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Aspects of Analysis

Curvature Criterion

Isoperimetry

Evolution Equations

MATRIX PRESS
MATHEMATICAL NOTEBOOKS
Vol III

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Cataloging in Publication Data

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Aspects of Analysis Curvature Criterion, Isoperimetry, Evolution Equations.

Version 1.41

ISBN 1-905760-03-5

Mathematical Notebooks Vol.3, Editor B.Zegarliniski

2000 Mathematics Subject Classification : 35K05 60J60 26D10 35Q30

KeyWords: *Curvature Criterion, Isoperimetry, Product Measures*
Navier-Stocks Equations.

InternetSearchKey: **MathX**

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**Evolution Equations and Boundary Problems
with Application to the Navier-Stokes Equations**

Lectures by
Zhongmin Qian

Notes taken by
James Inglis

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1 Introduction

We will be concerned with the existence in small time of solutions of initial boundary value problems for evolution equations. We will place particular emphasis on the Navier-Stokes equations for incompressible fluids in a domain $\Omega \subset \mathbb{R}^3$. In general, we will consider evolution equations in a separable Hilbert space X of the following form

$$\begin{cases} \frac{\partial}{\partial t}u + Au + Q(u) = 0 \\ u(0) = u_0 \in X. \end{cases} \quad (1.1)$$

where A is a self-adjoint linear operator (typically $-\Delta$) and Q is a non-linear term. We give two important examples of such equations.

Example 1.0.1.

(i)

$$\frac{\partial}{\partial t}u + (-\Delta u) + p(u, \nabla u) = 0$$

where $p(x, y)$ is a polynomial, and u is a scalar.

(ii) The Navier-Stokes equations: *The incompressible flow of a fluid is governed by the Navier-Stokes equations:*

$$\begin{cases} \frac{\partial}{\partial t}u + (-\Delta u) + (u \cdot \nabla)u = -\nabla p \\ \nabla \cdot u = 0 \end{cases} \quad (1.2)$$

Here u represents the velocity vector field of the fluid flow (so that the equation is not scalar), p is a scalar pressure function (up to a function of time t) which maintains the incompressibility of the fluid (so that p is divergence free), and the requirement $\nabla \cdot u = 0$ is the incompressibility condition. This condition follows from the fact that we assume the density of the fluid to be constant.

If we are working in a domain $\Omega \subset \mathbb{R}^3$, then we can write $u = (u^1, u^2, u^3)$, and the equations (1.2) become

$$\begin{cases} \frac{\partial}{\partial t}u^i - \Delta u^i + \left(\sum_{j=1}^3 u^j \nabla_j \right) u^i = -\nabla_i p \\ \sum_{j=1}^3 \nabla_j u^j = 0 \end{cases}$$

in $\Omega \subset \mathbb{R}^3$ for $i \in \{1, 2, 3\}$.

We can reformulate these equations in a useful way, and show that they are actually equivalent to a functional evolution equation (and thus they behave very differently to a standard system of PDEs). We use the Helmholtz decomposition (see section 3.1 for details), which asserts the existence of an operator P_∞ that projects a vector field u (subject to boundary conditions) onto its divergence free part. Indeed, for any vector field u , we can write

$$u = P_\infty u + \nabla q$$

where $\nabla \cdot (P_\infty u) = 0$ so that $\Delta q = \operatorname{div}(u)$, which is supplemented by boundary conditions that depend on those of u .

Now suppose u satisfies (1.2). Then since $\nabla \cdot u = 0$ and p is divergence free

$$P_\infty u = u, \quad P_\infty \frac{\partial u}{\partial t} = \frac{\partial u}{\partial t}, \quad P_\infty \nabla p = 0,$$

so that if we apply P_∞ to (1.2), we arrive at

$$\frac{\partial}{\partial t} u - P_\infty(\Delta u) + P_\infty((u \cdot \nabla)u) = 0. \quad (1.3)$$

We thus have an evolution equation of the form (1.1) with $A = -P_\infty \circ \Delta$ and $Q(u) = P_\infty((u \cdot \nabla)u)$. Of course, in order to explore solutions, this equation must first be supplemented with suitable initial and boundary conditions. This is the main thrust of chapter 3.

2 General theory

Recall that we are interested in evolution equations of the form

$$(Eq.I) \quad \begin{cases} \frac{\partial}{\partial t} u + Au + Q(u) = 0 \\ u(0) = u_0. \end{cases}$$

where u takes values in some separable Hilbert space X , with inner product $(\cdot, \cdot)_X$. The aim of this chapter is to describe some conditions and spaces in which a solution (in a mild sense) to such an equation exists in small time. The main reference for this chapter is [3], though the reader might also like to refer to [6].

2.1 Preliminaries and assumptions

Let $A : X \rightarrow X$ be a self-adjoint operator, and suppose that A has a spectral gap $\lambda_0 > 0$. If not, we can shift it by a bounded operator, which we can then incorporate into Q . We can thus assume that A^{-1} is a bounded operator. Indeed, by spectral decomposition we can write

$$A = \int_{\lambda_0}^{\infty} \lambda dE_{\lambda}$$

(where $\{E_{\lambda} : \lambda \in \mathbb{R}\}$ is a spectral resolution of the identity for A), so that by the spectral theorem,

$$A^{\alpha} = \int_{\lambda_0}^{\infty} \lambda^{\alpha} dE_{\lambda},$$

with domain

$$\mathcal{D}(A^{\alpha}) = \left\{ x \in X : \int_{\lambda_0}^{\infty} \lambda^{2\alpha} d(E_{\lambda}x, x)_X < \infty \right\}$$

for all $\alpha \in \mathbb{R}$. Thus A^{α} is a bounded operator as long as $\alpha < 0$. We also define

$$P_t = e^{-tA} := \int_{\lambda_0}^{\infty} e^{-t\lambda} dE_{\lambda}. \quad (2.1)$$

Lemma 2.1.1. *For all $\alpha > 0, t > 0$ and $x \in X$*

$$\|A^{\alpha} P_t x\|_X \leq \frac{C(\alpha)}{t^{\alpha}} \|x\|_X.$$

Proof. By the spectral theorem

$$\|A^\alpha P_t x\|_X^2 = \int_{\lambda_0}^{\infty} \lambda^{2\alpha} e^{-2t\lambda} d\|E_\lambda x\|_X^2$$

and taking supremum of the integrand over $\lambda > 0$ the result follows. \square

In looking for a solution to (Eq.I), it is restrictive to assume that $u \in \mathcal{D}(A)$. We therefore start to look for weaker solutions. Our starting point is to note that

$$\begin{cases} \frac{\partial}{\partial t} = -Au \\ u(0) = u_0 \end{cases} \quad (2.2)$$

has a solution $u(t) = e^{-tA}u_0$. We can therefore regard (Eq.I) as a perturbed linear equation, treating the term $Q(u)$ as a non-linear perturbation.

Suppose $u(t)$ is a "good" solution to (Eq.I). Consider $f(s) := P_{t-s}u(s)$ for $0 \leq s \leq t$, so that $f(t) = u(t)$ and $f(0) = P_t u_0$. Then

$$\begin{aligned} f'(s) &= AP_{t-s}u(s) + P_{t-s} \frac{\partial u}{\partial s} \\ &= P_{t-s} \left(Au(s) + \frac{\partial}{\partial s} u \right) \\ &= -P_{t-s} (Q(u(s))) \\ \Rightarrow f(t) - f(0) &= \int_0^t \frac{d}{ds} f(s) ds = - \int_0^t P_{t-s} (Q(u(s))) ds, \end{aligned}$$

since P_{t-s} commutes with A by definition. Thus

$$(Eq.II) \quad u(t) = P_t u_0 - \int_0^t A^\tau P_{t-s} A^{-\tau} Q(u(s)) ds \quad (2.3)$$

for $\tau \in (0, 1)$. Instead of working directly with equation (Eq.I), we actually look for solutions to (Eq.II). Looking for solutions to this equation no longer requires us to assume that $u \in \mathcal{D}(A)$ (in fact, as we will see, we will just have to assume $u \in \mathcal{D}(A^\varepsilon)$ for some $\varepsilon < 1$).

It is worth noting that, by Lemma 2.1.1, $A^\tau P_{t-s}$ is bounded since $\int_0^t \frac{1}{t^\tau} dt < \infty$ for $\tau \in (0, 1)$. Moreover, $A^{-\tau}$ is bounded, since we have assumed that A has a spectral gap.

In view of the above, define

$$(\mathbb{L}u)(t) := P_t u_0 - \int_0^t A^\tau P_{t-s} H(u(s)) ds$$

where $H = A^{-\tau}Q$. It is clear that u is a fixed point for \mathbb{L} , if and only if u is a solution of (Eq.II). We will therefore look for a fixed point for the operator \mathbb{L} .

Remark 2.1.2. *In the case of the Navier-Stokes equations described in the introduction, a solution to (Eq.II) has enough regularity to construct a strong solution (see [3]).*

We will make the following important assumptions, which we will label **(A)**.

Assumptions (A): Suppose that there exist $\delta_1 \in [0, 1]$ and $\delta_2 \in [0, \delta_1)$ and such that

(I) $\mathcal{D}(H)$ consists of all $x \in \mathcal{D}(A^{\delta_1})$ such that there exists a sequence $(x_n)_{n \geq 0} \subset \mathcal{D}(A^{\delta_1})$ such that

(i) $\{\|A^{\delta_1}x_n\|_X : n \geq 0\}$ is bounded,

(ii) $x_n \rightarrow x$ in A^{δ_2} i.e. $\|A^{\delta_2}(x_n - x)\|_X \rightarrow 0$,

so that $\mathcal{D}(A^{\delta_1}) \subset \mathcal{D}(H) \subset \mathcal{D}(A^{\delta_2})$;

(II) $H|_{\mathcal{D}(A^{\delta_1})}$ is A^{δ_1} -bounded i.e.

$$\|H(x)\|_X \leq K_2 (\|A^{\delta_1}x\|_X)$$

for all $x \in \mathcal{D}(A^{\delta_1})$, where $K_2 : [0, \infty) \rightarrow [0, \infty)$ is an increasing function depending on the non-linearity of $H(x)$;

(III) $H|_{\mathcal{D}(A^{\delta_1})}$ is weakly differentiable in the following sense: for $x \in \mathcal{D}(A^{\delta_1})$,

there exists a bounded linear operator $DH(x)$ on $\mathcal{D}(A^{\delta_2})$ such that

(i) $\frac{d}{d\varepsilon} H(x + \varepsilon\xi)|_{\varepsilon=0} = DH(x)\xi$,

(ii) $\|DH(x)\xi\|_X \leq K_1 (\|A^{\delta_1}x\|_X) \|A^{\delta_2}\xi\|_X$
for some increasing function $K_1 : [0, \infty) \rightarrow [0, \infty)$.

These assumptions will be made throughout the rest of this chapter.

Lemma 2.1.3. *Under these assumptions H is locally Lipschitz in the following sense: for all $x, y \in \mathcal{D}(A^{\delta_1})$*

$$\|H(x) - H(y)\|_X \leq K_1 (\|A^{\delta_1}x\|_X + \|A^{\delta_1}y\|_X) \|A^{\delta_2}(x - y)\|_X.$$

Proof. This follows from the mean value theorem for the function $H(sx + (1-s)y)$, $s \in [0, 1]$ and **(III)**. \square

2.2 A first solution space

Definition 2.2.1. Let $T > 0$ and $\zeta_1 \geq 0$ be some fixed number to be chosen later. We will say that a path $u : [0, T] \rightarrow X$ belongs to \mathbb{H}_T if

- (i) for all $t \in (0, T]$, $u(t) \in \mathcal{D}(A^{\delta_1})$ and $t \mapsto A^{\delta_1}u(t)$ is continuous on $(0, T]$ such that

$$\sup_{t \in (0, T]} \|t^{\zeta_1} A^{\delta_1} u(t)\|_X < \infty;$$

- (ii) $u \in \mathcal{C}([0, T], X)$.

If we define for $u \in \mathbb{H}_T$

$$\|u\|_{\mathbb{H}_T} := \sup_{t \in [0, T]} \|u(t)\|_X + \sup_{t \in (0, T]} \|t^{\zeta_1} A^{\delta_1} u(t)\|_X,$$

\mathbb{H}_T becomes a Banach space.

We will show that, under our assumptions **(A)**, there exists a solution $u \in \mathbb{H}_T$ to (Eq.II) for properly chosen $\delta_1, \delta_2, \tau, \zeta_1$ and small enough $T > 0$.

Lemma 2.2.2. Let $u \in \mathbb{H}_T$, with $\|u\|_{\mathbb{H}_T} \leq \beta$. Let $\alpha \in [0, 1)$. Then

- (i) for all $t \in (0, T]$ the map $s \mapsto A^\alpha P_{t-s} H(u(s))$, is left-continuous on $(0, t)$ (note that there is a singularity at t , and at $s = 0$ there is no guarantee that $P_t H(u(s)) \in \mathcal{D}(A^\alpha)$);
- (ii) for all $t \in (0, T]$, $\int_0^t A^\alpha P_{t-s} H(u(s)) ds$ exists and

$$\left\| \int_0^t A^\alpha P_{t-s} H(u(s)) ds \right\|_X \leq C(\alpha) t^{1-\alpha} \int_0^1 (1-s)^{-\alpha} K_2(\beta(ts)^{-\zeta_1}) ds.$$

Proof. Fix $t \in (0, T]$. For $s \in (0, t)$, let $f(s) := A^\alpha P_{t-s} H(u(s))$. Let $s_1, s_2 \in (0, t)$, and without loss of generality, suppose $s_1 > s_2$. Then

$$\begin{aligned} \|f(s_1) - f(s_2)\|_X &= \|A^\alpha P_{t-s_1} H(u(s_1)) - A^\alpha P_{t-s_2} H(u(s_2))\|_X \\ &\leq \left\| A^\alpha P_{t-s_1} \left(H(u(s_1)) - H(u(s_2)) \right) \right\|_X \\ &\quad + \left\| \left(A^\alpha P_{t-s_1} - A^\alpha P_{t-s_2} \right) H(u(s_2)) \right\|_X \\ &= \left\| A^\alpha P_{t-s_1} \left(H(u(s_1)) - H(u(s_2)) \right) \right\|_X \\ &\quad + \|A^\alpha P_{t-s_1} (1 - P_{s_1-s_2}) H(u(s_2))\|_X. \end{aligned} \tag{2.4}$$

Now

$$\begin{aligned} \left\| A^\alpha P_{t-s_1} \left(H(u(s_1)) - H(u(s_2)) \right) \right\|_X &\leq \frac{C(\alpha)}{(t-s_1)^\alpha} \|H(u(s_1)) - H(u(s_2))\|_X \\ &\leq \frac{C(\alpha)}{(t-s_1)^\alpha} K_1 \left(\|A^{\delta_1} u(s_1)\|_X + \|A^{\delta_1} u(s_2)\|_X \right) \|A^{\delta_2}(u(s_1) - u(s_2))\|_X, \end{aligned}$$

where we have used Lemma 2.1.1 and Lemma 2.1.3, which converges to 0 as $s_2 \rightarrow s_1$ by part **(I)** of our assumptions.

To deal with the second term of (2.4), we simply note that, again using Lemma 2.1.1,

$$\|A^\alpha P_{t-s_1}(1 - P_{s_1-s_2})H(u(s_2))\|_X \leq \frac{C(\alpha)}{(t-s_1)^\alpha} \|(1 - P_{s_1-s_2})H(u(s_2))\|_X$$

which also converges to 0 as $s_2 \rightarrow s_1$. Thus we have shown (i).

For (ii) we calculate that

$$\begin{aligned} \|f(s)\|_X &\leq \frac{C(\alpha)}{(t-s)^\alpha} \|H(u(s))\|_X \\ &\leq \frac{C(\alpha)}{(t-s)^\alpha} K_2 \left(\|A^{\delta_1} u(s)\|_X \right) \\ &= \frac{C(\alpha)}{(t-s)^\alpha} K_2 \left(s^{-\zeta_1} \|s^{\zeta_1} A^{\delta_1} u(s)\|_X \right) \\ &\leq \frac{C(\alpha)}{(t-s)^\alpha} K_2 \left(s^{-\zeta_1} \beta \right), \end{aligned}$$

where we have used assumption **(II)**, $\|u\|_{\mathbb{H}_T} \leq \beta$ and the fact that the function K_2 is increasing. Then

$$\begin{aligned} \left\| \int_0^t f(s) ds \right\|_X &\leq \int_0^t \|f(s)\|_X ds \\ &\leq C(\alpha) \int_0^t \frac{1}{(t-s)^\alpha} K_2(s^{-\zeta_1} \beta) ds \\ &= C(\alpha) \int_0^1 \frac{t^{1-\alpha}}{(1-s)^\alpha} K_2((ts)^{-\zeta_1} \beta) ds, \end{aligned}$$

by making a substitution $s \mapsto ts$. □

Example 2.2.3. Suppose the function that appears in the case of the Navier-Stokes equation is given by $K_2(r) = kr^2$ for some positive constant k . Then the right-hand side of the bound (ii) of Lemma 2.2.2 will read

$$C(\alpha)t^{1-\alpha} \int_0^1 (1-s)^{-\alpha} K_2(\beta(ts)^{-\zeta_1}) ds = kC(\alpha)t^{1-\alpha-2\zeta_1}\beta^2 \int_0^1 \frac{1}{(1-s)^\alpha s^{2\zeta_1}} ds.$$

The integral on the right-hand side is the beta function of $\alpha + 1$ and $2\zeta_1 + 1$, and so is finite. In this case, we should choose $1 - \alpha - 2\zeta_1 > 0$ to ensure that we have a contraction in small time.

Lemma 2.2.4. Suppose $u, w \in \mathbb{H}_T$ with $\max\{\|u\|_{\mathbb{H}_T}, \|w\|_{\mathbb{H}_T}\} \leq \beta$ and $u_0 = w_0$. Then

(i)

$$\|\mathbb{L}u(t)\|_X \leq c_1\|u_0\|_X + c_2 h_1(t, \beta)$$

for some constants $c_1, c_2 \in (0, \infty)$, where

$$h_1(t, \beta) = t^{1-\tau} \int_0^1 (1-s)^{-\tau} K_2(\beta(ts)^{-\zeta_1}) ds;$$

(ii)

$$\|\mathbb{L}u(t) - \mathbb{L}w(t)\|_X \leq c_3 h_2(t, \beta) \|u - w\|_{\mathbb{H}_T}$$

for some constant $c_3 \in (0, \infty)$, where

$$h_2(t, \beta) = t^{1-\tau-\zeta_1} \int_0^1 (1-s)^{-\tau} s^{-\zeta_1} K_1(2\beta(ts)^{-\zeta_1}) ds.$$

Proof. By definition

$$\begin{aligned} \|\mathbb{L}u(t)\|_X &= \left\| P_t u_0 - \int_0^t A^\tau P_{t-s} H(u(s)) ds \right\|_X \\ &\leq c_1 \|u_0\|_X + \int_0^t \|A^\tau P_{t-s} H(u(s))\|_X ds \end{aligned}$$

where the constant c_1 comes from the spectral representation of P_t (in the same way as in Lemma 2.1.1). Part (i) follows from an application of part (ii) of Lemma 2.2.2.

For (ii), we note

$$\begin{aligned} \|\mathbb{L}u(t) - \mathbb{L}w(t)\|_X &= \left\| - \int_0^t A^\tau P_{t-s} (H(u(s)) - H(w(s))) ds \right\|_X \\ &\leq C(\tau) \int_0^t (t-s)^{-\tau} \|H(u(s)) - H(w(s))\|_X ds \\ &\leq C(\tau) \int_0^t (t-s)^{-\tau} K_1(\|A^{\delta_1} u(s)\|_X + \|A^{\delta_1} w(s)\|_X) \|A^{\delta_2}(u(s) - w(s))\|_X ds \end{aligned}$$

where we have used the fact that \bar{H} is locally Lipschitz i.e. Lemma 2.1.3. Now

$$\|A^{\delta_2}x\|_X = \|A^{-(\delta_1-\delta_2)}A^{\delta_1}x\|_X \leq c\|A^{\delta_1}x\|_X \quad (2.5)$$

for some constant c , since $A^{-(\delta_1-\delta_2)}$ is a bounded operator for $\delta_2 \leq \delta_1$. Thus

$$\begin{aligned} & \|\mathbb{L}u(t) - \mathbb{L}w(t)\|_X \\ & \leq cC(\tau) \int_0^t (t-s)^{-\tau} K_1 \left(\|A^{\delta_1}u(s)\|_X + \|A^{\delta_1}w(s)\|_X \right) s^{-\zeta_1} \|s^{\zeta_1}A^{\delta_1}(u(s) - w(s))\|_X ds \\ & \leq \tilde{C}(\tau)\|u - w\|_{\mathbb{H}_T} \int_0^t (t-s)^{-\tau} s^{-\zeta_1} K_1 (2\beta s^{-\zeta_1}) ds \end{aligned}$$

where we have used the fact that $u, w \in \mathbb{H}_T$ and the fact that K_1 is increasing. Using the substitution $s \mapsto st$ then yields (ii). \square

We will also need the following:

Lemma 2.2.5. *Suppose $u, w \in \mathbb{H}_T$ with $\max\{\|u\|_{\mathbb{H}_T}, \|w\|_{\mathbb{H}_T}\} \leq \beta$ and $u_0 = w_0$. Then*

(i)

$$\|t^{\zeta_1}A^{\delta_1}\mathbb{L}u(t)\|_X \leq c_1\|A^{\delta_1-\zeta_1}u_0\|_X + c_2h_3(t, \beta)$$

for some constants $c_1, c_2 \in (0, \infty)$, where

$$h_3(t, \beta) = t^{1-\tau-(\delta_1-\zeta_1)} \int_0^1 (1-s)^{-(\tau+\delta_1)} K_2 (\beta(ts)^{-\zeta_1}) ds;$$

(ii)

$$\left\| t^{\zeta_1}A^{\delta_1}(\mathbb{L}u(t) - \mathbb{L}w(t)) \right\|_X \leq c_3h_4(t, \beta)\|u - w\|_{\mathbb{H}_T}$$

for some constant $c_3 \in (0, \infty)$, where

$$h_4(t, \beta) = t^{1-\tau-\delta_1} \int_0^1 (1-s)^{-(\delta_1+\tau)} s^{-\zeta_1} K_1 (2\beta(ts)^{-\zeta_1}) ds.$$

Proof. By using Lemma 2.1.1,

$$\begin{aligned}
 \|t^{\zeta_1} A^{\delta_1} \mathbb{L}u(t)\|_X &= \left\| t^{\zeta_1} A^{\delta_1} \left(P_t u_0 - \int_0^t A^\tau P_{t-s} H(u(s)) ds \right) \right\|_X \\
 &\leq t^{\zeta_1} \|A^{\delta_1} P_t u_0\|_X + t^{\zeta_1} \int_0^t \|A^{\delta_1+\tau} P_{t-s} H(u(s))\|_X ds \\
 &\leq t^{\zeta_1} \|A^{\zeta_1} P_t A^{\delta_1-\zeta_1} u_0\|_X + C(\delta_1 + \tau) t^{\zeta_1} \int_0^t (t-s)^{-(\delta_1+\tau)} \|H(u(s))\|_X ds \\
 &\leq t^{\zeta_1} \frac{C(\zeta_1)}{t^{\zeta_1}} \|A^{\delta_1-\zeta_1} u_0\|_X + C(\delta_1 + \tau) t^{\zeta_1} \int_0^t (t-s)^{-(\delta_1+\tau)} K_2(\beta s^{-\zeta_1}) ds \\
 &= C(\zeta_1) \|A^{\delta_1-\zeta_1} u_0\|_X + C(\delta_1 + \tau) t^{1+\zeta_1-(\delta_1+\tau)} \int_0^1 (1-s)^{-(\delta_1+\tau)} K_2(\beta(ts)^{-\zeta_1}) ds
 \end{aligned}$$

so that (i) holds.

For (ii), write

$$\begin{aligned}
 \|t^{\zeta_1} A^{\delta_1} (\mathbb{L}u(t) - \mathbb{L}w(t))\|_X &= t^{\zeta_1} \left\| \int_0^t A^{\delta_1+\tau} P_{t-s} (H(u(s)) - H(w(s))) ds \right\|_X \\
 &\leq t^{\zeta_1} \int_0^t \|A^{\delta_1+\tau} P_{t-s} (H(u(s)) - H(w(s)))\|_X ds \\
 &\leq t^{\zeta_1} C(\delta_1 + \tau) \int_0^t (t-s)^{-(\delta_1+\tau)} \|H(u(s)) - H(w(s))\|_X ds \\
 &\leq t^{\zeta_1} C(\delta_1 + \tau) \int_0^t (t-s)^{-(\delta_1+\tau)} K_1(\|A^{\delta_1} u(s)\|_X + \|A^{\delta_1} w(s)\|_X) \|A^{\delta_2}(u(s) - w(s))\|_X ds \\
 &\leq t^{\zeta_1} cC(\delta_1 + \tau) \int_0^t (t-s)^{-(\delta_1+\tau)} K_1(2\beta s^{-\zeta_1}) s^{-\zeta_1} \|s^{\zeta_1} A^{\delta_1}(u(s) - w(s))\|_X ds \\
 &\leq t^{\zeta_1} cC(\delta_1 + \tau) \|u - w\|_{\mathbb{H}_T} \int_0^t (t-s)^{-(\delta_1+\tau)} s^{-\zeta_1} K_1(2\beta s^{-\zeta_1}) ds \\
 &= t^{1-\delta_1-\tau} cC(\delta_1 + \tau) \|u - w\|_{\mathbb{H}_T} \int_0^1 (1-s)^{-(\delta_1+\tau)} s^{-\zeta_1} K_1(2\beta(ts)^{-\zeta_1}) ds
 \end{aligned}$$

where we have used Lemma 2.1.1, Lemma 2.1.3, (2.5) and the definition of \mathbb{H}_T . \square

Combining Lemmas 2.2.4 and 2.2.5, we arrive at the following:

Lemma 2.2.6. *Suppose $u, w \in \mathbb{H}_T$ with $\max\{\|u\|_{\mathbb{H}_T}, \|w\|_{\mathbb{H}_T}\} \leq \beta$ and $u_0 = w_0$. Then*

$$\begin{aligned}
 \text{(i)} \quad \|\mathbb{L}u\|_{\mathbb{H}_T} &\leq c_1 (\|u_0\|_X + \|A^{\delta_1-\zeta_1} u_0\|_X) + c_2 \sup_{t \in (0, T]} h_5(t, \beta)
 \end{aligned}$$

for some constants $c_1, c_2 \in (0, \infty)$, where $h_5(t, \beta) = h_1(t, \beta) + h_3(t, \beta)$, with h_1 as in Lemma 2.2.4 and h_3 as in Lemma 2.2.5;

(ii)

$$\|\mathbb{L}u - \mathbb{L}w\|_{\mathbb{H}_T} \leq c_3 \left(\sup_{t \in (0, T]} h_6(t, \beta) \right) \|u - w\|_{\mathbb{H}_T}$$

for some constant $c_3 \in (0, \infty)$, where $h_6 = h_2 + h_4$ with h_2 as in Lemma 2.2.4 and h_4 as in Lemma 2.2.5.

Proof. Using both Lemma 2.2.4 and Lemma 2.2.5, we see that

$$\begin{aligned} \|\mathbb{L}u\|_{\mathbb{H}_T} &= \sup_{t \in (0, T]} \|\mathbb{L}u(t)\|_X + \sup_{t \in (0, T]} \|t^{\zeta_1} A^{\delta_1} u(t)\|_X \\ &\leq c_1 (\|u_0\|_X + \|A^{\delta_1 - \zeta_1} u_0\|_X) + c_2 \sup_{t \in (0, T]} (h_1(t, \beta) + h_3(t, \beta)) \end{aligned}$$

which proves (i). Similarly

$$\begin{aligned} \|\mathbb{L}u - \mathbb{L}w\|_{\mathbb{H}_T} &\leq \sup_{t \in (0, T]} \|\mathbb{L}u(t) - \mathbb{L}w(t)\|_X + \sup_{t \in (0, T]} \|t^{\zeta_1} A^{\delta_1} (\mathbb{L}u(t) - \mathbb{L}w(t))\|_X \\ &\leq c_3 \sup_{t \in (0, T]} (h_2(t, \beta) + h_4(t, \beta)) \|u - w\|_{\mathbb{H}_T}. \end{aligned}$$

□

Corollary 2.2.7. Suppose $K_j(r) \leq k_j(r^j + 1)$ for $j = 1, 2$, $r \geq 0$ and constants k_j . Choose τ such that $\delta_1 + \tau < 1$ and $\zeta_1 \in [0, 1 - \delta_1 - \tau)$ such that $2\zeta_1 + \tau < 1$.

Then

$$\begin{aligned} \|\mathbb{L}u\|_{\mathbb{H}_T} &\leq c_1 (\|u_0\|_X + \|A^{\delta_1 - \zeta_1} u_0\|_X) \\ &\quad + c_2 (T^{1-\tau-2\zeta_1} + T^{1-\tau-\delta_1-\zeta_1} + \beta^2 T^{1-\tau-\delta_1-\zeta_1} (T^{\delta_1-\zeta_1} + 1)) \end{aligned}$$

for some constants $c_1, c_2 \in (0, \infty)$, and

$$\|\mathbb{L}u - \mathbb{L}w\|_{\mathbb{H}_T} \leq c_3 b_0(T, \beta) \|u - w\|_{\mathbb{H}_T}$$

for some constant $c_3 \in (0, \infty)$, where

$$b_0(T, \beta) = T^{1-\tau-\zeta_1} + T^{1-\tau-\delta_1} + \beta T^{1-\tau-\delta_1-\zeta_1} (T^{\delta_1-\zeta_1} + 1).$$

Proof. Under the assumptions, we have

$$\begin{aligned}
 \sup_{(0,T]} h_5(t, \beta) &= \sup_{t \in (0,T]} \left(t^{1-\tau} \int_0^1 (1-s)^{-\tau} K_2 (\beta(ts)^{-\zeta_1}) ds \right. \\
 &\quad \left. + t^{1-\tau-(\delta_1-\zeta_1)} \int_0^1 (1-s)^{-(\tau+\delta_1)} K_2 (\beta(ts)^{-\zeta_1}) ds \right) \\
 &= \sup_{t \in (0,T]} \left(k_2 t^{1-\tau} \int_0^1 (1-s)^{-\tau} ds + k_2 \beta^2 t^{1-\tau-2\zeta_1} \int_0^1 (1-s)^{-\tau} s^{-2\zeta_1} ds \right. \\
 &\quad \left. + k_2 t^{1-\tau-(\delta_1-\zeta_1)} \int_0^1 (1-s)^{-(\tau+\delta_1)} ds \right. \\
 &\quad \left. + k_2 \beta^2 t^{1-\tau-\delta_1-\zeta_1} \int_0^1 (1-s)^{-(\tau+\delta_1)} s^{-2\zeta_1} ds \right) \\
 &\leq c_2 \sup_{t \in (0,T]} \left(t^{1-\tau} + t^{1-\tau-(\delta_1-\zeta_1)} + \beta^2 (t^{1-\tau-2\zeta_1} + t^{1-\tau-\delta_1-\zeta_1}) \right)
 \end{aligned}$$

where we have used the fact that $\delta_1 + \tau < 1$, which ensures all the integrals are finite. Since $\zeta_1 \in [0, 1 - \delta_1 - \tau)$ and $2\zeta_1 + \tau < 1$, all the exponents are > 0 . Thus

$$\sup_{(0,T]} h_4(t, \beta) \leq c_2 \left(T^{1-\tau} + T^{1-\tau-(\delta_1-\zeta_1)} + \beta^2 (T^{1-\tau-2\zeta_1} + T^{1-\tau-\delta_1-\zeta_1}) \right)$$

which proves the first estimate.

Moreover,

$$\begin{aligned}
 \sup_{(0,T]} h_6(t, \beta) &= \sup_{(0,T]} \left(t^{1-\tau-\zeta_1} \int_0^1 (1-s)^{-\tau} s^{-\zeta_1} K_1 (2\beta(ts)^{-\zeta_1}) ds \right. \\
 &\quad \left. + t^{1-\tau-\delta_1} \int_0^1 (1-s)^{-(\delta_1+\tau)} s^{-\zeta_1} K_1 (2\beta(ts)^{-\zeta_1}) ds \right) \\
 &= \sup_{(0,T]} \left(k_1 t^{1-\tau-\zeta_1} \int_0^1 (1-s)^{-\tau} s^{-\zeta_1} ds + 2k_1 \beta t^{1-\tau-2\zeta_1} \int_0^1 (1-s)^{-\tau} s^{-2\zeta_1} ds \right. \\
 &\quad \left. + k_1 t^{1-\tau-\delta_1} \int_0^1 (1-s)^{-(\delta_1+\tau)} s^{-\zeta_1} ds \right. \\
 &\quad \left. + 2k_1 \beta t^{1-\tau-\delta_1-\zeta_1} \int_0^1 (1-s)^{-(\delta_1+\tau)} s^{-2\zeta_1} ds \right) \\
 &\leq c_3 \sup_{(0,T]} \left(t^{1-\tau-\zeta_1} + t^{1-\tau-\delta_1} + \beta (t^{1-\tau-2\zeta_1} + t^{1-\tau-\delta_1-\zeta_1}) \right).
 \end{aligned}$$

Once again, all the exponents are > 0 by our assumptions, so that

$$\sup_{(0,T]} h_6(t, \beta) \leq c_3 \left(T^{1-\tau-\zeta_1} + T^{1-\tau-\delta_1} + \beta (T^{1-\tau-2\zeta_1} + T^{1-\tau-\delta_1-\zeta_1}) \right),$$

which proves the second estimate. \square

Remark 2.2.8. *We have to be careful to choose the parameters in the right way to ensure that our estimates don't explode. We try to choose ζ_1 as big as possible, so we have less regularity assumptions on the initial data, but we are constrained by the other parameters to ensure the estimates don't blow up.*

Theorem 2.2.9. *Under the conditions of Corollary 2.2.7, there exists $T^* > 0$ such that*

$$\|\mathbb{L}u - \mathbb{L}w\|_{\mathbb{H}_{T^*}} \leq \frac{1}{2}\|u - w\|_{\mathbb{H}_{T^*}}$$

for all $u, w \in \mathbb{H}_{T^}$ such that*

$$\max\{\|u\|_{\mathbb{H}_{T^*}}, \|w\|_{\mathbb{H}_{T^*}}\} \leq 2c_1(\|u_0\|_X + \|A^{\delta_1 - \zeta_1}u_0\|_X)$$

for some constant c_1 . Thus the operator \mathbb{L} is a contraction on the ball, and so by the Banach fixed point theorem, \mathbb{L} admits a unique fixed point in \mathbb{H}_{T^} . This fixed point is the unique solution to (Eq.II) i.e. it is the unique element of \mathbb{H}_{T^*} such that*

$$u(t) = P_t u_0 - \int_0^t A^\tau P_{t-s} H(u(s)) ds.$$

Proof. This simply follows from Corollary 2.2.7 by taking $T > 0$ small enough. \square

Example 2.2.10. *Consider again the Navier-Stokes equations:*

$$\begin{cases} \frac{\partial}{\partial t} u + (-\Delta u) + (u \cdot \nabla) u = -\nabla p \\ \nabla \cdot u = 0 \end{cases} \quad (2.6)$$

on the space $X = L^2(\Omega)$ where $\Omega \subset \mathbb{R}^3$ is a bounded subset with smooth boundary. In addition we impose the boundary condition

$$u|_{\partial\Omega} = 0. \quad (2.7)$$

As briefly described in the introduction, we can reformulate this equation so that it is of the form (Eq.II). In this case we can take $\delta_1 = \delta_2 = \frac{1}{2}$, $\tau = \frac{1}{4}$ and $\zeta_1 \in [0, \frac{1}{4})$ (see[3] for details).

It should be noted that one can also treat the case $\zeta_1 = \frac{1}{4}$ too, though the proof is much more complicated. This will correspond to the case when the initial data is $\frac{1}{2}$ differentiable. An open question very much related to the Navier-Stokes millennium problem is whether one can go beyond this i.e. treat the case $\zeta_1 > \frac{1}{4}$.

We also remark that in most cases the boundary condition $u|_{\partial\Omega} = 0$ is a reasonable physical assumption for a flowing fluid, as it says the velocity of the fluid becomes instantaneously 0 on the boundary. However, when the fluid is moving very fast, experiments have shown that the physical boundary is not the actual boundary (this is the case for example in engines). Indeed, the physical boundary can be slightly outside the actual boundary. In this kind of situation the boundary conditions are quite different, and we therefore have to perform a more careful analysis. This situation is treated in detail in chapter 3.

2.3 A more regular solution space

Since the equation we are considering is of the form

$$\frac{\partial u}{\partial t} + Au + Q(u) = 0$$

we might think that if $u \in \mathcal{D}(A)$ then u will be differentiable in t . So perhaps our approach should involve the time derivative of u . In this section we briefly mention that, using similar techniques to those used in section 2.2 and under some additional assumptions, one can show that a solution in small time exists in an adjusted space, which also takes into account the time derivative of u .

Definition 2.3.1. Let $T > 0$ and $\zeta_2 \geq 0$ be some fixed number to be chosen later. We will say that a path $u : [0, T] \rightarrow X$ belongs to \mathbb{W}_T if

- (i) $u \in \mathbb{H}_T$ i.e. $u \in \mathcal{C}([0, T], X)$, $u(t) \in \mathcal{D}(A^{\delta_1})$ for all $t \in (0, T]$, $A^{\delta_1}u \in \mathcal{C}((0, T], X)$ and

$$\sup_{t \in (0, T]} \|t^{\zeta_1} A^{\delta_1} u(t)\|_X < \infty;$$

- (ii) $u \in \mathcal{C}^1((0, T], X)$, (so u is differentiable with respect to t and its derivative is a continuous path on $(0, T]$), such that

$$u'(t) \in \mathcal{D}(A^{\delta_2})$$

for all $t \in (0, T]$, and

$$\sup_{t \in (0, T]} \|t^{\zeta_2} A^{\delta_2} u'(t)\|_X < \infty.$$

If we define, for $u \in \mathbb{W}_T$

$$\|u\|_{\mathbb{W}_T} := \sup_{t \in [0, T]} \|u(t)\|_X + \sup_{t \in (0, T]} \|t^{\zeta_1} A^{\delta_1} u(t)\|_X + \sup_{t \in (0, T]} \|t^{\zeta_2} A^{\delta_2} u'(t)\|_X,$$

\mathbb{W}_T becomes a Banach space.

For $u \in \mathbb{W}_T$, we consider

$$\mathbb{L}u(t) = P_t u_0 - \int_0^t A^\tau P_{t-s} H(u(s))$$

as before.

In this case, using very similar ideas to those described in detail in section 2.2, we can prove the following:

Theorem 2.3.2. *Suppose $(x, \xi) \mapsto DH(x)\xi$ is weakly differentiable for all $x, \xi \in \mathcal{D}(A^{\delta_1})$ and such that*

$$\|(D^2 H(x)\xi) \eta\|_X \leq \begin{cases} K_0(\|A^{\delta_1} x\|_X) \|A^{\delta_1} \xi\|_X \|A^{\delta_2} \eta\|_X \\ K_0(\|A^{\delta_1} x\|_X) \|A^{\delta_2} \xi\|_X \|A^{\delta_1} \eta\|_X \end{cases}$$

for some non-decreasing function $K_0 : [0, \infty) \rightarrow [0, \infty)$. Suppose $K_j(r) \leq k_j(r^j + 1)$ for $j = 0, 1, 2$ and constants k_j . Then for $u, w \in \mathbb{W}_T$ such that $\|u\|_{\mathbb{W}_T}, \|w\|_{\mathbb{W}_T} \leq \beta$ and $u_0 = w_0$, we have

(i)

$$\|\mathbb{L}u\|_{\mathbb{W}_T} \leq c_1(\|A^\alpha u_0\|_X + \|H(u_0)\|_X) + c_2 T^{\zeta_1} \|H(u_0)\|_X + c_3 \kappa(T) \beta^2$$

for $\alpha = \max\{\delta_1 - \zeta_1, 1 + \delta_2 - \zeta_2\}$ and $\kappa(T) = T^{1-\tau-2\zeta_1} + T^{1-\zeta_2} + T^{1-\tau-\delta_2-\zeta_1}$;

(ii)

$$\|\mathbb{L}u - \mathbb{L}w\|_{\mathbb{W}_T} \leq c_4 \beta \kappa(T) \|u - w\|_{\mathbb{W}_T}$$

where $\kappa(T)$ is as in part (i).

Thus, once again, with a suitable choice of parameters we have by the Banach fixed point theorem that there exists a unique $u \in \mathbb{W}_{T^*}$ for some $T^* > 0$ which is a solution to (Eq.II).

Proof. There is technicality here, which does not appear in the proofs given in the previous section. This difference comes from the fact that we must differentiate in t i.e. we must formally take the derivative of $\mathbb{L}u(t)$ in t . If this is done straight away we will end up with a $A^{1+\tau}$ in front of P_{t-s} , which is too much. To get around this, we must first use integration by parts. Indeed, if we define $w(t) := H(u(t))$ we have

$$w'(t) = DH(u(t))u'(t).$$

Then

$$\begin{aligned} \int_0^t \frac{d}{ds} P_{t-s}(w(s)) ds &= \int_0^t P_{t-s}(Aw(s) + w'(s)) ds \\ \Rightarrow A \int_0^t P_{t-s}(w(s)) ds &= w(t) - P_t w_0 - \int_0^t P_{t-s}(w'(s)) ds. \end{aligned}$$

We have thus removed the operator A from the left-hand side, but have paid a price in the form of the term involving $w'(s)$. This is where the assumption on D^2H is used. Everything else is the same as in the previous section, but the proofs are quite long and tedious, and so we omit them. \square

3 The Navier-Stokes equations with kinematic and Navier boundary conditions

The contents of this chapter is primarily taken from the paper [1] of Gui-Qiang Chen and Zhongmin Qian. Our main goal is to prove the existence in small time of a strong solution to the Navier-Stokes equations in a bounded domain Ω with kinematic and Navier boundary conditions.

3.1 The Helmholtz decomposition

Suppose that $u = (u^1, u^2, u^3)$ is a vector field on a domain $\Omega \subset \mathbb{R}^3$ which has smooth boundary Γ . Suppose also that $u \in \mathcal{C}^2$. We first aim to show that we may decompose u as

$$u = P_\infty u + \nabla f$$

where $\operatorname{div} (P_\infty u) = 0$. If we have such a decomposition, it follows that $\operatorname{div} (u) = \Delta f$. Indeed, we *define* P_∞ to be

$$P_\infty(u) = u - \nabla f$$

where f is the unique (up to a constant) solution of the equation

$$\begin{cases} \operatorname{div} (u) = \Delta f \\ \langle \nabla f, n \rangle \equiv \frac{\partial f}{\partial n} = \langle u, n \rangle \text{ on } \Gamma \end{cases} \quad (3.1)$$

where n is a vector normal to surface Γ . Note that f depends on the value of u on the boundary Γ . Then $\operatorname{div} (P_\infty u) = 0$ and importantly

$$\langle P_\infty u, n \rangle = 0$$

on the boundary Γ .

This continues to work for any $u \in H^1(\Omega) = \{f \in L^2(\Omega) : Df \in L^2(\Omega)\}$ by standard Sobolev space theory. However, we can extend this decomposition even further to general $u \in L^2(\Omega)$ by using a Poincaré inequality. In the case where Ω is a bounded domain, such an inequality holds. Indeed, suppose $u \in L^2(\Omega)$. Then we can choose $(u_n)_{n \geq 0} \subset \mathcal{C}^\infty(\Omega)$ such that $u_n \rightarrow u$ in $L^2(\Omega)$. Now

$$u_n = P_\infty(u_n) + \nabla f_n.$$

We can suppose that $\int_{\Omega} f_n = 0$ for all n , since we can shift f_n by a constant (recall f_n is only unique up to a constant). To pass to the limit we need to control

$$\|\nabla(f_n - f_m)\|_{L^2(\Omega)}$$

Note that we have

$$\begin{cases} \Delta(f_n - f_m) = \operatorname{div}(u_n - u_m) \\ \frac{\partial(f_n - f_m)}{\partial n} = \langle u_n - u_m, n \rangle \end{cases}.$$

Since $\operatorname{div}(f \nabla f) = |\nabla f|^2 + f \Delta f$,

$$\begin{aligned} \|\nabla(f_n - f_m)\|_{L^2(\Omega)}^2 &= \int_{\Omega} \operatorname{div} \left((f_n - f_m) \nabla(f_n - f_m) \right) - \int_{\Omega} (f_n - f_m) \operatorname{div}(u_n - u_m) \\ &= \int_{\Gamma} (f_n - f_m) \frac{\partial(f_n - f_m)}{\partial n} + \int_{\Omega} \langle \nabla(f_n - f_m), u_n - u_m \rangle \\ &\quad - \int_{\Gamma} (f_n - f_m) \langle u_n - u_m, n \rangle \\ &= \int_{\Omega} \langle \nabla(f_n - f_m), u_n - u_m \rangle, \end{aligned}$$

using the divergence theorem and integration by parts. Thus

$$\begin{aligned} \|\nabla(f_n - f_m)\|_{L^2(\Omega)}^2 &\leq \|\nabla(f_n - f_m)\|_{L^2(\Omega)} \|u_n - u_m\|_{L^2(\Omega)} \\ \Rightarrow \|\nabla(f_n - f_m)\|_{L^2(\Omega)} &\leq \|u_n - u_m\|_{L^2(\Omega)}. \end{aligned}$$

Since Ω is assumed to be bounded, a Poincaré inequality holds. Thus there exists a constant C such that

$$\|(f_n - f_m)\|_{L^2(\Omega)} \leq C \|\nabla(f_n - f_m)\|_{L^2(\Omega)} \leq C \|u_n - u_m\|_{L^2(\Omega)}.$$

Therefore $f_n \rightarrow f$ and $\nabla f_n \rightarrow g$ in $L^2(\Omega)$ for some functions $f, g \in L^2(\Omega)$, so that $\nabla f_n \rightarrow \nabla f$ in $L^2(\Omega)$ since ∇ is a closed operator. We have therefore shown that we can extend the decomposition to L^2 on bounded domains i.e.

Lemma 3.1.1 (Helmholtz Decomposition). *Suppose $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary. Then there exists a unique projection operator $P_{\infty} : L^2(\Omega) \rightarrow L^2(\Omega)$ such that if $u \in H^1(\Omega)$ then*

$$\operatorname{div}(P_{\infty} u) = 0 \text{ in } \Omega, \quad \langle P_{\infty} u, n \rangle = 0 \text{ on } \Gamma.$$

Remark 3.1.2. *This statement actually remains true for unbounded domains, but the proof is more involved.*

3.2 The kinematic and Navier boundary conditions

Now that we have rigorously shown that the Helmholtz decomposition is valid in our situation, we can project the Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary Γ onto the Hilbert space $\mathcal{X}(\Omega) := P_\infty(L^2(\Omega))$. Indeed, by applying P_∞ to the equations

$$\begin{cases} \frac{\partial}{\partial t} u + (-\Delta u) + (u \cdot \nabla) u = -\nabla p \\ \nabla \cdot u = 0 \end{cases} \quad (3.2)$$

we arrive at the evolution equation

$$\frac{\partial}{\partial t} u + P_\infty(u \cdot \nabla u) = P_\infty \circ \Delta u \quad (3.3)$$

in $\mathcal{X}(\Omega) = P_\infty(L^2(\Omega))$.

To be able to solve such an equation in Ω , we need to impose some boundary conditions. In fluid dynamics, if the rigid surface Γ is at rest, the kinematic and no-slip conditions are often imposed. The kinematic condition means that the normal component of the velocity on the boundary vanishes, that is, the velocity u at the boundary is tangent to Γ :

$$u^\perp|_\Gamma := \langle u, n \rangle = 0.$$

On the other hand the no-slip condition demands the coincidence of the tangent component of the fluid velocity with that of the boundary Γ . These two boundary conditions lead to the Dirichlet boundary problem associated with the Navier-Stokes equations. There has been a large literature for the Navier-Stokes equations subject to the Dirichlet boundary conditions (see for example [4], [7] and the references therein). The fundamental problem of the global (in time) existence and uniqueness of a strong solution remains open (and is one of the millennium problems); but the Dirichlet boundary problem of the Navier-Stokes equations is well-posed at least for a small time, or for small data globally in time.

However, the no-slip assumption does not always match the experimental results. Navier first proposed the so-called Navier boundary condition in [5], which is essentially a "slip-with-friction" boundary condition. It states that the tangent part of the velocity u at the boundary should be proportional to that of the normal vector field of the stress tensor with proportional constant $\zeta > 0$. Such a boundary condition plays an important role in the case of fast moving fluids (see [2]). It is these important physical situations that motivate the study described below of the Navier-Stokes equations subject to the Navier boundary condition.

In [1] an important step is to reformulate the Navier boundary condition in terms of the vorticity $\omega := \nabla \times u$ ($= \text{curl } u$). This reformulation requires some geometry, and so we sketch the ideas here, and refer the reader to [1] for the details.

We will use the convention that repeated indices in a formula are understood to be summed up from 1 to 3. For the bounded domain $\Omega \subset \mathbb{R}^3$ with smooth compact boundary Γ we carry out local computations on the boundary in a moving frame compatible with Γ . More precisely, if n is the unit normal to Γ pointing outwards with respect to Ω , by a moving frame we mean any local orthonormal basis (e_1, e_2, e_3) of the tangent space $T\Omega$ such that $e_3 = n$ when restricted to Γ . If $u = \sum_{j=1}^3 u^j e_j$ is a vector field on Ω then, on the boundary,

$$u^\parallel = \sum_{j=1}^2 u^j e_j, \quad u^\perp = u^3 n$$

are the tangent part and the normal part respectively of u . The Christoffel symbols Γ_{ij}^l are determined by the directional derivatives $\nabla_i e_j = \Gamma_{ij}^k e_k$ (recall $\Gamma_{ij}^k e_k = \sum_{k=1}^3 \Gamma_{ij}^k e_k$ by convention) where ∇_i is the directional derivative in the direction e_i .

The tensor $(\pi_{ij})_{1 \leq i, j \leq 2}$ where $\pi_{ij} = -\Gamma_{ij}^3$ for $i, j = 1, 2$ is a symmetric tensor on Γ . We define

$$\pi(u^\parallel, v^\parallel) := \sum_{i, j=1, 2} \pi_{ij} u^i v^j$$

for any $u^\parallel, v^\parallel \in T\Gamma$. We will also identify π with the linear transformation defined as follows: if $u^\parallel = \sum_{j=1}^2 u^j e_j$ is tangent to Γ , then

$$\pi(u^\parallel) := \sum_{j=1}^2 \pi(u^\parallel)^j e_j = \sum_{i, j=1}^2 \pi_{ij} u^i e_j,$$

so that $\langle \pi(u^\parallel), v^\parallel \rangle = \pi(u^\parallel, v^\parallel)$.

The Navier boundary condition is usually formulated in a moving frame compatible to Γ as

$$u^k = -\zeta \left(\nabla_3 u^k + \nabla_k u^3 \right) \quad \text{on } \Gamma, \text{ for } k = 1, 2, \quad (3.4)$$

where ζ is the slip length which is a positive scalar function on Γ depending only on the nature of the fluid and the material of the rigid boundary. We refer the reader to [2] for the physical interpretation of this condition. In order to reformulate this condition in terms of the vorticity, we use the Hodge operator $*$. This operator is defined as

$$*(v^1, v^2) := (-v^2, v^1)$$

for any vector field (v^1, v^2) on Γ . The effect of this operator is to rotate a vector on Γ by 90° with respect to the normal vector pointing to the interior of Ω . It is independent of the choice of the moving frame, and may be defined via the identity

$$\langle v \times (*u^\parallel), n \rangle = \langle u^\parallel, v^\parallel \rangle \quad (3.5)$$

on Γ for any vector fields u, v . With all these definitions, it is then possible to reformulate the Navier boundary condition (3.4) as

$$\omega^\parallel \Big|_\Gamma = -\frac{1}{\zeta}(*u) + 2(*\pi(u)). \quad (3.6)$$

For a proof this see Proposition 2.1 of [1]. From now on we will refer to (3.6) as the *Navier boundary condition*. This condition can be rewritten in terms of coordinates as

$$(\nabla \times u)^1 = \frac{1}{\zeta}u^2 - 2 \sum_{j=1}^2 \pi_{j2}u^j, \quad (\nabla \times u)^2 = -\frac{1}{\zeta}u^1 + 2 \sum_{j=1}^2 \pi_{j1}u^j \quad (3.7)$$

in a moving frame compatible with Γ , for constant $\zeta > 0$.

3.3 The Stokes operator with Navier boundary condition

As per the discussion in the previous section, we are interested in a solution to the equation

$$(NS) \quad \begin{cases} \partial_t u + (u \cdot \nabla)u = \Delta u - \nabla p \\ \nabla \cdot u = 0 \end{cases} \quad (3.8)$$

(or more precisely the projection of these equations onto $\mathcal{X}(\Omega)$ as described above) in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth compact boundary Γ , subject to the boundary conditions

$$(BC) \quad \begin{cases} u^\perp|_\Gamma = 0 \\ (\nabla \times u)^\parallel \Big|_\Gamma = -\frac{1}{\zeta} * u + 2 * \pi(u) \end{cases} \quad (3.9)$$

for a small constant $\zeta > 0$ and where $\pi : T^*\Gamma \rightarrow T^*\Gamma$, is as described in section 3.1. The initial data will be given by

$$(I) \quad u|_{t=0} = u_0.$$

We suppose that we are working in a moving frame compatible with Γ .

Define the operator A by

$$\mathcal{D}_{\zeta,0}(A) = \{u \in \mathcal{X}(\Omega) \cap \mathcal{C}^\infty(\bar{\Omega}) : u \text{ satisfies } (BC)\}$$

and

$$A = -P_\infty \circ \Delta$$

3.3 The Stokes operator with Navier boundary condition

where P_∞ is the projection from the Helmholtz decomposition. Then $(A, \mathcal{D}_{\zeta,0}(A))$ is a densely defined linear operator on the Hilbert space $\mathcal{X}(\Omega)$. If $u \in \mathcal{D}_{\zeta,0}(A)$, $Au = -P_\infty(\Delta u)$ and using a standard vector calculus identity

$$\begin{aligned}\Delta u &= -\nabla \times (\nabla \times u) + \nabla(\nabla \cdot u) \\ &= -\nabla \times (\nabla \times u)\end{aligned}\tag{3.10}$$

since $\nabla \cdot u = 0$. Now define the bi-linear form

$$\mathcal{E}(u, w) := - \int_{\Omega} \langle P_\infty \circ \Delta u, w \rangle \tag{3.11}$$

for any $u, w \in \mathcal{D}_{\zeta,0}(A)$. Using the fact that P_∞ is a projection, we then have for any $u, w \in \mathcal{D}_{\zeta,0}(A)$

$$\begin{aligned}\mathcal{E}(u, w) &= - \int_{\Omega} \langle \Delta u, P_\infty w \rangle \\ &= - \int_{\Omega} \langle \Delta u, w \rangle \\ &= \int_{\Omega} \langle \nabla \times (\nabla \times u), w \rangle \\ &= \int_{\Omega} \langle \nabla \times u, \nabla \times w \rangle + \int_{\Gamma} \langle (\nabla \times u)^\parallel \times w, n \rangle\end{aligned}$$

where we have also used (3.10), integration by parts and the fact that u, w satisfy (BC) . In particular, since u, w satisfy (BC) we have $u^\perp = w^\perp = 0$ on Γ so that $u = u^\parallel, w = w^\parallel$ on Γ . Therefore, by the definition of π and $*$ given in section 3.2, we have

$$\begin{aligned}\langle (\nabla \times u)^\parallel \times w, n \rangle &= \left\langle \left(-\frac{1}{\zeta} * u + 2 * \pi(u) \right) \times w, n \right\rangle \\ &= -\frac{1}{\zeta} \left\langle \begin{pmatrix} -u^2 \\ u^1 \\ 0 \end{pmatrix} \times \begin{pmatrix} w^1 \\ w^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \\ &\quad + \left\langle \begin{pmatrix} -\sum_i \pi_{i2} u^i \\ \sum_i \pi_{i1} u^i \\ 0 \end{pmatrix} \times \begin{pmatrix} w^1 \\ w^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \\ &= \frac{1}{\zeta} (u^1 w^1 + u^2 w^2) - 2 \sum_{i,j} \pi_{ij} u^i w^j \\ &= \frac{1}{\zeta} \langle u, w \rangle - 2\pi(u, w)\end{aligned}$$

on Γ so that

$$\mathcal{E}(u, w) = \int_{\Omega} \langle \nabla \times u, \nabla \times w \rangle + \frac{1}{\zeta} \int_{\Gamma} \langle u, w \rangle - 2 \int_{\Gamma} \pi(u, w).$$

Therefore \mathcal{E} is symmetric and bi-linear. Moreover, since $u^\perp|_{\Gamma} = w^\perp|_{\Gamma} = 0$

$$\int_{\Omega} \langle \nabla u, \nabla w \rangle = \int_{\Omega} \langle \nabla \times u, \nabla \times w \rangle - \int_{\Gamma} \pi(u, w).$$

We have therefore proved the following:

Lemma 3.3.1. *The bi-linear form $(\mathcal{E}, \mathcal{D}_{\zeta,0}(A))$ on $\mathcal{X}(\Omega)$ given by (3.11) is symmetric and such that*

$$\mathcal{E}(u, w) = \int_{\Omega} \langle \nabla u, \nabla w \rangle - \int_{\Gamma} \pi(u, w) + \frac{1}{\zeta} \int_{\Gamma} \langle u, w \rangle. \quad (3.12)$$

The next Lemma states some further useful properties of the bi-linear form \mathcal{E} .

Lemma 3.3.2. (i) *For any $\varepsilon \in (0, 1)$, there exists a constant $C(\varepsilon, \Omega)$ such that*

$$\mathcal{E}(u, u) \geq (1 - \varepsilon) \|\nabla u\|_2^2 - C(\varepsilon, \Omega) \|u\|_2^2$$

for any $u \in \mathcal{D}_{\zeta,0}(A)$.

(ii) *$(\mathcal{E}, \mathcal{D}_{\zeta,0}(A))$ is closable on $\mathcal{X}(\Omega)$ and its closure will be denoted by $(\mathcal{E}, \mathcal{D}_{\zeta}(\mathcal{E}))$. Moreover, identity (3.12) remains true for any $u, w \in \mathcal{D}_{\zeta}(\mathcal{E})$.*

(iii) *If $\pi \leq \frac{1}{\zeta}$, then*

$$\mathcal{E}(u, u) \geq \|\nabla u\|_2^2$$

for any $u \in \mathcal{D}_{\zeta}(\mathcal{E})$.

(iv) *$\mathcal{D}_{\zeta}(\mathcal{E}) = \mathcal{X}(\Omega) \cap H^1(\Omega)$, which is thus independent of ζ and denoted by $\mathcal{D}(\mathcal{E})$.*

Proof. For (i), let λ_1 be an upper bound for π i.e. $\pi \leq \lambda_1$. Then by (3.12), we have

$$\mathcal{E}(u, u) \geq \|\nabla u\|_2^2 - \lambda_1 \int_{\Gamma} |u|^2 \geq (1 - \varepsilon) \|\nabla u\|_2^2 - \frac{C}{\varepsilon} \|u\|_2^2$$

for some $C = C(\Omega) > 0$ and any $\varepsilon \in (0, 1)$, where we have used the Sobolev embedding inequality:

$$\int_{\Gamma} |u|^2 \leq \varepsilon \|\nabla u\|_2^2 + \frac{C}{\varepsilon} \|u\|_2^2 \quad \forall \varepsilon \in (0, 1).$$

For (ii), suppose $\{u_n\}_{n \geq 1} \subset \mathcal{D}_{\zeta,0}(A)$ are such that $\mathcal{E}(u_n - u_m, u_n - u_m) \rightarrow 0$ and $\|u_n - u_m\|_2 \rightarrow 0$. Then by part (i), there exists a constant C such that

$$\frac{1}{2} \|\nabla(u_n - u_m)\|_2^2 \leq C \|u_n - u_m\|_2^2 + \mathcal{E}(u_n - u_m, u_n - u_m)$$

so that $\|\nabla(u_n - u_m)\|_2^2 \rightarrow 0$. Thus, $\{u_n\}_{n \geq 1}$ is a Cauchy sequence in $H^1(\Omega)$, and hence there exists a unique $u \in H^1(\Omega)$ such that

$$\|u_n - u\|_2^2 + \|\nabla(u_n - u)\|_2^2 \rightarrow 0.$$

It follows by the Sobolev embedding theorem that

$$\lim_{n \rightarrow \infty} \int_{\Gamma} \langle u_n, u_n \rangle = \int_{\Gamma} |u|^2, \quad \lim_{n \rightarrow \infty} \int_{\Gamma} \pi(u_n, u_n) = \int_{\Gamma} \pi(u, u)$$

so that

$$\lim_{n \rightarrow \infty} \mathcal{E}(u_n, u_n) = \int_{\Omega} |\nabla u|^2 + \frac{1}{\zeta} \int_{\Gamma} |u|^2 - \int_{\Gamma} \pi(u, u),$$

and u belongs to the closure of $(\mathcal{E}, \mathcal{D}_{\zeta,0}(A))$.

Part (iii) follows easily from identity (3.12).

Finally, for part (iv), we remark that Navier's ζ -boundary condition has to be satisfied for any $u \in \mathcal{D}_{0,\zeta}$, which will be forgotten when passing to the limit in $H^1(\Omega)$ (in which the boundary values of the first derivative can not be retained). Therefore $\mathcal{D}_{0,\zeta}(A)$ is dense in $\mathcal{X}(\Omega) \cap H^1(\Omega)$ (see [1] for details). \square

Corollary 3.3.3. *$(\mathcal{E}, \mathcal{D}_{\zeta}(\mathcal{E}))$ is a densely defined closed symmetric form on the Hilbert space $\mathcal{X}(\Omega)$ which is bounded from below. Moreover,*

$$\mathcal{E}(u, u) = \|\nabla u\|_2^2 + \frac{1}{\zeta} \|u\|_{L^2(\Gamma)}^2 - \int_{\Gamma} \pi(u, u)$$

for any $u \in \mathcal{D}_{\zeta}(\mathcal{E})$, and there exist constants $\Lambda, M(\varepsilon, \zeta)$ such that

$$0 \leq (\mathcal{E} + \Lambda I)(u, u) \leq (1 + \varepsilon) \|\nabla u\|_2^2 + M(\varepsilon, \zeta) \|u\|_2^2$$

for all $u \in \mathcal{D}_{\zeta}(\mathcal{E})$ and any $\varepsilon > 0$.

Proof. The first inequality follows directly from part (i) of Lemma 3.3.2. For the second inequality, note that $\pi \geq -C_0$ for some $C_0 > 0$ since Γ is smooth and compact. Thus by (3.12)

$$\mathcal{E}(u, u) \leq \|\nabla u\|_2^2 + \left(\frac{1}{\zeta} + C_0 \right) \|u\|_{L^2(\Gamma)}^2.$$

The Sobolev embedding theorem then yields that, for every $\varepsilon \in (0, 1)$ there exists $C_0 > 0$ such that

$$\left(\frac{1}{\zeta} + C_0\right) \|u\|_{L^2(\Gamma)}^2 \leq \varepsilon \|\nabla u\|_2^2 + M(\varepsilon, \zeta) \|u\|_2^2.$$

□

Definition 3.3.4. Let $\zeta > 0$. Then the unique self-adjoint operator on $\mathcal{X}(\Omega)$ associated with the closed symmetric form $(\mathcal{E}, \mathcal{D}_\zeta(\mathcal{E}))$ is denoted again by A , with domain $\mathcal{D}_\zeta(A)$. It is called the Stokes operator with the Navier ζ -boundary condition.

Remark 3.3.5. According to the definition, $(A, \mathcal{D}_\zeta(A))$ is the unique self-adjoint operator on $\mathcal{X}(\Omega)$ such that

$$\mathcal{E}(u, w) = - \int_{\Omega} \langle Au, w \rangle$$

for any $u, w \in \mathcal{D}_\zeta(A)$ and

$$\mathcal{D}_{0,\zeta}(A) \subset \mathcal{D}_\zeta(A) \subset H^1(\Omega) \cap \mathcal{X}(\Omega).$$

Moreover, if $u \in \mathcal{D}_\zeta(A)$, then $u \in H^1(\Omega)$ with $\nabla \cdot u = 0$ and $u^\perp|_\Gamma = 0$. In particular there exists $\Lambda \geq 0$ such that $-A + \Lambda I$ is positive definite.

3.4 Spectral theory and useful estimates

In this section we very briefly mention some estimates that are required for the existence results in section 3.5. We do not prove these technical results, and refer the reader once again to [1] for the details. We instead give some flavour of the estimates needed.

Let $\Lambda > 0$ be the constant such that $-A + \Lambda I \geq 0$. Then, for $\lambda > \Lambda$ let $R_\lambda := (\lambda I - A)^{-1}$, which is a bounded linear operator on $\mathcal{X}(\Omega)$.

Theorem 3.4.1. For any $\lambda > \Lambda$, R_λ is a compact operator on $\mathcal{X}(\Omega)$.

Corollary 3.4.2. The spectrum of the Stokes operator $(A, \mathcal{D}_\zeta(A))$ with Navier's ζ -boundary condition is discrete and belongs to $(-\infty, \Lambda]$. The eigenvalues $\lambda_j \leq \Lambda$ can be ordered as

$$\Lambda \geq \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_n \geq \dots$$

with $\lambda_n \rightarrow -\infty$. Moreover, there are eigenfunctions $\{a_k\}_{k \geq 0} \subset \mathcal{D}_{\zeta,0}(A)$ which form a complete orthonormal basis of $\mathcal{X}(\Omega)$.

Let $\{a_k\}_{k \geq 0}$ be the orthonormal basis of $\mathcal{X}(\Omega)$ consisting of eigenfunctions of $(A, \mathcal{D}_\zeta(A))$ given by the above corollary. For an integer N let X_N be the Hilbert space spanned by $\{a_k : k \leq N\}$. Thus $\bigcup_N X_N = \mathcal{X}(\Omega)$. Let $P_N : L^2(\Omega) \rightarrow X_N$ be the projection such that for $u \in L^2(\Omega)$,

$$P_N u = \sum_{k=0}^N a_k \int_{\Omega} \langle a_k, u \rangle.$$

Of course $P_{\infty} u = \sum_{k=0}^{\infty} a_k \int_{\Omega} \langle a_k, u \rangle$ is the projection from $L^2(\Omega)$ onto $\mathcal{X}(\Omega)$ as above.

Let $u \in \mathcal{D}_\zeta(A)$ and define $\omega := \nabla \times u$ and $\psi := \nabla \times \omega = -\Delta u$. We will also use the notation that for $g_1, \dots, g_m \in L^2(\Omega)$

$$\|(g_1, \dots, g_m)\|_2^2 := \sum_{j=1}^m \|g_j\|_2^2.$$

Lemma 3.4.3. *For every $\varepsilon > 0$, there exists $M(\varepsilon) > 0$ such that*

$$\int_{\Gamma} \partial_n (|\psi|^2) \leq \varepsilon \|\nabla^3 u\|_2^2 + M \|(\psi, u)\|_2^2$$

for any $u \in \bigcup_N X_N$.

Theorem 3.4.4. *Let $u \in \bigcup_N X_N$. Then*

$$\|\nabla^3 u\|_2 \leq M (\|\nabla \psi\|_2 + \|u\|_{H^2})$$

so that

$$\|u\|_{H^3} \leq M \|(\nabla \psi, \psi, u)\|_2,$$

where $M > 0$ is a constant depending only on ζ and the domain Ω , which may be different in each occurrence.

Corollary 3.4.5. *There exists a constant M such that*

$$M \|(\nabla \psi, \psi, u)\|_2 \leq \|u\|_{H^3} \leq M^{-1} \|(\nabla \psi, \psi, u)\|_2$$

for any $u \in \bigcup_N X_N$.

Lemma 3.4.6. *For any $\varepsilon > 0$, there exists $M > 0$ such that*

$$\|(\nabla \times P_N(u), \nabla P_N(u))\|_2^2 \leq M \|(\nabla \times u, u)\|_2^2$$

for any $u \in H^2(\Omega)$ and integer N .

3.5 Existence of a strong solution to (NS) subject to (BC) and (I) in small time

The idea is to first construct a weak solution globally in time, and then, using the estimates above, show that for small time such a solution is in fact a strong solution.

3.5.1 Weak solutions

We introduce the notion of a weak solution to the initial-boundary problem (NS) with (BC) and (I) in the following way. The minimal requirement on the initial data is that $u_0 \in \mathcal{X}(\Omega)$.

Definition 3.5.1. A vector field $u(t, x)$ on Ω is said to be a weak solution of (NS) with (BC) and (I) if

- (i) for each $t > 0$, $u(t, \cdot) \in \mathcal{X}(\Omega)$ and $u \in L^2([0, T], H^1(\Omega))$ for any $T > 0$;
- (ii) for any smooth vector field $\varphi(t, x)$ with $\varphi(t, \cdot) \in \mathcal{X}(\Omega)$ for $t \in [0, T]$

$$\begin{aligned} & \int_{\Omega} \langle u(T, \cdot), \varphi(T, \cdot) \rangle \\ &= \langle u_0, \varphi_0 \rangle + \int_0^T \int_{\Omega} \langle u(t, \cdot), \partial_t \varphi(t, \cdot) \rangle \\ & \quad - \int_0^T \int_{\Omega} \langle \nabla \times u, (u \times \varphi + \nabla \times \varphi) \rangle - \frac{1}{\zeta} \int_0^T \int_{\Gamma} \langle u, \varphi \rangle + 2 \int_0^T \int_{\Gamma} \pi(u, \varphi) \end{aligned} \quad (3.13)$$

for any $T > 0$;

(iii)

$$\|u(T, \cdot)\|_2^2 + 2 \int_0^T \|\nabla u\|_2^2 + 2 \int_0^T \int_{\Gamma} \left(\frac{1}{\zeta} |u|^2 - \pi(u, u) \right) \leq \|u_0\|_2^2$$

for any $T > 0$, which is the energy inequality.

Remark 3.5.2. Equation (3.13) is obtained by integrating (NS) and using integration by parts together with (BC).

We start the construction of such a weak solution by recalling that, by Corollary 3.4.2, the operator $(A, \mathcal{D}_{\zeta}(A))$ as defined in section 3.3 has a discrete spectrum consisting of eigenvalues $\{\lambda_n\}_{n \geq 0}$ such that

$$\Lambda \geq \lambda_0 \geq \lambda_1 \geq \dots$$

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and the eigenfunctions $\{a_n\}_{n \geq 0} \subset \mathcal{D}_{0,\zeta}(A)$ form an orthonormal basis of $\mathcal{X}(\Omega)$. Then a_n are subject to (BC) and such that

$$\Delta a_n - \nabla p_n = \lambda_n a_n.$$

Suppose

$$u(t, \cdot) = \sum_{k=0}^{\infty} c_k(t) a_k, \quad c_k(t) = \int_{\Omega} \langle a_k, u \rangle$$

is a solution of (NS) with initial data u_0 . Then

$$\partial_t c_k = \int_{\Omega} \langle a_k, \Delta u \rangle - \int_{\Omega} \langle a_k, u \cdot \nabla u \rangle = \lambda_k c_k - \sum_{i,j=0}^{\infty} c_i c_j \int_{\Omega} \langle a_k, a_i \cdot \nabla a_j \rangle.$$

In view of this, for each integer N , we solve the Cauchy problem

$$\begin{cases} \frac{d}{dt} c_k = \lambda_k c_k - \sum_{i,j=0}^N c_i c_j \int_{\Omega} \langle a_k, a_i \cdot \nabla a_j \rangle \\ c_k|_{t=0} = \int_{\Omega} \langle a_k, u_0 \rangle \end{cases} \quad (3.14)$$

and then define

$$u^N(t, \cdot) := \sum_{k=0}^N c_k(t) a_k.$$

Thus $u^N(t, \cdot) \in \mathcal{D}_{0,\zeta}(A)$ for $t > 0$ and u^N satisfies the evolution equation

$$\partial_t u^N = A u^N - \sum_{k=0}^N a_k \int_{\Omega} \langle a_k, u^N \cdot \nabla u^N \rangle \quad (3.15)$$

subject to boundary conditions (BC). We now make energy estimates, and (with the help of the Sobolev embedding theorem) we arrive at

$$\|u^N(T, \cdot)\|_2^2 + \int_0^T \|\nabla u^N\|_2^2 + \frac{2}{\zeta} \int_0^T \int_{\Gamma} |u^N|^2 \leq \|u_0\|_2^2 + C \int_0^T \|u^N(s, \cdot)\|_2^2$$

for some constant C independent of N and t . This estimate also ensures that, for each N , the system (3.14) has a unique solution for all $t > 0$. Finally, by the Gronwall lemma, we have that $\|u^N(t, \cdot)\|_2^2$ and $\|\nabla u^N\|_2^2$ are uniformly bounded in t and N . Thus we arrive at the following existence result:

Theorem 3.5.3. *Let $u_0 \in \mathcal{X}(\Omega)$. Then, for any $T > 0$, the family $\{u^N(t, x)\}$, for $0 \leq t \leq T$, is weakly compact in $L^2([0, T], \mathcal{X}(\Omega))$ so that it has a convergent sub-sequence that converges to a vector $u \in L^2([0, T], \mathcal{X}(\Omega))$. This limit function is a weak solution to (NS) with (BC) and (I).*

3.5.2 Strong solutions in small time

Theorem 3.5.4. *Let $u_0 \in \mathcal{X}(\Omega) \cap H^2(\Omega)$. Then there exists $T^* > 0$ depending only on ζ, Ω and $\|u_0\|_{H^2(\Omega)}$ such that there is a strong solution $u(t, x)$ of the initial boundary value problem (NS) with (BC) and (I) up to T^* .*

To prove this theorem we look for a uniform bound on the second order derivatives of u^N defined in the previous section, which will yield the result in the same way as Theorem 3.5.3. Indeed, we have the following:

Theorem 3.5.5. *Let $u_0 \in \mathcal{X}(\Omega) \cap H^2(\Omega)$. Then there exist $T^* > 0$ and $M > 0$ depending only on ζ, Ω and $\|u_0\|_{H^2(\Omega)}$ (independent of N) such that*

$$\|u^N(t, \cdot)\|_{H^2(\Omega)}^2 + \|\partial_t u^N(t, \cdot)\|_2^2 \leq M.$$

Proof. Let u^N be as above. In what follows we will drop the N for notational sake, so that $u = u^N$, since it will not cause any confusion. Let $\omega = \nabla \times u$ and $\psi = -\Delta u$ as usual. Define $F := \|(\psi, u, u_t)\|_2^2$. We first aim to show that

$$\frac{d}{dt}F \leq M_1 F + M_2 F^2. \quad (3.16)$$

for some constants $M_1, M_2 > 0$. To do this we must estimate the three terms $\frac{d}{dt}\|u\|_2^2$, $\frac{d}{dt}\|u_t\|_2^2$ and $\frac{d}{dt}\|\psi\|_2^2$.

Recall that u satisfies the evolution equation (3.15) i.e.

$$\partial_t u = Au - \sum_{k=0}^N a_k \int_{\Omega} \langle a_k, u \cdot \nabla u \rangle \quad (3.17)$$

where $u(t, \cdot) \in \mathcal{D}_{0,\zeta}(A)$ for all $t > 0$. Thus we have that

$$\frac{d}{dt}\|u\|_2^2 = -2 \int_{\Omega} |\nabla u|^2 - \int_{\Omega} \langle u, \nabla(|u|^2) \rangle - \frac{2}{\zeta} \int_{\Gamma} |u|^2 + 2 \int_{\Gamma} \pi(u, u).$$

Since $\nabla \cdot u = 0$ and $u^\perp = 0$, $\int_{\Omega} \langle u, \nabla(|u|^2) \rangle = 0$, so that we arrive at the energy balance identity:

$$\frac{d}{dt}\|u\|_2^2 + 2 \int_{\Omega} |\nabla u|^2 = -\frac{2}{\zeta} \|u\|_{L^2(\Gamma)}^2 + 2 \int_{\Gamma} \pi(u, u). \quad (3.18)$$

Now, by the Sobolev embedding inequality, we have

$$\int_{\Gamma} \pi(u, u) \leq \frac{1}{2} \|\nabla u\|_2^2 + C \|u\|_2^2$$

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for some constant C . Thus by (3.18) we see that

$$\frac{d}{dt}\|u\|_2^2 \leq 2C\|u\|_2^2. \quad (3.19)$$

To estimate $\frac{d}{dt}\|u_t\|$, we note that from (3.17) it also follows that

$$\partial_t u_t = Au_t - \sum_{k=1}^N a_k \int_{\Omega} \langle a_k, u_t \cdot \nabla u \rangle - \sum_{k=1}^N a_k \int_{\Omega} \langle a_k, u \cdot \nabla u_t \rangle,$$

and $u_t(t, \cdot) \in \mathcal{D}_{0,\zeta}(A)$ for all $t > 0$. Therefore $\nabla \cdot u_t = 0$ and u_t again satisfies the same boundary conditions as u . Therefore

$$\begin{aligned} \frac{d}{dt}\|u_t\|_2^2 &= 2 \int_{\Omega} \langle Au_t, u_t \rangle - 2 \int_{\Omega} \langle u_t, u_t \cdot \nabla u \rangle - 2 \int_{\Omega} \langle u_t, u \cdot \nabla u_t \rangle \\ &= 2 \int_{\Omega} \langle Au_t, u_t \rangle - 2 \int_{\Omega} \langle u_t, u_t \cdot \nabla u \rangle \\ &= -2\|\nabla u_t\|^2 - \frac{2}{\zeta} \int_{\Gamma} |u_t|^2 + 2 \int_{\Gamma} \pi(u_t, u_t) - 2 \int_{\Omega} \langle u_t, u_t \cdot \nabla u \rangle \end{aligned} \quad (3.20)$$

using Lemma 3.3.1. Thus, again using the Sobolev embedding inequality,

$$\begin{aligned} \frac{d}{dt}\|u_t\|_2^2 &= -2\|\nabla u_t\|^2 - \frac{2}{\zeta} \|u_t\|_{L^2(\Gamma)}^2 + 2 \int_{\Gamma} \pi(u_t, u_t) - 2 \int_{\Omega} \langle u_t, u_t \cdot \nabla u \rangle \\ &\leq -\|\nabla u_t\|^2 - \frac{2}{\zeta} \|u_t\|_{L^2(\Gamma)}^2 + 2C\|u_t\|_2^2 - 2 \int_{\Omega} \langle u_t, u_t \cdot \nabla u \rangle \\ &\leq -\|\nabla u_t\|^2 - \frac{2}{\zeta} \|u_t\|_{L^2(\Gamma)}^2 + 2C\|u_t\|_2^2 + 2\|u_t\|_2^2 \|\nabla u\|_{\infty} \\ &\leq -\|\nabla u_t\|^2 - \frac{2}{\zeta} \|u_t\|_{L^2(\Gamma)}^2 + 2C\|u_t\|_2^2 + \|u_t\|_2^4 + \|\nabla u\|_{\infty}^2 \\ &\leq -\|\nabla u_t\|^2 - \frac{2}{\zeta} \|u_t\|_{L^2(\Gamma)}^2 + \varepsilon \|\nabla^3 u\|_2^2 + C_1 (\|(\psi, u, u_t)\|_2^2 + \|u_t\|_2^4) \end{aligned} \quad (3.21)$$

for all $\varepsilon > 0$ and some constant $C_1 = C_1(\varepsilon)$. Now, by Theorem 3.4.4 and Corollary 3.4.5 we have that there exist constants C_2, C_3 such that

$$\begin{aligned} \|\nabla^3 u\|_2^2 &\leq C_2 (\|\nabla \psi\|_2^2 + \|u\|_{H^2}^2) \\ &\leq C_2 (\|\nabla \psi\|_2^2 + \|u\|_{H^3}^2) \\ &\leq C_2 (\|\nabla \psi\|_2^2 + C_3 \|(\nabla \psi, \psi, u)\|_2^2) \\ &\leq C_4 (\|\nabla \psi\|_2^2 + \|(\psi, u)\|_2^2) \end{aligned} \quad (3.22)$$

for $C_4 = C_2 + C_3$. Using this in (3.21) we see that

$$\frac{d}{dt}\|u_t\|_2^2 \leq \varepsilon C_4 \|\nabla \psi\|_2^2 + C_5 (\|(\psi, u, u_t)\|_2^2 + \|u_t\|_2^4) \quad (3.23)$$

where $C_5 = C_1 + \varepsilon C_4$ for all $\varepsilon > 0$.

We lastly must estimate $\frac{d}{dt}\|\psi\|_2^2$. In order to do this, we first note that (3.17) may be re-written as

$$\partial_t u = Au - P_N(u \cdot \nabla u)$$

where $P_N u = \sum_{k=0}^N a_k \int_{\Omega} \langle a_k, u \rangle$ as in section 3.4. Since we can write $u \cdot \nabla u = \frac{1}{2}|u|^2 - u \times \omega$ (where we recall $\omega = \nabla \times u$), we have

$$\partial_t u = Au + P_N(u \times \omega). \quad (3.24)$$

By taking the curl of both sides once and then twice yields

$$\partial_t \omega = \Delta \omega + \nabla \times P_N(u \times \omega)$$

and

$$\partial_t \psi = \Delta \psi + \nabla \times \nabla \times P_N(u \times \omega),$$

since $\nabla \times Au = \nabla \times (\Delta u)$. It follows that

$$\frac{d}{dt}\|\psi\|_2^2 = 2 \int_{\Omega} \langle \Delta \psi, \psi \rangle + 2 \int_{\Omega} \langle \nabla \times \nabla \times P_N(u \times \omega), \psi \rangle.$$

Integration by parts leads to

$$2 \int_{\Omega} \langle \Delta \psi, \psi \rangle = -2\|\nabla \psi\|_2^2 + \int_{\Gamma} \partial_n(|\psi|^2),$$

and

$$\begin{aligned} & \int_{\Omega} \langle \nabla \times \nabla \times P_N(u \times \omega), \psi \rangle \\ &= \int_{\Omega} \langle \nabla \times P_N(u \times \omega), \nabla \times \psi \rangle + \frac{1}{\zeta} \int_{\Gamma} \langle \psi, P_N(u \times \omega) \rangle - 2 \int_{\Gamma} \pi(\psi, P_N(u \times \omega)). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{d}{dt}\|\psi\|_2^2 &= -2\|\nabla \psi\|_2^2 + \int_{\Gamma} \partial_n(|\psi|^2) + 2 \int_{\Omega} \langle \nabla \times P_N(u \times \omega), \nabla \times \psi \rangle \\ &\quad + \frac{2}{\zeta} \int_{\Gamma} \langle \psi, P_N(u \times \omega) \rangle - 4 \int_{\Gamma} \pi(\psi, P_N(u \times \omega)). \end{aligned}$$

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By Hölder's inequality, one obtains

$$\begin{aligned} \frac{d}{dt} \|\psi\|_2^2 &\leq -2\|\nabla\psi\|_2^2 + \int_{\Gamma} \partial_n(|\psi|^2) + 2\|\nabla \times P_N(u \times \omega)\|_2 \|\nabla \times \psi\|_2 \\ &\quad + C_6 \|\psi\|_{L^2(\Gamma)} \|P_N(u \times \omega)\|_{L^2(\Gamma)} \end{aligned}$$

for some constant C_6 . By Lemma 3.4.3, we have that for all $\varepsilon > 0$ there exists a constant $C_7 = C_7(\varepsilon)$ such that

$$\begin{aligned} \int_{\Gamma} \partial_n(|\psi|^2) &\leq \varepsilon \|\nabla^3 u\|_2^2 + C_7 \|(\psi, u)\|_2^2 \\ &\leq \varepsilon C_4 (\|\nabla\psi\|_2^2 + \|(\psi, u)\|_2^2) + C_7 \|(\psi, u)\|_2^2 \\ &\leq \varepsilon C_4 \|\nabla\psi\|_2^2 + C_8 \|(\psi, u)\|_2^2 \end{aligned}$$

where we have used (3.22) and $C_8 = C_7 + \varepsilon C_4$. Thus

$$\begin{aligned} \frac{d}{dt} \|\psi\|_2^2 &\leq -2\|\nabla\psi\|_2^2 + \varepsilon C_4 \|\nabla\psi\|_2^2 + C_8 \|(\psi, u)\|_2^2 \\ &\quad + 2\|\nabla \times P_N(u \times \omega)\|_2 \|\nabla \times \psi\|_2 + C_6 \|\psi\|_{L^2(\Gamma)} \|P_N(u \times \omega)\|_{L^2(\Gamma)}. \end{aligned} \quad (3.25)$$

for all $\varepsilon > 0$. Now by the Sobolev embedding theorem, for all $\varepsilon > 0$ there exist constants $C_9, C_{10} = C_{10}(\varepsilon)$ such that

$$\begin{aligned} C_6 \|\psi\|_{L^2(\Gamma)} \|P_N(u \times \omega)\|_{L^2(\Gamma)} &\leq \left(\|\nabla\psi\|_2 + C_9^{\frac{1}{2}} \|\psi\|_2 \right) \left(\varepsilon^{\frac{1}{2}} \|\nabla P_N(u \times \omega)\|_2 + C_{10}^{\frac{1}{2}} \|u \times \omega\|_2 \right) \\ &\leq \|\nabla\psi\|_2^2 + C_9 \|\psi\|_2^2 + \varepsilon \|\nabla P_N(u \times \omega)\|_2^2 + C_{10} \|u \times \omega\|_2^2 \\ &\leq \|\nabla\psi\|_2^2 + C_9 \|\psi\|_2^2 + \varepsilon \|\nabla P_N(u \times \omega)\|_2^2 + C_{11} \|(\psi, u)\|_2^2 \end{aligned}$$

for some constant C_{11} , where we have used the fact that there exists a constant M such that

$$\|u \times \omega\|_2 \leq M \|u\|_{H^1}^2 = M \|(\psi, u)\|_2^2.$$

Now we can use Lemma 3.4.6 to see that there exists a constant C_{12} such that

$$\begin{aligned} \|(\nabla \times P_N(u \times \omega), \nabla P_N(u \times \omega))\|_2^2 &\leq C_{12} \|(\nabla \times u \times \omega, u \times \omega)\|_2^2 \\ &= C_{12} \|\nabla \times u \times \omega\|_2^2 + C_{12} \|u \times \omega\|_2^2 \\ &\leq C_{13} \|(\omega \cdot \nabla u, u \cdot \nabla \omega, u \times \omega)\|_2^2 \\ &\leq C_{14} \|(\psi, u)\|_2^2 \end{aligned}$$

for some constants C_{13}, C_{14} . Hence

$$C_6 \|\psi\|_{L^2(\Gamma)} \|P_N(u \times \omega)\|_{L^2(\Gamma)} \leq \|\nabla\psi\|_2^2 + C_{15} \|(\psi, u)\|_2^2$$

where $C_{15} = C_9 + C_{11} + C_{14}$. Using this in (3.25) yields

$$\frac{d}{dt} \|\psi\|_2^2 \leq -(1 - \varepsilon C_4) \|\nabla \psi\|_2^2 + \varepsilon C_{16} \|(\psi, u)\|_2^2 \quad (3.26)$$

$$+ 2 \|\nabla \times P_N(u \times \omega)\|_2 \|\nabla \times \omega\|_2, \quad (3.27)$$

where $C_{16} = C_{15} + C_8$ for all $\varepsilon > 0$. Finally, by another application of Lemma 3.4.6, we see that there exists a constant C_{17} such that

$$\frac{d}{dt} \|\psi\|_2^2 \leq -(1 - \varepsilon C_4) \|\nabla \psi\|_2^2 + C_{17} (\|(\psi, u)\|_2^2 + \|(\psi, u)\|_2^4). \quad (3.28)$$

We can now combine (3.19), (3.23) and (3.28) to see that, by taking ε small enough, there exist positive constants M_1, M_2 , such that

$$\frac{d}{dt} F \leq M_1 F + M_2 F^2,$$

which is (3.16).

To conclude the proof, we note that from (3.24)

$$\begin{aligned} \|u_t\|_2 &\leq \|Au\|_2 + \|P_N(u \times \omega)\|_2 \\ &\leq 2\|\Delta u\|_2 + \|u \times \omega\|_2 \\ &\leq 2\|\Delta u\|_2 + \|u\|_2 \|\omega\|_2, \end{aligned}$$

so that

$$F(0) \leq C \|u_0\|_{H^2}^2$$

for some constant C . Let ρ be the solution on $[0, T^*)$ to the ordinary differential equation:

$$\rho' = M_1 \rho + M_2 \rho^2, \quad \rho(0) = C \|u_0\|_{H^2}^2,$$

where $T^* > 0$ is the blowup time of ρ . Finally, the differential inequality (3.16) together with the fact that $F(0) \leq \rho(0)$ implies that $F(t) \leq \rho(t)$ on $[0, T^*)$. \square

Bibliography

- [1] G.-Q. CHEN AND Z. QIAN, *A study of the Navier-Stokes equations with the kinematic and Navier boundary conditions*, Indiana Univ. Math. J. 59 (2010), no. 2, 721–760.
- [2] D. EINZEL, P. PANZER, AND M. LIU, *Boundary condition for fluid flow: curved or rough surfaces*, Physical Review Letters, 64 (1990), pp. 2269–2272.
- [3] H. FUJITA AND T. KATO, *On the Navier-Stokes initial value problem I.*, Archive for Rational Mechanics and Analysis, 16 (1964), pp. 269–315.
- [4] H. O. KREISS AND J. LORENZ, *Initial-Boundary Value Problems and the Navier-Stokes Equations*, Academic Press, 1989.
- [5] C. L. M. NAVIER, *Mémoire sur les lois du mouvement des fluides*, Mémoires de l'Académie Royale des Sciences de l'Institut de France, 1 (1816).
- [6] H. SOHR, *The Navier-Stokes Equations: An elementary functional analytic approach*, Birkhäuser Advanced Texts, Birkhäuser, 2001.
- [7] R. TEMAM, *Navier-Stokes Equations*, North Holland, 1984.

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