_0(x) < 0$$

It follows that the condition (13) is satisfied by the vector field G. One can easily verify that G is also homogeneous of degree 0. ∎

Appendix B. REFERENCES

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