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Remarks on sufficient information for adaptive nonlinear regulation

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Abstract

The present work is an attempt to design "universal regulators" for nonlinear systems under some minimal informations. It presents a controller which is a generalisation of one proposed previously for the linear case, and based on the knowledge of a finite dimensionnal family of controllers one of which is known to be a stabilizing one.

1 Problem statement

Consider a control system

$$\dot{x} = f(x,u) \tag{1}$$

with state $x \in I\!R^n$ and input $u \in I\!R^m$. We assume that the complete state x is available for control (i.e. is measured and can be taken as the input of a controller), but we consider that the map $f : I\!R^n \times I\!R^m \longrightarrow I\!R^n$ itself is not known explicitly. For the sake of simplicity, we suppose that it is smooth.

The objective is to design a dynamic controller

$$\begin{array}{rcl} \dot{\zeta} & = & \text{a certain function of } \zeta \text{ and } x \\ u & = & \text{a certain (other) function of } \zeta \text{ and } x \end{array} \tag{2}$$

where ζ is the state of this controller, such that this controller "regulates" x in the sense that the closed-loop system (its state is (x,ζ)) has the following property : for any initial condition, $(x(t),\zeta(t))$ is bounded and $x(t)$ goes to zero.

Of course, this is absolutely impossible without any information at all on the function f : if it were possible, the obtained controller would achieve the desired objective for *any* f, and, for example, would also work in the case $f(x,u) = x$, i.e. render the solutions of $\dot{x} = x$ bounded.

An interesting question is : what information on f is necessary to achieve the desired objective ? First of all, to avoid absurd situations as the one mentioned above ($f(x,u) = x$), it is necessary to be certain that f is such that the objective could be achieved if it were known, i.e. there exists a controller achieving the objective at least for this precise f. It is however very doubtfull that the following information on f: "f is smooth and there exists a controller of the form (2) which regulates x" is sufficient, i.e. it seems there most probably exists no controller of the form (2) satisfying the objective for *any* possible f satisfying these two properties. Note that this non-existence is only a conjecture.

A more reasonable framework is the case where not only is it known that a controller exists for f, but also this controller is known to belongs to a **known finite-dimensional** family of controllers. In addition, and to simplify the statements, we shall assume that this is a family of static controllers :

$$\left.\begin{array}{l} \text{There exists a } \textbf{known} \text{ smooth map } U \\ \text{from } I\!R^r \times I\!R^n \text{ to } I\!R^m \\ \text{such that, for a certain } \theta^\star \text{ in } I\!R^r, \\ 0 \text{ is an asymptotically stable equilibrium} \\ \text{of } \dot{x} = f(x, U(\theta^\star, x)). \end{array}\right\} \tag{3}$$

In the linear case, i.e. if f is a linear mapping, (3) is no more restrictive than the existence of a controller of the form (2) which makes x go to zero since this is equivalent to the existence of a *linear static* controller $u = K^\star x$ ($K^\star$ being an $n \times m$ matrix) making 0 an asymptotically stable equilibrium of $\dot{x} = f(x, K^\star x)$: then K plays the role of θ, r is nm and $U(K,x) = Kx$. It is well known (see e.g. [5]) that, in the linear case, i.e. if one adds to (3) the –important– information that f is linear, then an adaptive controller of the form (2) may be constructed.

Of course, if we remove the information "f is a linear mapping", we are left with a more general and difficult problem, which is actually the problem of nonlinear adaptive control, stated in a "direct" manner. See section 2 for an equivalent "indirect" setting, which is the one usually adopted in nonlinear adaptive control. One may ask the question : is the knowledge of the mapping U satisfying (3) sufficient to build a controller of the form (2) ? To our knowledge, no answer to this question is known, and it is of course a question of interest to control theory to find an answer and, if it is negative, to understand what information can be added to make such a construction possible ? or, ideally, what is the "minimal information" to be added ?

This problem has never really be tackled, and all the studies in nonlinear adaptive control consist in finding a controller of the form (2) in a "realistic" situation, where far more information than (3) is available; of course from an engineering point of view, this is quite important since it brings some practical solution for many situations. However, even from the engineering point of view, some necessary conditions, i. e. some informations on the systems proved to be necessary for regulation would also have much interest.

As mentionned above, the linear case in this state-space setting is well-known; with output measurement, solutions have long been known only under some minimum-phase assumptions (see e.g. [5]), but the question of finding the minimal information necessary has been studied, looking for "universal controllers". In [3], it is proved that an upperbound on the order of a (dynamic linear) stabilizing controller (note that this implies the knowledge of a finite dimensionnal of controllers one of which is a stabilizing one) is sufficient information for regulation.

In the present paper, we give a "generalisation" of the controller proposed in [3] to the nonlinear case, but with full state measurement. Of course, we have to add some more information to (3), and we have no idea if they are necessary, but at least, they are not at all of the same nature as the ones usually considered. In particular, no assumption on linearity in the parameters is made. Of course, this does not give any necessary conditions for adaptive regulation. The controller given in [3] was an answer to the question of necessary informationtion, in the wold of linear systems : there is no necessary condition beside an upperbound ofthe order of a stabilizing controller; we do not bring such a result to adaptive nonlinear control, but only a first attempt to build a "universal controller".

2 The indirect setting

Note that the above formulation of the adaptive regulation problem is purely "direct" in the sense that no parametrization of the *system* is considered. A more classical –and "indirect"– formulation of this problem is to suppose that the system (1) to be controlled belongs to a known finite-dimensional family of systems

$$\dot{x} \quad = \quad F(p,x,u) \quad , \tag{4}$$

parametrized by $p \in \mathbb{R}^l$. This means that for a certain $p^\star$ in $\mathbb{R}^l$,

$$F(p^\star,x,u) \quad = \quad f(x,u) \quad \forall x \, \forall u \quad . \tag{5}$$

The traditionnal framework is then to be in a precise situation where (as in [1, 2, 4, 8]), or to suppose that (as in [6, 7]), there exists, for all p, a controller which stabilizes (4), and that this controllers depends smoothly on p, which means that :

$$\left.\begin{array}{l}\text{There is a } \textbf{known} \text{ smooth map } u_{\rm nom} \\ \text{from } \mathbb{R}^l \times \mathbb{R}^n \text{ to } \mathbb{R}^m \text{ such that, for all } p \text{ in } \mathbb{R}^l, \\ 0 \text{ is an asymptotically stable equilibrium} \\ \text{of } \dot{x} = F(p,x,u_{\rm nom}(p,x)).\end{array}\right\} \tag{6}$$

Of course, this implies (3) with $l = r$, $\theta = p$, $\theta^\star = p^\star$ and $U = u_{\rm nom}$. Note that this is more than (3) where nothing is assumed for θ different from $\theta^\star$. With the exception of [8], which actually only needs the equilibium point of $\dot{x} = F(p,x,u_{\rm nom}(p,x))$ to be stable when $p = p^\star$, (6) is met in all the situations where a nonlinear adaptive controller is known to work.

Here, we only make use of the information (3), which is implied by information (6) from the above remarks. The knowledge of the function F is simply ignored in the controller we are giving here.

In all the situations where a solution is known, the function F depends linearly in p; of course, since we even do not condider F, we do not have to make such an assumption.

Let us mention that, in this framework, our problem can be seen as a control problem with output measurement since it may be rewritten $\dot{x} = F(p,x,u)$, $\dot{p} = 0$, where the state is (x,p) and the ouput is x, the state p being unmeasured: its initial condition $p^\star$ is unknown. Note that some necessary and sufficient conditions for the existence a (very abstract) stabilizing dynamic output "controller" are given in [9]. His conditions are not necessary here because we do not require convergence of the extra dynamics (ζ) whereas by stabilization, he means convergence of the total closed-loop state (here (x,ζ)).

3 Assumptions

Actually, we shall need some more restrictive requirements on the finite-dimensionnal family of controllers than the one in (3) :

Information 1 : *A map*

$$U \; : \; \mathbb{R}^r \times \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad (7)$$

is **known** *such that there exists an open subset* $\Omega^\star$ *of* $\mathbb{R}^r$ *and two functions* V *and* ρ *from* $\mathbb{R}^n$ *to* $[0,+\infty)$ *such that*

$$V(x) = 0 \quad \Leftrightarrow \quad x = 0 \qquad (8)$$

$$\rho(x) = 0 \quad \Leftrightarrow \quad x = 0 \qquad (9)$$

$$\{x\,,\; V(x) < M\} \text{ is bounded} \qquad \forall M > 0 \quad (10)$$

$$\frac{\partial V}{\partial x}(x)\, F(x, U(\theta, x)) \;\leq\; -\,\rho(x) \qquad \begin{array}{l}\forall \theta \in \Omega^\star \\ \forall x \in \mathbb{R}^n\end{array} \quad (11)$$

This contains a rather restrictive assumption, i.e., compared with the formulation (3), the global stability of the equilibrium point in $\dot{x} = f(x, U(\theta^\star, x))$ is robust in the sense that it is kept when $p^\star$ is changed into $p^\star + \varepsilon$ with ε small enough. This is obviously true in the linear case since when you change slightly the coefficients of a linear stabilizing controller, you obtain a controller which is still stabilizing.

Recall that U is assumed to be known, but not the functions ρ and V. However, we require the following bounds, which are indirect information on the functions ρ, V and f.

Information 2 : *Two positive functions* μ *and* R *are known and have the following properties :*

$$\left. \begin{array}{l} R(x) = 0 \quad \Leftrightarrow \quad x = 0 \\ \{x\,,\; R(x) < M\} \text{ is bounded for any } M > 0 \end{array} \right\} \qquad (12)$$

and, for all x *and* u*,*

$$\|\frac{\partial V}{\partial x}(x).f(x,u)\| \;\leq\; c_1\, \mu(x,u) \qquad (13)$$

$$R(x) \;\leq\; c_2\, \rho(x) \qquad (14)$$

$$\mu(x, U(\theta, x)) \;\leq\; c_3\, \rho(x) \quad \forall \theta \in \Omega^\star \qquad (15)$$

Note that (15) should be understood as : "there exists an open subset of $\Omega^\star$ such that $\mu(x, U(\theta, x)) \leq c_3\, \rho(x)$ for θ in this open set"; this open set is still called $\Omega^\star$.

Note also that, in the linear case, since f is a linear mapping, and V may always be take a quadratic form in x, μ may always be chosen to be $\mu(x,u) = \|x\|^2 + \|u\|^2$ and R to be $R(x) = \|x\|^2$.

This paper does not contain a comparison with the usual conditions for adaptive nonlinear control. The only very clear difference is that all the known algorithms require a linear parametrization of F in the indirect setting described in section 2, and we have no requirement of this nature. An interesting question is, of course : in all the situations where a solution is known, is there enough information available to build a U, a μ and a R as above ?

4 The controller and main result

Let us define a curve

$$\Theta \; : \; [0, +\infty[\longrightarrow \mathbb{R}^r \qquad (16)$$

with bounded "speed" :

$$\frac{\partial \Theta}{\partial h}(h) \;\leq\; 1 \quad \forall h \geq 0 \qquad (17)$$

which has the property to be dense in $\mathbb{R}^r$ in the following sense :

$$\forall \theta \in \mathbb{R}^r,\ \forall \varepsilon > 0,\ \forall H > 0,\ \exists h > H \text{ such that } \|\Theta(h) - \theta\| < \varepsilon\,. \qquad (18)$$

An explicit construction of such a curve Θ is given in [3].

Now, let us define

$$h \; : \; [0, +\infty[\longrightarrow [0, +\infty[\qquad (19)$$

such that

$$\frac{\partial h}{\partial k} \;>\; 0\,, \qquad (20)$$

$$\frac{\partial^2 h}{\partial k^2} \;<\; 0\,, \qquad (21)$$

$$h(0) \;=\; 0\,, \qquad (22)$$

$$\lim_{k \to +\infty} h(k) \;=\; +\infty\,, \qquad (23)$$

$$\lim_{k \to +\infty} k\,\frac{\partial h}{\partial k}(k) \;=\; 0\,. \qquad (24)$$

An explicit choice for h which meets all these requirements is

$$h(k) \;=\; \mathrm{Log}(1 + \mathrm{Log}(1 + k))\,. \qquad (25)$$

Let now the controller be

$$u \;=\; U(\Theta(h(k)), x) \qquad (26)$$

$$\dot{k} \;=\; \mu(x,u) + R(x,u) \qquad (27)$$

It is clear that the only information on the system used in this controller is the knowledge of some functions U, R and μ such that the properties given in "information" 1 and 2. These informations happen to be enough for regulation :

Theorem 1 *Controller (26)-(27) achieves regulation of* x *in the sense that, on any solution* $(x(t), k(t))$ *of the closed-loop system (26)-(27)-(1),* $x(t)$ *and* $k(t)$ *are bounded and* $x(t)$ *goes to zero.*

Appendix: Proof of theorem 1

The following proof is a generalization to the nonlinear case of the one given in [3]. When possible, we have kept the same notations as in that paper.

Since $\dot{k}$ is positive, k either goes to infinity or tends to a finite limit k_∞. The proof is in two steps. First, we prove that if k tends to a finite limit k_∞, x goes to zero, and then we prove by contradiction that k cannot go to infinity.

Step 1 :

Let us suppose that

$$\lim_{t\to+\infty} k(t) \;=\; k_\infty \;<\; +\infty \,. \tag{28}$$

From (13) and (27), we have, for all $t > 0$,

$$\dot{V} \;\leq\; c_1\,\dot{k} \tag{29}$$

and therefore

$$V(t) \;\leq\; V(0) \;+\; k_\infty \;-\; k(0) \quad \forall t > 0 \,. \tag{30}$$

This implies, from (10), that x is bounded. In addition, since $R \leq \dot{k}$, the function $R(t)$ is in $L^1([0,+\infty[)$; since $\dot{R}(t)$ is obviously bounded (it can be expressed as a continuous function of x and k, which are bounded), we may deduce that $R(t)$ goes to zero and therefore, from (12), that x goes to zero.

Step 2 :

Let us prove by contradiction that k does go to a finite limit k_∞. For this, we suppose that k goes to infinity. This implies, from (23), that h goes to infinity and therefore that $\Theta(h(k))$ goes all the way on the dense curve. Let $O^\star$ be an open set *strictly* contained in $\Omega^\star$. (18) implies that $\Theta(h(k(t)))$ goes infinitely many times in and out of $O^\star$ and therefore that there exists infinitely many time-intervals

$$I_\nu \;=\;]\,t_\nu^-\,,\,t_\nu^+\,[\qquad \nu \in I\!N \tag{31}$$

such that $\theta(h(k(t)))$ enters $\Omega_\star$ at time t_ν^-, exits at time t_ν^+, and goes inside $O^\star$ at one time in the interval, i.e., denoting $\Theta(h(k(t)))$ by $\Theta(t)$ for short,

$$\begin{array}{c} t \in I_\nu \;\Rightarrow\; \Theta(t) \in \Omega^\star \\ \exists t \in I_\nu\,,\; \Theta(t) \in O^\star \\ \Theta(t_\nu^-) \notin \Omega^\star\,; \quad \Theta(t_\nu^+) \notin \Omega^\star \,. \end{array} \tag{32}$$

Since $\frac{\partial\Theta}{\partial h}$ is bounded, and the length of the curve described by Θ while t is in I_ν is at least twice the distance between $O^\star$ and outside $\Omega^\star$, there exists a positive number D (which may depend on the trajectory but does not depend on ν) such that

$$h(t_\nu^+) \;-\; h(t_\nu^-) \;>\; D \;>\; 0 \,. \tag{33}$$

Now, let us compute $h(t_\nu^+) - h(t_\nu^-)$ in another manner. Since

$$h(t_\nu^+) \;-\; h(t_\nu^-) \;=\; \int_{t_\nu^-}^{t_\nu^+} \frac{\partial h}{\partial k}\,\dot{k}\,\mathrm{d}t \;, \tag{34}$$

we have, from (27), (14), (15) and (21),

$$h(t_\nu^+) - h(t_\nu^-) \;\leq\; \frac{\partial h}{\partial k}(k(t_\nu^-)) \int_{t_\nu^-}^{t_\nu^+} (\,c_2 + c_3\,)\,[-\dot{V}(t)]\,\mathrm{d}t \;, \tag{35}$$

which yields, since $V(t)$ is non-increasing,

$$h(t_\nu^+) \;-\; h(t_\nu^-) \;\leq\; (c_2+c_3)\,\frac{\partial h}{\partial k}(k(t_\nu^-))\,V(t_\nu^-) \tag{36}$$

and finally, since, from (13) and (27),

$$V(t_\nu^-) \;\leq\; V(0) \;+\; c_1\,(\,k(\,t_\nu^-) \;-\; k(0)\,)\;, \tag{37}$$

$$\begin{array}{rl} h(t_\nu^+) \;-\; h(t_\nu^-) \;\leq\; (c_2+c_3)\,\frac{\partial h}{\partial k}(k(t_\nu^-))\,[\,V(0) & \\ -\,c_1 k(0) \;+\; c_1\,(k(\,t_\nu^-)\,] \,. \end{array} \tag{38}$$

This proves, from (24), that $h(t_\nu^+) - h(t_\nu^-)$ goes to zero when ν goes to infinity. This is a clear contradiction with (33). $k(t)$ cannot therefore be unbounded on any trajectory. This ends the proof of theorem 1. ∎

5 Conclusion

We have given here an adaptive controller, which is of a different nature as the ones usually designed for some specific situations. The required assumptions are also of a rather different nature. We hope that this contribution initiates some work on the minimum knowledge necessary for regulation.

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