# Prioritized optimization by Nash games : towards an adaptive multi-objective strategy

*Application to a problem of flight mechanics*

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# Outline

# What is a 'complex system' from an engg. optimization standpoint?

*Prototype example: shape optimization of an aircraft wing*

## A situation that is

- multi-disciplinary

  *aerodynamics, structural design, acoustics, stealth, ...*

- multi-objective (in each discipline)

  *and for aerodynamics alone:*

  *3 forces and 3 moments, and several perspectives: energy, maneuverabilty, ...*

- multi-point

  *and for aerodynamics alone:*

  *cruise (transonic, small AoA) vs take-off and landing (subsonic, large AoA, deployed flaps), other critical configurations of the flight envelope, ...*

# Complex system from an engg. optimization viewpoint (foreword)

## A situation involving cost functions that are

- functionals
  (not in closed-form expressions), involving several complex simulation codes
  compressible 3D EULER (wave drag) vs RANS equations (deployed flaps), elasticity, wave equation (acoustics), Maxwell (stealth)...
- many,
  some of which being revealed more critical than anticipated after a first campaign of multi-objective optimization, and a first analysis of the concept.

*Let us simplify...*

# Outline

# Problematics

Suppose a design-point $\mathbf{x}_A^\star \in \mathbb{R}^n$ that is Pareto-optimal w.r.t. the minimization of the following $m$ primary cost functions

$$\left\{f_j(\mathbf{x})\right\} \quad (j = 1, \dots, m)$$

subject to the $K$ equality constraints

$$c_k(\mathbf{x}) = 0 \quad (k = 1, \dots, K)$$

is known (by a preliminary multiobjective process).
Can we construct a continuum of neighboring design-points $\{\bar{\mathbf{x}}_\varepsilon\}$ parameterized by a small parameter $\varepsilon$ in such a way that:

1. "consistency":
$$\bar{\mathbf{x}}_0 = \mathbf{x}_A^\star$$
2. the Pareto optimality of the primary cost functions is degraded by an-$O(\varepsilon^2)$ term only
3. the constraint are satisfied throughout ($c_k(\bar{\mathbf{x}}_\varepsilon) = 0,\ \forall k,\ \forall \varepsilon$)
4. additional secondary cost functions
$$\left\{f_j(\mathbf{x})\right\} \quad (j = m+1, \dots, M)$$
are reduced at least linearly in $\varepsilon$?

# Outline

# Construction

## Preparation phase

- Primary steering function $f_A^+(\mathbf{x})$, convexity fix
- Territory splitting
- Secondary steering function $f_B(\mathbf{x})$

## Nash games

# Preparation phase

*Momentarily assume that all differential elements are available*

## Primary steering function $f_A^+(\mathbf{x})$, convexity fix

- Gram-Schmidt orthogonalization of constraint gradients $\{\nabla c_k^\star\}$ $(k = 1, \ldots, K)$ assumed to be linearly independent by "constraint qualification":

$$\nabla \mathbf{c}^\star = \begin{pmatrix} \vdots & & \vdots \\ \nabla c_1^\star & \ldots & \nabla c_K^\star \\ \vdots & & \vdots \end{pmatrix} = \mathbf{QR}, \quad \mathbf{Q} = \begin{pmatrix} \vdots & & \vdots \\ q^1 & \ldots & q^K \\ \vdots & & \vdots \end{pmatrix}.$$

- Define the projection operator onto the subspace tangent to the constraint manifold:

$$\mathbf{P} = \mathbf{I}_n - \sum_{k=1}^{K} \left[\mathbf{q}^k\right] \left[\mathbf{q}^k\right]^t .$$

- Compute the projected (logarithmic) gradients of the primary cost functions:

$$\mathbf{g}_j^\star = \mathbf{P} \frac{\nabla f_j^\star}{f_j^\star} = \frac{\nabla f_j^\star}{f_j^\star} - \sum_{k=1}^{K} \left(\frac{\nabla f_j^\star}{f_j^\star}, \mathbf{q}^k\right) \mathbf{q}^k \qquad (j = 1, \ldots, m,\ \mathbf{g}_j^\star \in \mathrm{R}^n).$$

# Primary steering function $f_A^+(\mathbf{x})$ (end)

- MGDA: recursive exploration of the boundary of an $n$-polytope with $m$ vertices to identify the coefficients $\{\alpha_j^\star\}$ appearing in the expression of Pareto-optimality at $\mathbf{x} = \mathbf{x}_A^\star$:

$$\omega_A^\star = \operatorname{arg\,min}_{\mathbf{u}\in\overline{\mathbf{U}}_A} ||\mathbf{u}|| = \textstyle\sum_{j=1}^m \alpha_j^\star \mathbf{g}_j^\star = 0$$

$$\overline{\mathbf{U}}_A = \left\{\mathbf{u} = \textstyle\sum_{j=1}^m \alpha_j \mathbf{g}_j^\star \text{ s.t. } \alpha_j \geq 0\ (\forall j),\ \sum_{j=1}^m \alpha_j = 1\right\}.$$

- Define the "agglomerated primary cost function":

$$f_A(\mathbf{x}) = \sum_{j=1}^m \alpha_j^\star \frac{f_j(\mathbf{x})}{f_j^\star}$$

so that $\mathbf{P}\nabla f_A^\star = 0 \Longleftrightarrow \nabla f_A^\star + \sum_{k=1}^K \lambda_k \nabla c_k^\star = 0 \quad (\lambda_k :$ Lagrange multiplier).

- **"Primary steering function"**: augment $f_A(\mathbf{x})$ by a convexity-fix term:

$$\boxed{f_A^+(\mathbf{x}) = f_A(\mathbf{x}) + \frac{c}{2}\left\|\mathbf{x} - \mathbf{x}_A^\star\right\|^2}$$

($c \geq 0$, and sufficiently large): convex and minimum at $\mathbf{x}_A^\star$ under the constraints.

# Preparation continued

## Territory splitting

### Split design space $R^n$ into two supplementary subspace $U$ and $V$

- Compute and diagonalize the reduced Hessian matrix

$$\mathbf{H}'_A = \mathbf{P}\left(\nabla^2 f_A^{+,\star}\right)\mathbf{P} = \mathbf{\Omega}\mathcal{H}\mathbf{\Omega}^t, \quad \mathbf{\Omega}^t\mathbf{\Omega} = \mathbf{I}_n, \quad \mathcal{H} = \mathbf{Diag}(h'_k) = \begin{pmatrix} \mathcal{H}_\mathbf{u} & 0 \\ 0 & \mathcal{H}_\mathbf{v} \end{pmatrix},$$

  where the ordering is such that: $h'_1 = \cdots = h'_K = 0;\ h'_{K+1} \geq h'_{K+2} \geq \cdots \geq h'_n > 0$.

- Choose dimension $p$, and split the matrix $\mathbf{\Omega}$ as follows:

$$\mathbf{\Omega} = (\mathbf{\Omega_u}\ \mathbf{\Omega_v}), \quad \mathbf{\Omega_u} = \begin{pmatrix} \mathbf{\Omega_{uu}} \\ \mathbf{\Omega_{vu}} \end{pmatrix}, \quad \mathbf{\Omega_v} = \begin{pmatrix} \mathbf{\Omega_{uv}} \\ \mathbf{\Omega_{vv}} \end{pmatrix},$$

  ($\mathbf{\Omega_u}:\ n \times (n-p)$; $\mathbf{\Omega_v}:\ n \times p$) corresponding to a split of $R^n$ into two supplementary subspaces

$$R^n = U \oplus V \quad (\dim U = n-p,\ \dim V = p,\ U \perp V)$$

  where $U$ is spanned by the first $n-p$ eigenvectors of matrix $\mathbf{H}'_A$, the first $K$ of which spanning the null space of $\mathbf{P}$, and $V$ by the last $p$ eigenvectors.

- Introduce the change of variables:

$$\boxed{\mathbf{x} = \mathbf{x}_A^\star + \mathbf{\Omega}\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \mathbf{x}_A^\star + \mathbf{\Omega_u u} + \mathbf{\Omega_v v} := \mathbf{X}(\mathbf{u},\mathbf{v}) \quad (\mathbf{u} \in R^{n-p},\ \mathbf{v} \in R^p)}$$

# Preparation (end)

## Secondary steering function, $f_B(\mathbf{x})$

- New $p$-dimensional variable (alternative to $\mathbf{v}$), $\mathbf{w} = \mathbf{S}^{\frac{1}{2}}\,\mathbf{v}$ where $\mathbf{S} = \nabla^2_{\mathbf{vv}} f_A^{+,\star} = \mathcal{H}_{\mathbf{v}}$ (positive-definite diagonal matrix)
- Logarithmic gradients of secondary cost functions w.r.t. the variable $\mathbf{w}$ (for fixed $\mathbf{u}$):

$$\mathbf{g}_j^\star = \frac{\nabla_{\mathbf{w}} f_j^\star}{f_j^\star} = \frac{\mathbf{S}^{-\frac{1}{2}}\,\boldsymbol{\Omega}_{\mathbf{v}}^t \nabla f_j^\star}{f_j^\star} \quad (j = m+1, \ldots, M).$$

- MGDA: identify minimum-Euclidean norm element $\boldsymbol{\omega}_B^\star = \sum_{j=m+1}^M \alpha_j^\star \mathbf{g}_j^\star$ in the convex hull $\overline{\mathbf{U}}_B$ of the above gradients:

$$\overline{\mathbf{U}}_B = \left\{ \mathbf{u} = \sum_{j=m+1}^{M} \alpha_j \mathbf{g}_j^\star \text{ s.t. } \alpha_j \geq 0 \ (\forall j),\ \sum_{j=m+1}^{M} \alpha_j = 1 \right\}.$$

- **"Secondary steering function"**:

$$\boxed{f_B(\mathbf{x}) = \sum_{j=m+1}^M \alpha_j^\star \frac{f_j(\mathbf{x})}{f_j^\star}}$$

so that: $\nabla_{\mathbf{w}} f_B^\star = \boldsymbol{\omega}_B^\star, \quad (\nabla_{\mathbf{w}} f_B^\star, \boldsymbol{\omega}_B^\star) = \sigma_B = \left\| \boldsymbol{\omega}_B^\star \right\|^2 \geq 0.$

- If $\sigma_B$ is too small: abandon, or formulate the problem differently; otherwise, proceed.

# Outline

# Nash games[1]

## Formulation

- Define a continuation parameter $\varepsilon$ $(0 \leq \varepsilon \leq 1)$, and the auxiliary cost function

$$f_{AB} = (1-\varepsilon)f_A^+ + \varepsilon f_B.$$

- For each fixed value of $\varepsilon$, a Nash equilibrium point

$$\boxed{\bar{\mathbf{x}}_\varepsilon = \mathbf{X}(\bar{\mathbf{u}}_\varepsilon, \bar{\mathbf{v}}_\varepsilon)}$$

  is sought such that:
  - the sub-vector $\bar{\mathbf{u}}_\varepsilon$ minimizes $f_A^+(\mathbf{X}(\mathbf{u}, \bar{\mathbf{v}}_\varepsilon))$ w.r.t. $\mathbf{u}$ subject to the constraints

$$c_k(\mathbf{X}(\mathbf{u}, \bar{\mathbf{v}}_\varepsilon)) = 0, \ \forall k$$

  - the sub-vector $\bar{\mathbf{v}}_\varepsilon$ minimizes $f_{AB}(\mathbf{X}(\bar{\mathbf{u}}_\varepsilon, \mathbf{v}))$ w.r.t. $\mathbf{v}$ subject to no constraints.

# Nash games[2]

## Theoretical result

- The Nash equilibrium point $\bar{\mathbf{x}}_\varepsilon$ exists
  for all sufficiently small $\varepsilon$, and for $\varepsilon = 0$:

$$\bar{\mathbf{x}}_0 = \mathbf{x}_A^\star$$

  (consistency).
- As $\varepsilon$ increases
  (and remains sufficiently small):
  - The primary steering function $f_A^+(\mathbf{X}(\bar{\mathbf{u}}_\varepsilon, \bar{\mathbf{v}}_\varepsilon))$ augments
    from the value 1, with an initial derivative equal to 0, and the equilibrium point departs from the Pareto-optimality of the primary cost functions $\{f_j\}$ $(j = 1, \ldots, m)$ by a term $O(\varepsilon^2)$ only.
  - The secondary steering function $f_B(\mathbf{X}(\bar{\mathbf{u}}_\varepsilon, \bar{\mathbf{v}}_\varepsilon))$ diminishes linearly with $\varepsilon$, as $(-\sigma_B)\varepsilon$
    (where the positive constant $\sigma_B$ is calculated a priori). The secondary cost functions $\{f_j\}$ $(j = m+1, \ldots, M)$ decrease at the same rate, or faster.

*"Horn-shaped pattern in function space"*

# Nash games[3]

## Computational implementation

## Metamodels

- Global (i.e. fixed) quadratic metamodels for steering functions $f = f_A^+$ or $f_B$:

$$\tilde{f}(\mathbf{x}) = f^\star + (\mathbf{x} - \mathbf{x}_A^\star, \nabla f^\star + \frac{1}{2}\mathbf{H}^\star(\mathbf{x} - \mathbf{x}_A^\star))$$

  (gradients by finite differences, Hessians by least squares)

- Local (i.e. regularly upgraded along the process) quadratic constraint metamodels:

$$\tilde{c}_k(\mathbf{x}) = c_k(\bar{\mathbf{x}}) + (\mathbf{x} - \bar{\mathbf{x}}, \overline{\nabla c_k} + \frac{1}{2}\overline{\mathbf{H}_{c_k}}(\mathbf{x} - \bar{\mathbf{x}})) \quad (k = 1, \ldots, K)$$

## Nash-Game Implementation

- **u-subproblem:** minimization of a quadratic form subject to a set of quadratic constraints; the optimality conditions are solved by Newton's method.
- **v-subproblem:** unconstrained minimization of a quadratic form; the optimality conditions are linear and solved by direct inversion.

**Coordination by Schwarz-method-type algorithm:** The upper bound $\varepsilon_{max}$ guaranteeing convexity is known a priori.

# Outline

# A testcase in flight mechanics

## Black-box software provided by Dassault Aviation

15 design variables subject to interval bounds

| symbol ($X_i$) | significance (unit) | lower bound $X_{i,\min}$ | upper bound $X_{i,\max}$ |
|---|---|---|---|
| z | cruise altitude (m) | 8000 | 18500 |
| xmach | cruise Mach number | 1.6 | 2.0 |
| S | wing reference surface ($m^2$) | 100 | 200 |
| phi0w | wing leading-edge sweep angle ($^o$) | 40 | 70 |
| phi100w | wing trailing-edge sweep angle ($^o$) | -10 | 20 |
| xlw | wing taper ratio | 0.05 | 0.50 |
| t_cw | wing relative thickness | 0.04 | 0.08 |
| phi0t | vertical-tail leading-edge sweep angle ($^o$) | 40 | 70 |
| phi100t | vertical-tail trailing-edge sweep angle ($^o$) | 0 | 10 |
| xlt | vertical-tail taper ratio | 0.05 | 0.50 |
| t_ct | vertical-tail relative thickness | 0.05 | 0.08 |
| dfus | fuselage diameter (m) | 2.0 | 2.5 |
| wfuel | fuel mass (kg) | 15,000 | 40,000 |
| alpha | landing maximum angle of attack ($^o$) | 10 | 15 |
| xfac | mlw/tow, landing to take-off mass ratio | 0.85 | 0.95 |

For each design variable $X_i$, let:

$$X_i = \frac{X_{i,\max}+X_{i,\min}}{2} + \frac{X_{i,\max}-X_{i,\min}}{2}\sin x_i$$

$\mathbf{x} = \{x_i\} \in \mathrm{R}^{15}$: preliminary optimization variable

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## Optimization formulation

## Functions of interest

- mass, $M$ (kg) (to be minimized)
- range, $R$ (m) (to be maximized)
- take-off (t.o.) distance, $D$ (m) (to be maintained below 1828 m)
- approach speed, $V$ (m/s) (to be reduced)

Function values, no gradients.

## Cost functions and constraint

Primary: $$f_1(\mathbf{x}) = \exp\left[\gamma\left(\frac{M}{M^\star} - 1\right)\right], \quad f_2(\mathbf{x}) = \exp\left[\gamma\left(\frac{R^\star}{R} - 1\right)\right]$$

Secondary: $$f_3(\mathbf{x}) = \exp\left[\gamma\left(\frac{V}{V^\star} - 1\right)\right]$$

Constraint: $$c_1(\mathbf{x}) = \frac{D}{1828} - 1 + x_{16}^2 = 0 \quad (x_{16} : \text{ slack variable})$$

(starred symbols: values at $\mathbf{x}_A^\star$)

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## Two-step numerical experiment

### First step

Identify some starting point $\mathbf{x}_A^\star$ on the mass-range $(f_1, f_2)$ Pareto front subject to constraint on take-off distance $c_1(\mathbf{x}) = 0$ (and interval bounds on all 15 physical variables)

Procedure: *Pareto Archived Evolution Strategy*

### Second step

Reduce approach speed (through $f_3$) while maintaining:

- quasi mass-range Pareto-optimality,
- and constraint on take-off distance.

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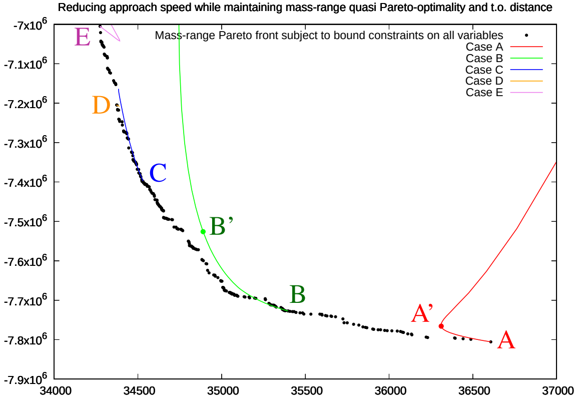
Results

## Discrete mass-range Pareto front and five continua of Nash equilibria to reduce approach speed




(-range) in m vs mass in kg

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## Set-up for the numerical experimention

## Numerical parameters

| case | $\gamma$ [a] | hdiff [b] | hbox [c] | $\kappa$ [d] | $\ell_{max}$ [e] | $TOL$ [f] | $\lambda_{max}$ [g] | $\mu_{max}$ [h] |
|---|---|---|---|---|---|---|---|---|
| A | 10 | $10^{-2}$ | 0.5 | 10 | 100 ; 96 | $10^{-4}$ | 15 | 5 |
| B | 10 | $10^{-2}$ | 0.5 | 5 | 100 ; 94 | $10^{-4}$ | 15 | 5 |
| C | 10 | $10^{-4}$ | $0.5\times10^{-2}$ | 10 | 100 ; 99 | $10^{-4}$ | 15 | 5 |
| D | $10^3$ | $10^{-5}$ | $0.5\times10^{-3}$ | 10 | 100 ; 99 | $10^{-4}$ | 15 | 5 |
| E | 40 | $10^{-4}$ | $0.5\times10^{-2}$ | 1.5 | 100 ; 99 | $10^{-4}$ | 15 | 5 |

[a] constant to control magnitude of initial gradients
[b] stepsize in central differencing
[c] box size for global metamodels
[d] parameter controling the convexity fix
[e] number of discretization subintervals of $[0, \varepsilon_{max}]$
[f] accuracy tolerance in $(\mathbf{u}, \mathbf{v})$ coordination outer loop
[g] maximum number of $(\mathbf{u}, \mathbf{v})$ coordination iterations
[h] maximum number of iterations by Newton's method in each $\mathbf{u}$ subproblem

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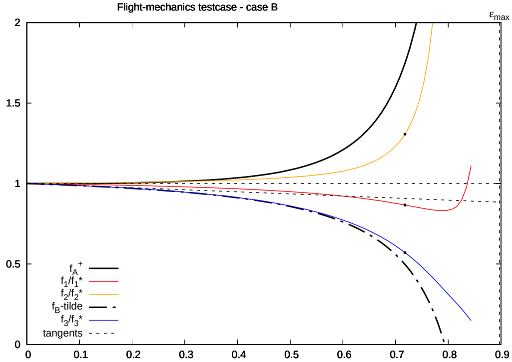
## Case B

## Horn-shaped or lily-shaped cost function pattern



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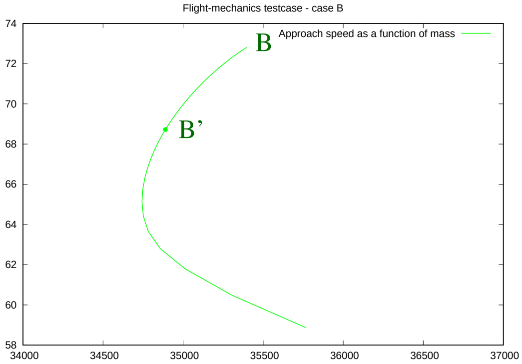
Case B

## Actual reduction of approach speed



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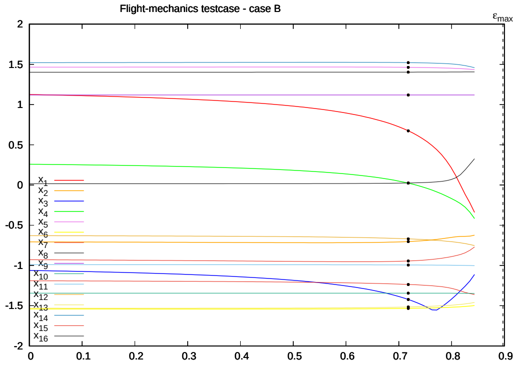
## Case B

## The 16 optimization variables along the continuum



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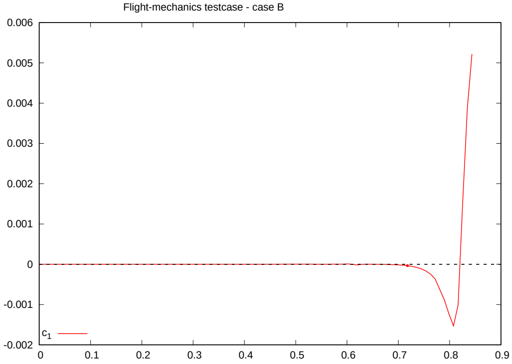
Case B

## The constraint along the continuum




At point B': $c_1 = -3.5 \times 10^{-5}$.

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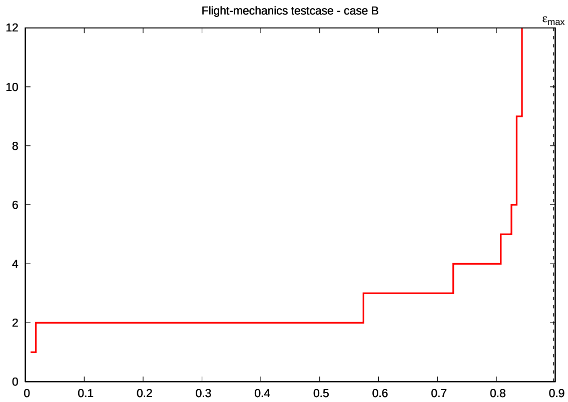
## Case B




Number $\lambda$ of outer iterations necessary to coordinate the subvectors $\mathbf{u}$ and $\mathbf{v}$ at a given $\varepsilon = \varepsilon_\ell$ as a function of $\ell = 1, \ldots, 94$ (Case B).

# A testcase in flight mechanics

## Main conclusions[1]

- Testcase of actual interest revealed physical scales raising numerical difficulties (PAES).
- Successful Nash-game approach able to identify a continuum of equilibria in $\mathrm{R}^{16}$ (15 physical + 1 slack variables) initiated from a large part of the mass-range Pareto front subject to the constraint on take-off distance to preserve properly the Pareto optimality.
- Along (more than 80% of) the continuum, the variation of the cost functions exhibit a horn-shaped (or lily-shaped) pattern:
  - the primary steering function increases moderately with the continuation parameter $\varepsilon$ (initial derivative equal to 0), and the Pareto-optimality condition relating the primary cost functions and the constraint is degraded of only $O(\varepsilon^2)$;
  - the secondary steering function decreases linearly with $\varepsilon$ and the initial derivative $(-\sigma)$ is given by the theory, as well as the limit of convexity $\varepsilon_{\text{max}}$;
  - the constraint is strictly satisfied
  - the objective of reducing the secondary cost function related to approach speed is attained.

# A testcase in flight mechanics

## Main conclusions[2]

- Quadratic metamodels are adequate; however, the constraint metamodels must be upgraded throughout.
- For a given $\varepsilon$, are sufficient:
  - 3 iterations by Newton's method to solve each $\mathbf{u}$-subproblem, for fixed $\mathbf{v}$;
  - a moderate number of $(\mathbf{u}, \mathbf{v})$ coordination iterations except at the approach of the limit of convexity $\varepsilon_{\max}$.
- The interactive running time is about 45 s for a given case on a standard laptop. This includes: code assembly and compilation, 15 stages of preparation of the Nash games involving in particular the construction of metamodels, the application of the QR algorithm on the Hessian and several matrix diagonalizations and system inversions (by Lapack), the computation of 100 Nash equilibria, and the elaboration of graphics.
- The software platform is general: the user provides
  - a Fortran-compatible source file for the evaluation of cost functions and constraints (no gradients required),
  - a datafile specifying 14 methodological parameters including the coordinates of the Pareto-optimal starting point $\mathbf{x}_A^\star$.

# Outline

# Conclusion

## Inria Research Reports 9290 & 9291

- Platform for prioritized multi-objective optimization by metamodel-assisted Nash games
- Direct and adaptive approaches to multi-objective optimization

accessible on the HAL open archive
`https://hal.inria.fr`

Software platform `http://mgda.inria.fr` currently being remodeled.

*Thank you!*