30th IEEE Conf. on Decision and Control (CDC 1991), vol. 2, pp. 1811-1812
http://dx.doi.org/10.1109/CDC.1991.261722

Some Results on Dynamic Output Feedback Regulation of Nonlinear Systems

W. Cebuhar, R. Hirschorn and J.-B. Pomet
Queen's University at Kingston
Kingston, Ontario K7L 3N6 Canada

Abstract

In this paper preliminary results are presented on the problem of regulating nonlinear systems by output feedback, using Lyapunov based techniques. Sufficient conditions for the global stabilization of the observed states via dynamic output feedback are obtained, assuming that such stabilization is possible using full state feedback. Some simple examples are included to illustrate our approach.

1 Introduction

This paper is concerned with the problem of output feedback regulation of nonlinear systems. Many authors have considered such problems, using various approaches [4]-[7].

Some necessary and sufficient conditions for regulation via static output feedback are established in Tsinias and Kalouptsidis [7], by extending previous results of Arstein [1] and Sontag [6]. However, these conditions are not explicit and involve the existence of a special "control Lyapunov function." Furthermore, as it is well known, most feedback stabilizable systems are not static output feedback stabilizable (e.g. $\dot{x}_1 = x_2,\ \dot{x}_2 = u,\ y = x_1$).

For linear systems, a more general problem has been solved, among others, by Francis [2]. These results were extended to nonlinear systems by Isidori and Byrnes [4], for the case in which the nonobserved dynamics evolve independently, as a so-called "exosystem". This exosystem is assumed to be Poisson stable. In addition, for the case in which the state of the exosystem is not available to the controller, detectability of the linear approximation of the combined system is required.

The most "natural" approach to output control is to try to build an observer. But for nonlinear systems it is unclear how this can be done. Even in the cases where it is possible to design an observer, it might not solve the problem of output control because the "separation principle", which is valuable in the linear case, does not hold in general. In [3], Gauthier and Kupka have proved that this principle holds for a certain class of bilinear systems and very particular observers.

We, on the other hand, follow an approach that does not involve explicitly building an observer. Out main assumptions are that the observed states are stabilizable by full state feedback and that the nonobserved states enter the system equations linearly. These assumptions, together with some more technical Lyapunov conditions allow us to obtain global results. Our methods are an extension of those commonly used in adaptive stabilization (see, for example [5]). It should be noted that nonlinear adaptive stabilization is a particular case of our problem, where the nonobserved states are constant (i.e. unknown parameters).

2 Problem Statement

We consider nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \ ; \ x \in \mathbb{R}^n \\ y &= h(x) \end{aligned}$$

which, in new coordinates $(x_1 = h(x), x_2)$ can be written as

$$\begin{aligned} \dot{x}_1 &= a(x,u) + A(x_1,u)x_2 \\ \dot{x}_2 &= b(x_1,u) + B(x_1,u)x_2 \\ y &= x_1 \end{aligned} \tag{1}$$

where $x_1 \in \mathbb{R}^k$, $x_2 \in \mathbb{R}^\ell$, $u \in \mathbb{R}^m$, $n = k + \ell$, and a, A, b and B are smooth.

Our aim is to have $y = x_1$ go to zero, and x remain bounded, on any trajectory, the only available measurement being y.

We will assume that this would be possible if the full state x were measured, i.e. that there exists a control law $u = u_{\text{nom}}(x_1, x_2)$ such that the closed loop system $\dot{x} = f(x) + g(x)u_{\text{nom}}(x_1, x_2)$ satisfies $x_1 \to 0$ and x bounded. We make this precise the following way:

Basic Assumption A1

There exist two positive functions $V_1(x)$ and $V_2(x)$ such that

1) $V_1'(x)[f(x) + g(x)u_{\text{nom}}(x)] \le -\rho(x)$ where $\rho(x) \ge 0$ and is 0 if and only if $h(x)$ is zero.

2) $V_2'(x) \cdot f(x) \le 0 \quad$ and $\quad V_2'(x) \cdot g(x) \equiv 0 \quad \forall x$.

3) the function V from $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$V(x) = V_1(x) + V_2(x)$$

is proper (i.e. the preimage of a compact set is compact).

Actually, we will further restrict ourselves as follows:

Assumption A2

There exists a positive definite $\ell \times \ell$ matrix Q such that, for any x_1 and u,

$$B(x_1,u)^T Q + Q B(x_1,u)$$

is symmetric negative semidefinite.

Assumption A3

The function $V_1(x)$ has the following form:

$$V_1(x) = U_1(x_1) + \frac{1}{2} x_2^T M x_2$$

where U_1 is smooth, M is symmetric positive semidefinite, and $B(x_1,u)^T M + M B(x_1,u)$ is negative semidefinite for all x_1 and u.

3 Main Result

Consider the dynamic controller:

$$u = u_{\text{nom}}(x_1, \hat{x}_2)$$

$$\dot{\hat{x}}_2 = Q^{-1}\left[\hat{A}^T\left(\frac{\partial U_1}{\partial x_1}\right)^T + M\hat{b} + (M\hat{B} + \hat{B}^T M)\hat{x}_2\right] + \hat{b} + \hat{B}\hat{x}_2 \tag{2}$$

where

$\hat{a}$ stands for $a(x_1, u_{\text{nom}}(x_1, \hat{x}_2))$
$\hat{b}$ stands for $b(x_1, u_{\text{nom}}(x_1, \hat{x}_2))$
$\hat{A}$ stands for $A(x_1, u_{\text{nom}}(x_1, \hat{x}_2))$
$\hat{B}$ stands for $B(x_1, u_{\text{nom}}(x_1, \hat{x}_2))$.

Theorem 1 *Under assumptions A1, A2 and A3, this dynamic controller produces a closed-loop system with the following properties: $x_1(t), x_2(t)$ and $\hat{x}_2(t)$ are bounded, and x_1 goes to zero.*

Proof: Let W be the following function of time:

$$W(t) = V_1(x(t)) + V_2(x(t)) + \frac{1}{2}[x_2(t) - \hat{x}_2(t)]^T Q [X_2 - \hat{x}_2(t)]. \tag{3}$$

Then

$$\dot W(t) = \frac{\partial V_1}{\partial x_1}(x)\left[\hat a + \hat A x_2\right] + \frac{\partial V_1}{\partial x_2}(x)\left[\hat b + \hat B x_2\right] + \frac{\partial V_2}{\partial x}(x)\left[f(x) + g(x)\hat u_{\mathrm{nom}}\right] + (x_2-\hat x_2)^T Q\left[\hat b + \hat B x_2 - \dot{\hat x}_2\right].$$

A straightforward computation shows that

$$\dot W(t) \le -\rho(x_1,\hat x_2) + (x_2-\hat x_2)^T\left[\hat A^T\left(\frac{\partial U_1}{\partial x_1}\right)^T + M\hat b + (M\hat B + \hat B^T M)\hat x_2 + Q\hat b + Q\hat B\hat x_2 - Q\dot{\hat x}_2\right] + \frac{1}{2}(x_2-\hat x_2)^T\left[Q\hat B + \hat B^T Q + M\hat B + \hat B^T M\right](x_2-\hat x_2)$$

which, from the expression of $\dot{\hat x}_2$ and assumptions A2, A3, yields

$$\dot W(t) \le -\rho(x_1,\hat x_2)$$

This implies that $(x_1,x_2,\hat x_2)$ is bounded because W is, and that x_1 goes to zero because ρ is zero if and only if x_1 is zero.

4 Examples and extensions

Example 1. Let us consider the system ($k=1$, $\ell=1$)

$$\begin{aligned}\dot x_1 &= x_2 e^{x_1} + u\\ \dot x_2 &= x_1^2\,.\end{aligned}$$

If we use

$$\begin{aligned}u_{\mathrm{nom}}(x_1,x_2) &= -x_1x_2 - x_2e^{x_1} - x_1\\ V_1(x) &= \frac{1}{2}\left(x_1^2+x_2^2\right)\\ V_2(x) &= 0\,,\end{aligned}$$

then assumption A1 is met with $\rho(x) = -x_1^2$ and A2 is met using any $Q>0$ as $B(x_1,u)=0$. Clearly assumption A3 is also satisfied.

In this example, using $Q=1$, our dynamic controller takes the form

$$u(x_1,\hat x_2) = u_{\mathrm{nom}}(x_1,\hat x_2) = -x_1\hat x_2 - \hat x_2 e^{x_1} - x_1$$
$$\dot{\hat x}_2 = \left[e^{x_1}x_1 + x_1^2\right] + x_1^2.$$

Let us have a look back at our assumptions: the linearity of the state equations with respect to x_2 and A1 are crucial to our method. A2 and A3 are more technical. A3 specializes the form of the Lyapunov function, which restricts the class of systems studied, yet allows, for example, bilinear systems with a skew symmetric drift matrix. A2 on the other hand is stronger since $Q\hat B + \hat B^T Q$ being non positive does, in itself, almost imply x_2 being bounded independently of the control. This is natural if x_2 is considered as the state of an "exosystem" as in [4] and $\dot x_2$ depends on x_2 only. The following example displays a situation where $Q\hat B + \hat B^T Q$ is non positive for no constant matrix Q, but an extension of this method using more dynamics still allows stabilization using the output only.

Example 2. Consider the system ($k=1, \ell=2$) defined by (1) with

$$a(x_1,u) = u \qquad A(x_1,u) = (0,1)$$

$$b(x_1,u) = \begin{bmatrix} x_1\\ 0\end{bmatrix} \qquad B(x_1,u) = \begin{bmatrix} 0 & 0\\ 2x_1 & 0\end{bmatrix}.$$

Or, using for convenience $x_2 = (p,q)^T$,

$$\begin{cases}\dot x_1 &= q+u\\ \dot p &= x_1\\ \dot q &= 2px_1.\end{cases}$$

With

$$\begin{cases} u_{\mathrm{nom}}(x) &= u_{\mathrm{nom}}(x_1,p,q) = -x_1 - p - q\\ V_1(x) &= \frac{1}{2}x_1^2 + \frac{1}{2}p^2\\ V_2(x) &= (p^2-q)^2\\ \rho(x) &= -x_1^2\\ U_1(x_1) &= \frac{1}{2}x_1^2 \text{ and } M = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\end{cases}$$

assumptions A1 and A3 are met. On the other hand A2 cannot be satisfied: there is no positive definite Q such that $QB + B^TQ$ is negative semidefinite (since $\det(QB+B^TQ) < 0$ for any $Q>0$). Using any positive definite Q, controller (2) becomes

$$u = u_{\mathrm{nom}}(x_1,\hat p,\hat q)$$

$$\begin{bmatrix}\dot{\hat p}\\ \dot{\hat q}\end{bmatrix} = Q^{-1}\begin{bmatrix}x_1\\ x_1\end{bmatrix} + \begin{bmatrix}x_1\\ 2\hat p x_1\end{bmatrix}$$

and, taking W as in (3), we obtain

$$\dot W \le -\rho(x_1,\hat p,\hat q) + \frac{1}{2}\begin{bmatrix}p-\hat p\\ q-\hat q\end{bmatrix}^T (Q\hat B + \hat B^T Q)\begin{bmatrix}p-\hat p\\ q-\hat q\end{bmatrix}.$$

Since A2 is not met, W need not decrease along solutions of the closed-loop system.

However, we can extend our method by using a dynamic Q: Instead of taking Q constant, we take $Q(t)$ to satisfy

$$\dot Q = -Q\hat B - \hat B^T Q \tag{4}$$

with a positive definite initial condition $Q(0)$, so that $\dot W \le \rho(x_1,\hat p,\hat q)$. Thus we can conclude that $(x_1,p,q,\hat p,\hat q)$ remain bounded and $x_1\to 0$ provided that $Q(t)$ remains bounded and bounded away from singular matrices. Since the bottom-right entry of $Q\hat B + \hat B^T Q$ is zero, the corresponding entry in $Q(t)$ is constant (we will take it to be 1). With $Q = \begin{bmatrix}\lambda & \mu\\ \mu & 1\end{bmatrix}$, (4) becomes

$$\begin{aligned}\dot\mu &= -2x_1\\ \dot\lambda &= -4x_1\mu\end{aligned}$$

and (3) may be written as

$$W = \frac{1}{2}x_1^2 + \frac{1}{2}p^2 + (p^2-q)^2 + \frac{1}{2}[(\lambda-\mu^2)(p-\hat p)^2] + \frac{1}{2}[\mu(p-\hat p) + (q-\hat q)]^2.$$

Therefore x_1,p,q, and the quantities in square brackets are bounded. Noting that $\lambda-\mu^2$ is constant we can conclude that $\hat p$ is bounded. Since

$$\frac{1}{2}\mu^2 - (\lambda-\mu^2+1)\mu - 2(\lambda-\mu^2)\hat p$$

is also constant we can conclude that μ is bounded, and therefore $\hat q$ is bounded as well.

References

[1] Arstein, Z., "Stabilization with relaxed controls", *Nonl. Anal.*, TMA, Vol. 7, 1983.

[2] Francis, B., "The linear multivariable regulator problem ", *SIAM J. Contr. Opt.*, Vol. 15, No. 3, 1977.

[3] Gauthier, J.-P. and Kupka, I., "A separation principle for bilinear systems with dissipative drift", Internal Report, *LAGEP*, University of Lyon 1.

[4] Isidori, A. and Byrnes, C. I., "Output regulation of nonlinear systems", *IEEE Trans. Auto. Control*, Vol. **AC-35**, No. 2, 1990.

[5] Praly, L., Bastin, G., Pomet, J.-B. and Jiang, Z. P., "Adaptive Stabilization of Nonlinear Systems", to be published in the series: Lecture Notes in Information and Control, Springer-Verlag, 1991.

[6] Sontag, E. D., "A universal construction of Arstein's theorem on nonlinear stabilization", *Systems and Control Letters*, Vol. **13**, No. 3, 1989.

[7] Tsinias, J. and Kalouptsidis, N., "Output feedback stabilization", *IEEE Trans. Auto. Control*, Vol. **AC-35**, No. 8, 1990.