

# Ergodicity for infinite particle systems with locally conserved quantities. \*

J. INGLIS<sup>§</sup>, M. NEKLYUDOV<sup>‡</sup> and B. ZEGARLINSKI<sup>#†</sup>

<sup>§</sup>Department of Mathematics, Imperial College London, UK

<sup>‡</sup>Department of Mathematics, University of York, Heslington, UK

<sup>†</sup>CNRS, Toulouse, France

## Abstract

We analyse certain degenerate infinite dimensional sub-elliptic generators and obtain estimates on the long-time behaviour of the corresponding Markov semigroups. In particular, we establish ergodicity for invariant measures and show that the optimal rate of convergence to equilibrium for the Markov semigroup corresponding to a certain model of heat conduction is polynomial. Consequently, there is no spectral gap, but a Liggett-Nash type inequality holds.

Keywords: Hörmander type generators, locally conserved quantities, Liggett-Nash inequality, ergodicity.

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# 1 Introduction

In this paper we study a class of Markov semigroups  $(P_t)_{t \geq 0}$  whose generators are defined in Hörmander form by an infinite family of non-commuting fields as follows

$$\mathcal{L} \equiv \sum_{\gamma} \mathbf{X}_{\gamma}^2. \quad (1.1)$$

In particular we will be interested in the situation when we have “locally preserved quantities”, that is when any operator

$$\mathcal{L}_{\Lambda} \equiv \sum_{\gamma \in \Lambda} \mathbf{X}_{\gamma}^2,$$

defined with a finite set of indices  $\Lambda$ , has a non-trivial set of harmonic functions, while for the full generator  $\mathcal{L}$ , this is not the case. One should therefore expect that the corresponding semigroup is ergodic. We will assume that the fields  $\mathbf{X}_{\gamma}$  are homogeneous of the same degree, in the sense that there is a natural dilation generator  $D$  such that

$$[D, \mathbf{X}_{\gamma}] = \lambda \mathbf{X}_{\gamma},$$

with  $\lambda \in \mathbb{R}$  independent of  $\gamma$ . However, unlike in Hörmander theory, we admit a situation when a commutator of the degenerate fields of any order does not remove degeneration. To model such situation we consider an infinite product space and fields of the following form

$$\mathbf{X}_{\mathbf{ij}} \equiv \partial_{\mathbf{i}} V(x) \partial_{\mathbf{j}} - \partial_{\mathbf{j}} V(x) \partial_{\mathbf{i}},$$

with  $\partial_{\mathbf{i}}$  denoting the partial derivative with respect to the coordinate with index  $\mathbf{i}$ , and  $\partial_{\mathbf{i}} V(x)$  indicates some (polynomial) coefficients.

Generators of a similar type appear in the study of dissipative dynamics in which certain quantities are preserved. See for example [1], [3] and [9], where a

system of harmonic oscillators perturbed by conservative noise is considered. A special case of the system we investigate (cf. Example 3.4 below) can be thought of as a limiting case of the models considered in these works, when the conservative noise dominates the deterministic interaction between oscillators. It is interesting that our results (see Corollary 7.2) show ergodicity of the system even in such situations! A further example of a physical model very closely related to our setup is the heat conduction model discussed in [2] and [10]. For more information in this direction, in particular in connection with an effort to explain the so-called Fourier law of heat conduction, we refer to a nice review [7] as well as [6] and the references therein.

Another line of motivation stems from the works [4], [5] and [15], where an attempt was made to understand infinite systems from the point of view of functional inequalities. Although formally similar to our present situation, we notice that one can obtain a variety of different long-time behaviours depending on the underlying space.

One final motivation to study  $(P_t)_{t \geq 0}$  given by this particular generator comes from the fact that we can easily deduce that an invariant measure for the semigroup is formally given by “ $e^{-V} dx$ ” (since  $V$  is formally conserved under the action of  $P_t$ ). On the one hand, the semigroup  $(P_t)_{t \geq 0}$  is quite simple, since we can calculate many quantities we are interested in directly. On the other hand, standard methods from interacting particle theory [17, 18] do not help in this situation because they require some type of strong non-degeneracy condition such as Hörmander’s condition, which is not satisfied in this case. Another difficulty stems from the intrinsic difference between the infinite dimensional case we consider, and the finite dimensional case i.e. the case when  $V$  depends on only a finite number of variables, and instead of the lattice we use its truncation with a periodic boundary condition. Indeed, in the finite dimensional case we can notice that  $V$  is a non-trivial fixed point for  $P$ , and therefore the semigroup is strictly not ergodic. This reasoning turns out to be incorrect in the infinite dimensional case. The situation here is more subtle because the expression  $V$  is only formal (and would be equal to infinity on the support set of the invariant measure).

We give a detailed study of the case when the coefficients of the fields are linear, providing analysis of the corresponding spectral theory and showing that the system is ergodic with polynomial convergence rate to equilibrium, before finally deriving Liggett-Nash type inequalities.

The organisation of the paper is as follows. In Section 2 we introduce basic notation and state an infinite system of stochastic equations of interest to us. In Section 3 we show the existence of a mild solution and continue in Sections 4 and 5 with some discussion of general properties of the corresponding semigroup, such as the existence of an invariant measure, strong continuity, positivity and contractiv-

ity properties in  $L_p$ -spaces. Because of the special non-commutative features of the fields and the form of the generator these matters are slightly more combersome than otherwise. Section 6 provides a certain characterisation of invariant (Sobolev-type) subspaces, while Section 7 is devoted to demonstration of ergodicity with optimal rate of convergence to equilibrium. In Section 8 we use previously obtained information to derive Liggett-Nash type inequalities. In Section 9 we consider a generalised dynamics of a similar type with generators including a first order term  $-\beta D$  with some parameter  $\beta \in [0, \infty)$ . We show that in such families one observes a change in the behaviour of the decay to equilibrium from exponential to algebraic (when the additional control parameter  $\beta$  goes towards zero). Finally in the last section we provide some further application of our ergodicity results.

## 2 The system

Throughout this paper we will work in the following setting.

**The Lattice:** Let  $\mathbb{Z}^N$  be the  $N$ -dimensional square lattice for some fixed  $N \in \mathbb{N}$ . We equip  $\mathbb{Z}^N$  with the  $l_1$  lattice metric  $dist(\cdot, \cdot)$  defined by

$$dist(\mathbf{i}, \mathbf{j}) := |\mathbf{i} - \mathbf{j}|_1 \equiv \sum_{l=1}^N |i_l - j_l|$$

for  $\mathbf{i} = (i_1, \dots, i_N), \mathbf{j} = (j_1, \dots, j_N) \in \mathbb{Z}^N$ . For  $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^N$  we will write  $\mathbf{i} \sim \mathbf{j}$  whenever  $dist(\mathbf{i}, \mathbf{j}) = 1$ . When  $\mathbf{i} \sim \mathbf{j}$  we say that  $\mathbf{i}$  and  $\mathbf{j}$  are neighbours in the lattice.

**The Configuration Space:** Let  $\Omega \equiv (\mathbb{R})^{\mathbb{Z}^N}$ . Define the Hilbert spaces

$$E_\alpha = \left\{ x \in \Omega : \|x\|_{E_\alpha}^2 := \sum_{\mathbf{i} \in \mathbb{Z}^N} x_{\mathbf{i}}^2 e^{-\alpha |\mathbf{i}|_1} < \infty \right\}$$

for  $\alpha > 0$ , and

$$H = \left\{ (h^{(1)}, \dots, h^{(N)}) \in (\Omega)^N : \|(h^{(1)}, \dots, h^{(N)})\|_H^2 := \sum_{\mathbf{i} \in \mathbb{Z}^N} \sum_{k=1}^N \left(h_{\mathbf{i}}^{(k)}\right)^2 < \infty \right\},$$

with inner products given by

$$\langle x, y \rangle_{E_\alpha} := \sum_{\mathbf{i} \in \mathbb{Z}^N} x_{\mathbf{i}} y_{\mathbf{i}} e^{-\alpha |\mathbf{i}|_1}$$

for  $x, y \in E_\alpha$  and

$$\langle (g^{(1)}, \dots, g^{(N)}), (h^{(1)}, \dots, h^{(N)}) \rangle_H := \sum_{\mathbf{i} \in \mathbb{Z}^N} \sum_{k=1}^N g_{\mathbf{i}}^{(k)} h_{\mathbf{i}}^{(k)}$$

for  $(g^{(1)}, \dots, g^{(N)}), (h^{(1)}, \dots, h^{(N)}) \in H$  respectively.

**The Gibbs Measure:** Let  $\mu_{\mathbf{G}}$  be a Gaussian probability measure on  $(E_\alpha, \mathcal{B}(E_\alpha))$  with mean zero and covariance  $\mathbf{G}$ . We assume that the inverse  $\mathbf{G}^{-1}$  of the covariance is of finite range i.e.

$$\mathbf{M}_{\mathbf{i}, \mathbf{j}} := \mathbf{G}_{\mathbf{i}, \mathbf{j}}^{-1} = 0 \quad \text{if } \text{dist}(\mathbf{i}, \mathbf{j}) > R,$$

and that  $|\mathbf{M}_{\mathbf{i}, \mathbf{j}}| \leq M$  for all  $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^N$ .

**The System:** Let

$$W = \{(W^{(1)}, \dots, W^{(N)})\}$$

be a cylindrical Wiener process in  $H$  ([20], p.96).

Introduce the following notation: for  $\mathbf{i} = (i_1, \dots, i_N) \in \mathbb{Z}^N$  define for  $k \in \{1, \dots, N\}$

$$\mathbf{i}^\pm(k) := (i_1, \dots, i_{k-1}, i_k \pm 1, i_{k+1}, \dots, i_N).$$

We also define, for  $x \in E_\alpha$ ,  $\mathbf{i} \in \mathbb{Z}^N$ ,

$$V_{\mathbf{i}}(x) := \sum_{\mathbf{j} \in \mathbb{Z}^N} x_{\mathbf{i}} \mathbf{M}_{\mathbf{i}, \mathbf{j}} x_{\mathbf{j}}$$

(which is a finite sum, since  $\mathbf{M}_{\mathbf{i}, \mathbf{j}} = 0$  if  $\text{dist}(\mathbf{i}, \mathbf{j}) > R$ ), and for all finite subsets  $\Lambda \subset \mathbb{Z}^N$  set

$$V_\Lambda(x) := \sum_{\mathbf{i} \in \Lambda} V_{\mathbf{i}}(x).$$

Using the formal expression

$$V(x) := \frac{1}{2} \sum_{\mathbf{i} \in \mathbb{Z}^N} V_{\mathbf{i}}(x),$$

it will be convenient to simplify the notation for  $\partial_{\mathbf{i}} V_{\mathbf{i}}$  as follows

$$\partial_{\mathbf{i}} V(x) = \frac{1}{2} \partial_{\mathbf{i}} \left( \sum_{\mathbf{j}, \mathbf{l} \in \mathbb{Z}^N} x_{\mathbf{j}} \mathbf{M}_{\mathbf{j}, \mathbf{l}} x_{\mathbf{l}} \right) \equiv \sum_{\mathbf{j}} \mathbf{M}_{\mathbf{i}, \mathbf{j}} x_{\mathbf{j}} = \partial_{\mathbf{i}} V_{\mathbf{i}}.$$

We consider the following system of Stratonovich SDEs:

$$dY_{\mathbf{i}}(t) = \sum_{k=1}^N \left( \partial_{\mathbf{i}-(k)} V(Y(t)) \circ dW_{\mathbf{i}-(k)}^{(k)}(t) - \partial_{\mathbf{i}+(k)} V(Y(t)) \circ dW_{\mathbf{i}}^{(k)}(t) \right), \quad (2.1)$$

for  $\mathbf{i} \in \mathbb{Z}^N$ .

### 3 Existence of a mild solution

In this section we show that the system (2.1) has a mild solution  $Y(t)$  taking values in the Hilbert space  $E_\alpha$ .

For the existence of a mild solution, the first step is to write (2.1) in Itô form. Indeed we have

$$\begin{aligned} dY_{\mathbf{i}}(t) &= \sum_{k=1}^N \left( \partial_{\mathbf{i}-(k)} V(Y(t)) dW_{\mathbf{i}-(k)}^{(k)}(t) - \partial_{\mathbf{i}+(k)} V(Y(t)) dW_{\mathbf{i}}^{(k)}(t) \right) \\ &\quad + \frac{1}{2} \sum_{k=1}^N \left( d \left[ \partial_{\mathbf{i}-(k)} V(Y(\cdot)), W_{\mathbf{i}-(k)}^{(k)}(\cdot) \right]_t - d \left[ \partial_{\mathbf{i}+(k)} V(Y(\cdot)), W_{\mathbf{i}}^{(k)}(\cdot) \right]_t \right), \end{aligned} \quad (3.1)$$

where  $[\cdot, \cdot]_t$  is a quadratic covariation ([12], p. 61). Hence, by Itô's formula,

$$\begin{aligned} \left[ \partial_{\mathbf{i}-(k)} V(Y(\cdot)), W_{\mathbf{i}-(k)}^{(k)}(\cdot) \right]_t &= \left[ \sum_{\mathbf{j} \in \mathbb{Z}^N} \int_0^\cdot \partial_{\mathbf{j}} \partial_{\mathbf{i}-(k)} V(Y(s)) dY_{\mathbf{j}}(s), \int_0^\cdot dW_{\mathbf{i}-(k)}^{(k)}(s) \right]_t \\ &= \sum_{\mathbf{j} \in \mathbb{Z}^N} \left[ \int_0^\cdot \partial_{\mathbf{j}} \partial_{\mathbf{i}-(k)} V(Y(s)) \partial_{\mathbf{j}-(k)} V(Y(s)) dW_{\mathbf{j}-(k)}^k(s), \int_0^\cdot dW_{\mathbf{i}-(k)}^k(s) \right]_t \\ &\quad - \sum_{\mathbf{j} \in \mathbb{Z}^N} \left[ \int_0^\cdot \partial_{\mathbf{j}} \partial_{\mathbf{i}-(k)} V(Y(s)) \partial_{\mathbf{j}+(k)} V(Y(s)) dW_{\mathbf{j}}^k(s), \int_0^\cdot dW_{\mathbf{i}-(k)}^k(s) \right]_t \\ &= \int_0^t \partial_{\mathbf{i}, \mathbf{i}-(k)}^2 V(Y(s)) \partial_{\mathbf{i}-(k)} V(Y(s)) ds - \int_0^t \partial_{\mathbf{i}-(k)}^2 V(Y(s)) \partial_{\mathbf{i}} V(Y(s)) ds. \end{aligned}$$

By a similar calculation, and using this in (3.1) we see that

$$\begin{aligned}
dY_{\mathbf{i}}(t) = & \sum_{k=1}^N \left( \partial_{\mathbf{i}^-(k)} V(Y(t)) dW_{\mathbf{i}^-(k)}^{(k)}(t) - \partial_{\mathbf{i}^+(k)} V(Y(t)) dW_{\mathbf{i}^+(k)}^{(k)}(t) \right) \\
& - \frac{1}{2} \sum_{k=1}^N \left\{ \left( \partial_{\mathbf{i}^-(k)}^2 V(Y(t)) + \partial_{\mathbf{i}^+(k)}^2 V(Y(t)) \right) \partial_{\mathbf{i}} V(Y(t)) \right. \\
& \quad \left. - \partial_{\mathbf{i}, \mathbf{i}^-(k)}^2 V(Y(t)) \partial_{\mathbf{i}^-(k)} V(Y(t)) - \partial_{\mathbf{i}, \mathbf{i}^+(k)}^2 V(Y(t)) \partial_{\mathbf{i}^+(k)} V(Y(t)) \right\} dt.
\end{aligned} \tag{3.2}$$

Recall now that  $\partial_{\mathbf{j}} V(x) = \sum_{\mathbf{l} \in \mathbb{Z}^N} \mathbf{M}_{\mathbf{j}, \mathbf{l}} x_{\mathbf{l}}$  so that  $\partial_{\mathbf{i}, \mathbf{j}}^2 V(x) = \mathbf{M}_{\mathbf{i}, \mathbf{j}}$ . Thus the system (3.2) can be written as

$$\begin{aligned}
dY_{\mathbf{i}}(t) = & \sum_{k=1}^N \left( \partial_{\mathbf{i}^-(k)} V(Y(t)) dW_{\mathbf{i}^-(k)}^{(k)}(t) - \partial_{\mathbf{i}^+(k)} V(Y(t)) dW_{\mathbf{i}^+(k)}^{(k)}(t) \right) \\
& - \frac{1}{2} \sum_{k=1}^N \left\{ \left( \mathbf{M}_{\mathbf{i}^-(k), \mathbf{i}^-(k)} + \mathbf{M}_{\mathbf{i}^+(k), \mathbf{i}^+(k)} \right) \partial_{\mathbf{i}} V(Y(t)) \right. \\
& \quad \left. - \mathbf{M}_{\mathbf{i}, \mathbf{i}^-(k)} \partial_{\mathbf{i}^-(k)} V(Y(t)) - \mathbf{M}_{\mathbf{i}, \mathbf{i}^+(k)} \partial_{\mathbf{i}^+(k)} V(Y(t)) \right\} dt.
\end{aligned} \tag{3.3}$$

We now claim that we can write this system in operator form:

$$dY(t) = AY(t)dt + B(Y(t))dW(t) \tag{3.4}$$

where  $A$  is a bounded linear mapping from  $E_{\alpha}$  to  $E_{\alpha}$  given by

$$(Ax)_{\mathbf{i}} := \sum_{k=1}^N a_{\mathbf{i}}^{(k)}(x), \quad \mathbf{i} \in \mathbb{Z}^N, \tag{3.5}$$

where

$$\begin{aligned}
a_{\mathbf{i}}^{(k)}(x) = & -\frac{1}{2} \left\{ \left( \mathbf{M}_{\mathbf{i}^-(k), \mathbf{i}^-(k)} + \mathbf{M}_{\mathbf{i}^+(k), \mathbf{i}^+(k)} \right) \sum_{\mathbf{l} \in \mathbb{Z}^N} \mathbf{M}_{\mathbf{l}, \mathbf{i}} x_{\mathbf{l}} \right. \\
& \quad \left. - \mathbf{M}_{\mathbf{i}, \mathbf{i}^-(k)} \sum_{\mathbf{l} \in \mathbb{Z}^N} \mathbf{M}_{\mathbf{l}, \mathbf{i}^-(k)} x_{\mathbf{l}} - \mathbf{M}_{\mathbf{i}, \mathbf{i}^+(k)} \sum_{\mathbf{l} \in \mathbb{Z}^N} \mathbf{M}_{\mathbf{l}, \mathbf{i}^+(k)} x_{\mathbf{l}} \right\}
\end{aligned} \tag{3.6}$$

and  $B : E_{\alpha} \rightarrow L_{HS}(H, E_{\alpha})$  (here  $L_{HS}(H, E_{\alpha})$  denotes the space of Hilbert-Schmidt operators from  $H$  to  $E_{\alpha}$ ) is a bounded linear operator given by

$$(B(x)(h^{(1)}, \dots, h^{(N)}))_{\mathbf{i}} := \sum_{k=1}^N \left( \partial_{\mathbf{i}^-(k)} V(x) h_{\mathbf{i}^-(k)}^{(k)} - \partial_{\mathbf{i}^+(k)} V(x) h_{\mathbf{i}^+(k)}^{(k)} \right) \tag{3.7}$$

for  $x \in E_\alpha, (h^{(1)}, \dots, h^{(N)}) \in H$  and  $\mathbf{i} \in \mathbb{Z}^N$ .

Indeed, the fact that  $A : E_\alpha \rightarrow E_\alpha$  is a bounded linear operator follows from the fact that the constants  $\mathbf{M}_{\mathbf{i}, \mathbf{j}}$  are assumed to be uniformly bounded. To show that  $B \in L(E_\alpha, L_{HS}(H, E_\alpha))$ , first define for  $\mathbf{i} \in \mathbb{Z}^N, e(\mathbf{i}) \in \Omega$  by

$$(e(\mathbf{i}))_{\mathbf{j}} := \begin{cases} 1, & \text{if } \mathbf{j} = \mathbf{i} \\ 0, & \text{otherwise} \end{cases}$$

and for  $\mathbf{i} \in \mathbb{Z}^N, k \in \{1, \dots, N\}$ , let  $f_{\mathbf{i}}^k$  be the element in  $H$  given by

$$f_{\mathbf{i}}^k := (0, \dots, e(\mathbf{i}), \dots, 0),$$

where the  $e(\mathbf{i})$  occurs in the  $k$ -th coordinate. Then

$$\{f_{\mathbf{i}}^k : \mathbf{i} \in \mathbb{Z}^N, k \in \{1, \dots, N\}\}$$

is an orthonormal basis for  $H$ . Let  $x \in E_\alpha$ . Then

$$\|B(x)\|_{HS}^2 = \sum_{\mathbf{i} \in \mathbb{Z}^N} \sum_{k=1}^N |B(x)(f_{\mathbf{i}}^k)|_{E_\alpha}^2.$$

Now by definition

$$(B(x)(f_{\mathbf{i}}^k))_{\mathbf{j}} = \partial_{\mathbf{j}-(k)} V(x) (e(\mathbf{i}))_{\mathbf{j}-(k)} - \partial_{\mathbf{j}+(k)} V(x) (e(\mathbf{i}))_{\mathbf{j}}$$

so that

$$\begin{aligned} |B(x)(f_{\mathbf{i}}^k)|_{E_\alpha}^2 &= \sum_{\mathbf{j} \in \mathbb{Z}^N} \left( \partial_{\mathbf{j}-(k)} V(x) (e(\mathbf{i}))_{\mathbf{j}-(k)} - \partial_{\mathbf{j}+(k)} V(x) (e(\mathbf{i}))_{\mathbf{j}} \right)^2 e^{-\alpha|\mathbf{j}|_1} \\ &= (\partial_{\mathbf{i}} V(x))^2 e^{-\alpha|\mathbf{i}^+(k)|_1} + (\partial_{\mathbf{i}^+(k)} V(x))^2 e^{-\alpha|\mathbf{i}|_1} \\ &\leq C e^{(R+1)\alpha} \left[ \left( \sum_{\mathbf{l}: |\mathbf{l}-\mathbf{i}|_1 \leq R} x_{\mathbf{l}}^2 e^{-\alpha|\mathbf{l}|_1} \right) + \left( \sum_{\mathbf{l}: |\mathbf{l}-\mathbf{i}^+(k)|_1 \leq R} x_{\mathbf{l}}^2 e^{-\alpha|\mathbf{l}|_1} \right) \right] \end{aligned}$$

where  $C = ((2R)^N + 1)M^2$ . Thus

$$\begin{aligned} \|B(x)\|_{HS} &= \sum_{\mathbf{i} \in \mathbb{Z}^N} \sum_{k=1}^N |B(x)(f_{\mathbf{i}}^k)|_{E_\alpha}^2 \\ &\leq C e^{(R+1)\alpha} \sum_{k=1}^N \sum_{\mathbf{i} \in \mathbb{Z}^N} \left[ \left( \sum_{\mathbf{l}: |\mathbf{l}-\mathbf{i}|_1 \leq R} x_{\mathbf{l}}^2 e^{-\alpha|\mathbf{l}|_1} \right) + \left( \sum_{\mathbf{l}: |\mathbf{l}-\mathbf{i}^+(k)|_1 \leq R} x_{\mathbf{l}}^2 e^{-\alpha|\mathbf{l}|_1} \right) \right] \\ &= 2N((2R)^N + 1)C e^{(R+1)\alpha} \|x\|_{E_\alpha}^2, \end{aligned}$$

which proves our claim that  $B \in L(E_\alpha, L_{HS}(H, E_\alpha))$ .

We thus have the following existence theorem for the our system.



**Proposition 3.1.** *Consider the stochastic evolution equation*

$$dY(t) = AY(t)dt + B(Y(t))dW(t), \quad Y_0 = x \in E_\alpha \quad (3.8)$$

where  $A$  and  $B$  are given by (3.5) and (3.7) respectively, and  $(W(t))_{t \geq 0}$  is a cylindrical Wiener process in  $H$ . This equation has a mild solution  $Y$  taking values in the Hilbert space  $E_\alpha$ , unique up to equivalence among the processes satisfying

$$\mathbb{P} \left( \int_0^T |Y(s)|_{E_\alpha}^2 ds < \infty \right) = 1.$$

Moreover, it has a continuous modification.

*Proof.* We have shown above that  $A : E_\alpha \rightarrow E_\alpha$  is a bounded linear operator, so that it is the infinitesimal generator of a  $C_0$ -semigroup in  $E_\alpha$  ( $A$  can be thought of as a bounded linear perturbation of 0, which is trivially the generator of a  $C_0$ -semigroup). We have also shown that  $B \in L(E_\alpha, L_{HS}(H, E_\alpha))$ . Hence the result follows immediately from Theorem 7.4 of [20].  $\square$

**Lemma 3.2.** *The mild solution  $Y$  to (3.8) solves the martingale problem for the operator*

$$\mathcal{L} = \frac{1}{4} \sum_{\mathbf{i} \in \mathbb{Z}^N} \sum_{\mathbf{j} \in \mathbb{Z}^N : |\mathbf{i} - \mathbf{j}|_1 = 1} (\partial_{\mathbf{i}} V(x) \partial_{\mathbf{j}} - \partial_{\mathbf{j}} V(x) \partial_{\mathbf{i}})^2.$$

*Proof.* By Itô's formula, we have for any suitable function  $f$  that

$$\begin{aligned} f(Y(t)) &= f(Y(0)) + \sum_{\mathbf{i} \in \mathbb{Z}^N} \int_0^t \partial_{\mathbf{i}} f(Y(s)) dY_{\mathbf{i}}(s) \\ &\quad + \frac{1}{2} \sum_{\mathbf{i}, \mathbf{j} \in \mathbb{Z}^N} \int_0^t \partial_{\mathbf{i}, \mathbf{j}}^2 f(Y(s)) d[Y_{\mathbf{i}}, Y_{\mathbf{j}}]_s. \end{aligned}$$

We can then calculate from (3.3) that

$$d[Y_{\mathbf{i}}, Y_{\mathbf{j}}]_t := \begin{cases} -\partial_{\mathbf{i}} V(Y(t)) \partial_{\mathbf{i}^-(k)} V(Y(t)) dt, & \text{if } \mathbf{j} = \mathbf{i}^-(k) \\ \sum_{k=1}^N \left\{ (\partial_{\mathbf{i}^-(k)} V(Y(t)))^2 + (\partial_{\mathbf{i}^+(k)} V(Y(t)))^2 \right\} dt, & \text{if } \mathbf{j} = \mathbf{i}, \\ -\partial_{\mathbf{i}} V(Y(t)) \partial_{\mathbf{i}^+(k)} V(Y(t)) dt, & \text{if } \mathbf{j} = \mathbf{i}^+(k), \end{cases}$$

so that

$$\begin{aligned}
& \sum_{\mathbf{i}, \mathbf{j} \in \mathbb{Z}^N} \int_0^t \partial_{\mathbf{i}, \mathbf{j}}^2 f(Y(s)) d[Y_{\mathbf{i}}, Y_{\mathbf{j}}]_s \\
&= \sum_{\mathbf{i} \in \mathbb{Z}^N} \int_0^t \partial_{\mathbf{i}}^2 f(Y(s)) \sum_{k=1}^N \left\{ (\partial_{\mathbf{i}-(k)} V(Y(t)))^2 + (\partial_{\mathbf{i}+(k)} V(Y(t)))^2 \right\} dt \\
&- 2 \sum_{k=1}^N \sum_{\mathbf{i} \in \mathbb{Z}^N} \int_0^t \partial_{\mathbf{i}, \mathbf{i}-(k)}^2 f(Y(s)) \partial_{\mathbf{i}} V(Y(t)) \partial_{\mathbf{i}-(k)} V(Y(t)) dt.
\end{aligned}$$

Thus, using (3.2), the generator of the system is given by

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2} \sum_{\mathbf{i} \in \mathbb{Z}^N} \sum_{k=1}^N \left\{ (\partial_{\mathbf{i}-(k)} V(x))^2 + (\partial_{\mathbf{i}+(k)} V(x))^2 \right\} \partial_{\mathbf{i}}^2 \\
&- \sum_{\mathbf{i} \in \mathbb{Z}^N} \sum_{k=1}^N \partial_{\mathbf{i}} V(x) \partial_{\mathbf{i}-(k)} V(x) \partial_{\mathbf{i}, \mathbf{i}-(k)}^2 \\
&- \frac{1}{2} \sum_{\mathbf{i} \in \mathbb{Z}^N} \sum_{k=1}^N \left\{ \left( \partial_{\mathbf{i}-(k)}^2 V(x) + \partial_{\mathbf{i}+(k)}^2 V(x) \right) \partial_{\mathbf{i}} V(x) \right. \\
&\quad \left. - \partial_{\mathbf{i}, \mathbf{i}-(k)}^2 V(x) \partial_{\mathbf{i}-(k)} V(x) - \partial_{\mathbf{i}, \mathbf{i}+(k)}^2 V(x) \partial_{\mathbf{i}+(k)} V(x) \right\} \partial_{\mathbf{i}}.
\end{aligned}$$

One can then check by direct calculation that we have

$$\mathcal{L} = \frac{1}{4} \sum_{\mathbf{i} \in \mathbb{Z}^N} \sum_{\mathbf{j} \in \mathbb{Z}^N: |\mathbf{i}-\mathbf{j}|_1=1} (\partial_{\mathbf{i}} V(x) \partial_{\mathbf{j}} - \partial_{\mathbf{j}} V(x) \partial_{\mathbf{i}})^2.$$

□

For  $n \in \{0, 1, \dots\}$ , let  $\mathcal{UC}_b^n \equiv \mathcal{UC}_b^n(E_\alpha)$ ,  $\alpha > 0$  denote the set of all functions which are uniformly continuous and bounded, together with their Fréchet derivatives up to order  $n$ .

**Corollary 3.3.** *The semigroup  $(P_t)_{t \geq 0}$  acting on  $\mathcal{UC}_b(E_\alpha)$ ,  $\alpha > 0$  corresponding to the system (3.8) is Feller and can be represented by the formula*

$$P_t f(\cdot) = \mathbb{E} f(Y_t(\cdot)), \quad t \geq 0,$$

where  $Y_t(x)$  is a mild solution to the system (3.8) with initial condition  $x \in E_\alpha$ . Furthermore,  $(P_t)_{t \geq 0}$  satisfies Kolmogorov's backward equation, and solutions of the system are strong Markov processes.

*Proof.* The result follows immediately from Theorems 9.14 and 9.16 of [20].  $\square$

**Example 3.4.** Suppose

$$\mathbf{M}_{\mathbf{i},\mathbf{i}} = 1, \quad \mathbf{M}_{\mathbf{i},\mathbf{j}} = 0 \text{ if } \mathbf{i} \neq \mathbf{j}.$$

Then  $\partial_{\mathbf{i}}V(x) = x_{\mathbf{i}}$ , and the system (3.3) becomes

$$dY_{\mathbf{i}}(t) = - \sum_{k=1}^N Y_{\mathbf{i}}(t)dt + \sum_{k=1}^N \left( Y_{\mathbf{i}^-(k)}(t)dW_{\mathbf{i}^-(k)}^k(t) - Y_{\mathbf{i}^+(k)}(t)dW_{\mathbf{i}^+(k)}^k(t) \right),$$

which has generator

$$\mathcal{L} = \frac{1}{4} \sum_{\mathbf{i} \in \mathbb{Z}^N} \sum_{\mathbf{j} \in \mathbb{Z}^N: |\mathbf{i}-\mathbf{j}|_1=1} (x_{\mathbf{i}}\partial_{\mathbf{j}} - x_{\mathbf{j}}\partial_{\mathbf{i}})^2. \quad (3.9)$$

Very closely related generators are considered in the physical models for heat conduction described in [1, 2, 3, 9] and [10]. In this case the Gaussian measure  $\mu_{\mathbf{G}}$  on  $E_{\alpha}$  is the product Gaussian measure.

**Remark 3.5.** Let  $(r_{\mathbf{i},\mathbf{j}}, \theta_{\mathbf{i},\mathbf{j}})$  be a polar coordinates in the plane  $(x_{\mathbf{i}}, x_{\mathbf{j}})$ . Then

$$\frac{\partial}{\partial \theta_{\mathbf{i},\mathbf{j}}} = x_{\mathbf{i}}\partial_{\mathbf{j}} - x_{\mathbf{j}}\partial_{\mathbf{i}}.$$

Therefore

$$\mathcal{L} = \frac{1}{4} \sum_{\mathbf{i} \in \mathbb{Z}^N} \sum_{\mathbf{j} \in \mathbb{Z}^N: |\mathbf{i}-\mathbf{j}|_1=1} \frac{\partial^2}{\partial \theta_{\mathbf{i},\mathbf{j}}^2}.$$

Note that the operator  $-\frac{\partial^2}{\partial \theta_{\mathbf{i},\mathbf{j}}^2}$  is a Hamiltonian for the rigid rotor on the plane. Thus, the operator  $-\mathcal{L}$  is the Hamiltonian of a chain of coupled rigid rotors.

## 4 Invariant measure

Let  $\mu_{\mathbf{G}}, E_{\alpha}, H$  be as in section 2. Suppose  $(Y(t))_{t \geq 0}$  is the unique mild solution to the evolution equation (3.8) in the Hilbert space  $E_{\alpha}$  i.e.

$$dY(t) = AY(t)dt + B(Y(t))dW(t)$$

where  $A, B$  are given by (3.5) and (3.7) respectively, and  $(W(t))_{t \geq 0}$  is a cylindrical Wiener process in  $H$ . Let  $(P_t)_{t \geq 0}$  be the corresponding semigroup, defined as above.

For  $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^N$ , define

$$\mathbf{X}_{\mathbf{i}, \mathbf{j}} = \partial_{\mathbf{i}} V(x) \partial_{\mathbf{j}} - \partial_{\mathbf{j}} V(x) \partial_{\mathbf{i}}$$

so that by Lemma 3.2,

$$\mathcal{L} = \frac{1}{4} \sum_{\mathbf{i} \in \mathbb{Z}^N} \sum_{\mathbf{j} \in \mathbb{Z}^N: |\mathbf{i} - \mathbf{j}|_1 = 1} \mathbf{X}_{\mathbf{i}, \mathbf{j}}^2$$

is the generator of our system. We will need the following Lemma:

**Lemma 4.1.**

$$\mu_{\mathbf{G}}(f \mathbf{X}_{\mathbf{i}, \mathbf{j}} g) = -\mu_{\mathbf{G}}(g \mathbf{X}_{\mathbf{i}, \mathbf{j}} f)$$

for all  $f, g \in \mathcal{UC}_b^2(E_\alpha)$  and  $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^N$ .

*Proof.* For finite subsets  $\Lambda \subset \mathbb{Z}^N$  and  $\omega \in \mathbb{R}^{\mathbb{Z}^N}$ , denote by  $\mathbb{E}_\Lambda^\omega$  the conditional measure of  $\mu_{\mathbf{G}}$ , given the coordinates outside  $\Lambda$  coincide with those of  $\omega$ . Then we have that

$$\mathbb{E}_\Lambda^\omega(f) = \int_{\mathbb{R}^\Lambda} f(x_\Lambda \cdot \omega_{\Lambda^c}) \frac{e^{-\sum_{\mathbf{i} \in \Lambda} V_{\mathbf{i}}(x_\Lambda \cdot \omega_{\Lambda^c})}}{Z_\Lambda^\omega} dx_\Lambda$$

where  $x_\Lambda \cdot \omega_{\Lambda^c}$  is the element of  $\mathbb{R}^{\mathbb{Z}^N}$  given by

$$(x_\Lambda \cdot \omega_{\Lambda^c})_{\mathbf{i}} = \begin{cases} x_{\mathbf{i}}, & \text{if } \mathbf{i} \in \Lambda \\ \omega_{\mathbf{i}}, & \text{if } \mathbf{i} \in \Lambda^c \end{cases}$$

and  $Z_\Lambda^\omega$  is the normalisation constant. Now fix  $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^N$  and suppose that  $\Lambda$  is such that  $\{\mathbf{i}, \mathbf{j}\} \subset \Lambda$ . Then for  $f, g \in \mathcal{UC}_b^2(E_\alpha)$

$$\begin{aligned} \mathbb{E}_\Lambda^\omega(f \mathbf{X}_{\mathbf{i}, \mathbf{j}} g) &= \int_{\mathbb{R}^\Lambda} f(x_\Lambda \cdot \omega_{\Lambda^c}) \mathbf{X}_{\mathbf{i}, \mathbf{j}} g(x_\Lambda \cdot \omega_{\Lambda^c}) \frac{e^{-\sum_{\mathbf{i} \in \Lambda} V_{\mathbf{i}}(x_\Lambda \cdot \omega_{\Lambda^c})}}{Z_\Lambda^\omega} dx_\Lambda \\ &= -\mathbb{E}_\Lambda^\omega(g \mathbf{X}_{\mathbf{i}, \mathbf{j}} f) \\ &\quad + \mathbb{E}_\Lambda^\omega(fg [\partial_{\mathbf{i}} \partial_{\mathbf{j}} V(x) - \partial_{\mathbf{j}} \partial_{\mathbf{i}} V(x)]) \\ &\quad + \mathbb{E}_\Lambda^\omega(fg [\partial_{\mathbf{i}} V(x) \partial_{\mathbf{j}} V(x) - \partial_{\mathbf{j}} V(x) \partial_{\mathbf{i}} V(x)]) = -\mathbb{E}_\Lambda^\omega(g \mathbf{X}_{\mathbf{i}, \mathbf{j}} f) \end{aligned}$$

by integration by parts. Thus we have that

$$\mu_{\mathbf{G}}(f \mathbf{X}_{\mathbf{i}, \mathbf{j}} g) = \mu_{\mathbf{G}} \mathbb{E}_\Lambda(f \mathbf{X}_{\mathbf{i}, \mathbf{j}} g) = -\mu_{\mathbf{G}} \mathbb{E}_\Lambda(g \mathbf{X}_{\mathbf{i}, \mathbf{j}} f) = -\mu_{\mathbf{G}}(g \mathbf{X}_{\mathbf{i}, \mathbf{j}} f).$$

□

The following result shows that  $\mu_{\mathbf{G}}$  is reversible for the system (3.4).

**Theorem 4.2.** *For all  $f, g \in \mathcal{UC}_b^2(E_\alpha)$ , we have*

$$\mu_{\mathbf{G}}(fP_t g) = \mu_{\mathbf{G}}(gP_t f). \quad (4.1)$$

*Proof.* It is enough to show that (4.1) holds for  $f, g \in \mathcal{UC}_b^2(E_\alpha)$  depending only on a finite number of coordinates. Indeed, in general case we can find sequences of cylindrical functions  $\{f_n\}_{n=1}^\infty, \{g_n\}_{n=1}^\infty \subset \mathcal{UC}_b^2(E_\alpha)$  which approximate general  $f, g \in \mathcal{UC}_b^2(E_\alpha)$ . In view of this, suppose  $f(x) = f(\{x_i\}_{|i|_1 \leq n})$  and  $g(x) = g(\{x_i\}_{|i|_1 \leq n})$  for some  $n$ . Note that the generator  $\mathcal{L}$  can be rewritten as

$$\mathcal{L} = \frac{1}{2} \sum_{k=1}^N \sum_{\mathbf{i} \in \mathbb{Z}^N} \mathbf{X}_{\mathbf{i}, \mathbf{i}^+(k)}^2.$$

We decompose this operator further. Indeed, we can write

$$\mathcal{L} = \frac{1}{2} \sum_{k=1}^N \sum_{\mathbf{m} \in \{0, \dots, R+1\}^N} \left( \sum_{\mathbf{i} \in \otimes_{\sigma=1}^N ((R+2)\mathbb{Z} + m_\sigma)} \mathbf{X}_{\mathbf{i}, \mathbf{i}^+(k)}^2 \right)$$

and define for  $\mathbf{m} = (m_1, \dots, m_N) \in \{0, \dots, R+1\}^N$ ,  $k \in \{1, \dots, N\}$

$$\mathcal{L}_{\mathbf{m}}^{(k)} := \sum_{\mathbf{i} \in \otimes_{\sigma=1}^N ((R+2)\mathbb{Z} + m_\sigma)} \mathbf{X}_{\mathbf{i}, \mathbf{i}^+(k)}^2$$

so that

$$\mathcal{L} = \frac{1}{2} \sum_{k=1}^N \sum_{\mathbf{m} \in \{0, \dots, R+1\}^N} \mathcal{L}_{\mathbf{m}}^{(k)}.$$

Note that by construction, for fixed  $k \in \{1, \dots, N\}$  and  $\mathbf{m} \in \{0, \dots, R+1\}^N$ , we have for any  $\mathbf{i}, \mathbf{j} \in \otimes_{\sigma=1}^N ((R+2)\mathbb{Z} + m_\sigma)$  that

$$[\mathbf{X}_{\mathbf{i}, \mathbf{i}^+(k)}, \mathbf{X}_{\mathbf{j}, \mathbf{j}^+(k)}] = 0.$$

For  $\mathbf{i} = \mathbf{j}$  this is clear. If  $\mathbf{i} \neq \mathbf{j}$ , we have

$$[\mathbf{X}_{\mathbf{i}, \mathbf{i}^+(k)}, \mathbf{X}_{\mathbf{j}, \mathbf{j}^+(k)}] = [\partial_{\mathbf{i}} V(x) \partial_{\mathbf{i}^+(k)} - \partial_{\mathbf{i}^+(k)} V(x) \partial_{\mathbf{i}}, \partial_{\mathbf{j}} V(x) \partial_{\mathbf{j}^+(k)} - \partial_{\mathbf{j}^+(k)} V(x) \partial_{\mathbf{j}}]$$

and

$$\partial_{\mathbf{i}^+(k)} \partial_{\mathbf{j}} V(x) = 0.$$

Indeed,  $\partial_{\mathbf{j}} V(x)$  depends only on coordinates  $\mathbf{l}$  such that  $|\mathbf{j} - \mathbf{l}|_1 \leq R$ , and for all such  $\mathbf{l}$

$$\begin{aligned} |\mathbf{i}^+(k) - \mathbf{l}|_1 &\geq |\mathbf{i}^+(k) - \mathbf{j}|_1 - |\mathbf{j} - \mathbf{l}|_1 \\ &\geq R + 1 - R \\ &= 1 \end{aligned}$$

so that  $\partial_{\mathbf{j}} V(x)$  does not depend on coordinate  $\mathbf{i}^+(k)$  for any  $k$ . Similarly

$$\partial_{\mathbf{i}^+(k)} \partial_{\mathbf{j}^+(k)} V(x) = \partial_{\mathbf{i}} \partial_{\mathbf{j}} V(x) = \partial_{\mathbf{i}} \partial_{\mathbf{j}^+(k)} V(x) = 0,$$

which proves the claim. Thus for any  $k \in \{1, \dots, N\}$  and  $\mathbf{m} \in \{0, \dots, R+1\}^N$ ,

$$S_t^{(k, \mathbf{m})} := e^{t\mathcal{L}_{\mathbf{m}}^{(k)}} = \prod_{\mathbf{i} \in \otimes_{\sigma=1}^N ((R+2)\mathbb{Z} + m_{\sigma})} e^{t\mathbf{X}_{\mathbf{i}, \mathbf{i}^+(k)}^2}$$

i.e.  $S_t^{(k, \mathbf{m})}$  is a product semigroup.

We now claim that

$$\mu_{\mathbf{G}} \left( f S_t^{(k, \mathbf{m})} g \right) = \mu_{\mathbf{G}} \left( g S_t^{(k, \mathbf{m})} f \right) \quad (4.2)$$

for  $k \in \{1, \dots, N\}$  and  $\mathbf{m} \in \{0, \dots, R+1\}^N$ . Let  $k = 1$  and  $\mathbf{m} = \mathbf{0}$  (the other cases are similar). Since  $g$  depends on coordinates  $\mathbf{i}$  such that  $|\mathbf{i}|_1 \leq n$ , we have

$$S_t^{(1, \mathbf{0})} g(x) = \prod_{\substack{\mathbf{i} \in \otimes_{\sigma=1}^N ((R+2)\mathbb{Z} + m_{\sigma}) \\ |\mathbf{i}|_1 \leq n+R+2}} e^{t\mathbf{X}_{\mathbf{i}, \mathbf{i}^+(1)}^2} g(x),$$

which is a finite product. As a result of Lemma 4.1, we then have that

$$\begin{aligned} \mu_{\mathbf{G}} \left( f S_t^{(1, \mathbf{0})} g \right) &= \mu_{\mathbf{G}} \left( f \prod_{\substack{\mathbf{i} \in \otimes_{\sigma=1}^N ((R+2)\mathbb{Z} + m_{\sigma}) \\ |\mathbf{i}|_1 \leq n+R+2}} e^{t\mathbf{X}_{\mathbf{i}, \mathbf{i}^+(1)}^2} g \right) \\ &= \mu_{\mathbf{G}} \left( g \prod_{\substack{\mathbf{i} \in \otimes_{\sigma=1}^N ((R+2)\mathbb{Z} + m_{\sigma}) \\ |\mathbf{i}|_1 \leq n+R+2}} e^{t\mathbf{X}_{\mathbf{i}, \mathbf{i}^+(1)}^2} f \right) \\ &= \mu_{\mathbf{G}} \left( g S_t^{(1, \mathbf{0})} f \right) \end{aligned}$$

as claimed.

To finish the proof, we will need to use the following version of the Trotter product formula (see [21]):

**Theorem 4.3.** *Let  $\mathcal{H}$  and  $\mathcal{E}$  be two Hilbert spaces, and  $F_i \in \text{Lip}(\mathcal{E}, \mathcal{E})$ ,  $U_i \in \text{Lip}(\mathcal{E}, L_{HS}(\mathcal{H}, \mathcal{E}))$  for  $i = 1, 2, 3$ . Let  $(W_t)_{t \geq 0}$  be a cylindrical Wiener process in  $\mathcal{H}$ . Consider the SDEs, indexed by  $i = 1, 2, 3$ , given by*

$$dY_i(t) = F_i(Y_i(t))dt + U_i(Y_i(t))dW_t, \quad Y_i(0) = x \in \mathcal{E},$$

and let  $(\mathcal{P}_t^i)_{t \geq 0}$  be the corresponding semigroups on  $\mathcal{UC}_b(\mathcal{E})$ . Assume that

$$F_3 = F_1 + F_2, \quad U_3 U_3^* = U_1 U_1^* + U_2 U_2^*,$$

and that the first and second Fréchet derivatives of  $F_i$  and  $U_i$  are uniformly continuous and bounded on bounded subsets of  $\mathcal{E}$ . Then

$$\lim_{n \rightarrow \infty} \left( \mathcal{P}_{\frac{t}{n}}^1 \mathcal{P}_{\frac{t}{n}}^2 \right)^n f(x) = \mathcal{P}_t^3 f(x)$$

for all  $f \in \mathbb{K}$ , where  $\mathbb{K}$  is the closure of  $\mathcal{UC}_b^2(\mathcal{E})$  in  $\mathcal{UC}_b(\mathcal{E})$ , and the convergence is uniform in  $x$  on any bounded subset of  $\mathcal{E}$ .

By above, we have that the generator of our system can be decomposed as

$$\mathcal{L} = \frac{1}{2} \sum_{k=1}^N \sum_{\mathbf{m} \in \{0, \dots, R+1\}^N} \mathcal{L}_{\mathbf{m}}^{(k)}$$

where, for  $k \in \{1, \dots, N\}$  and  $\mathbf{m} \in \{0, \dots, R+1\}^N$ ,  $\mathcal{L}_{\mathbf{m}}^{(k)}$  is the generator of the semigroup  $S_t^{(k, \mathbf{m})}$ , and the associated SDE is given by

$$dY(t) = A_{\mathbf{m}}^{(k)} Y(t) dt + B_{\mathbf{m}}^{(k)}(Y(t)) dW_t$$

where  $A_{\mathbf{m}}^{(k)} : E_\alpha \rightarrow E_\alpha$  and  $B_{\mathbf{m}}^{(k)} \in L(E_\alpha, L_{HS}(E_\alpha, H))$  are such that

$$A = \sum_{k=1}^N \sum_{\mathbf{m} \in \{0, \dots, R+1\}^N} A_{\mathbf{m}}^{(k)}$$

and

$$BB^* = \sum_{k=1}^N \sum_{\mathbf{m} \in \{0, \dots, R+1\}^N} B_{\mathbf{m}}^{(k)} (B_{\mathbf{m}}^{(k)})^*.$$

We can then apply Theorem 4.3 iteratively to get the result. Indeed, order the set

$$\{1, \dots, N\} \times \{0, \dots, R+1\}^N = \{\iota_1, \dots, \iota_S\}$$

where  $S = N(R+2)^N$ . If  $\iota_l = (k, \mathbf{m}) \in \{1, \dots, N\} \times \{0, \dots, R+1\}^N$ , write

$$A_{\mathbf{m}}^{(k)} = A_{\iota_l}, \quad B_{\mathbf{m}}^{(k)} = B_{\iota_l}, \quad \mathcal{L}_{\mathbf{m}}^{(k)} = \mathcal{L}_{\iota_l}, \quad S_t^{k, \mathbf{m}} = S_t^{\iota_l}.$$

Then define for  $1 \leq l \leq S$

$$\tilde{A}_l := \sum_{j=1}^l A_{\iota_j}$$

and  $\tilde{B}_l \in L(E_\alpha, L_{HS}(E_\alpha, H))$  to be such that

$$\tilde{B}_l \tilde{B}_l^* := \sum_{j=1}^l B_{\iota_j} B_{\iota_j}^*.$$

Consider the SDE

$$d\tilde{Y}_l(t) = \tilde{A}_l \tilde{Y}_l(t) dt + \tilde{B}_l \left( \tilde{Y}_l(t) \right) dW_t,$$

which has generator  $\tilde{\mathcal{L}}_l = \sum_{j=1}^l \mathcal{L}_{\iota_j}$ . Let  $(\tilde{P}_t^l)_{t \geq 0}$  be the semigroup on  $\mathcal{UC}_b(E_\alpha)$  associated with  $\tilde{\mathcal{L}}_l$ . By a first application of Theorem 4.3, for all  $f \in \mathbb{K}$ , we have

$$\lim_{n \rightarrow \infty} \left( S_{\frac{t}{n}}^{\iota_1} S_{\frac{t}{n}}^{\iota_2} \right)^n f(x) = \tilde{P}_t^2 f(x)$$

where the convergence is uniform on bounded subsets. Moreover, by claim (4.2) above and the dominated convergence theorem, we have

$$\mu_{\mathbf{G}} \left( f \tilde{P}_t^2 g \right) = \lim_{n \rightarrow \infty} \mu_{\mathbf{G}} \left( f \left( S_{\frac{t}{n}}^{\iota_1} S_{\frac{t}{n}}^{\iota_2} \right)^n g \right) = \lim_{n \rightarrow \infty} \mu_{\mathbf{G}} \left( g \left( S_{\frac{t}{n}}^{\iota_1} S_{\frac{t}{n}}^{\iota_2} \right)^n f \right) = \mu_{\mathbf{G}} \left( g \tilde{P}_t^2 f \right) \quad (4.3)$$

for all  $f, g \in \mathcal{UC}_b^2(E_\alpha)$ . Similarly, for all  $f \in \mathbb{K}$ , we have

$$\lim_{n \rightarrow \infty} \left( \tilde{P}_{\frac{t}{n}}^2 S_{\frac{t}{n}}^{\iota_3} \right)^n f(x) = \tilde{P}_t^3 f(x)$$

where again the convergence is uniform on bounded sets, so that

$$\mu_{\mathbf{G}} \left( f \tilde{P}_t^3 g \right) = \lim_{n \rightarrow \infty} \mu_{\mathbf{G}} \left( f \left( \tilde{P}_{\frac{t}{n}}^2 S_{\frac{t}{n}}^{\iota_3} \right)^n g \right) = \lim_{n \rightarrow \infty} \mu_{\mathbf{G}} \left( g \left( \tilde{P}_{\frac{t}{n}}^2 S_{\frac{t}{n}}^{\iota_3} \right)^n f \right) = \mu_{\mathbf{G}} \left( g \tilde{P}_t^3 f \right)$$

where we have used identities (4.2) and (4.3). Continuing in this manner, we see that  $P_t = \tilde{P}_t^S$ , the semigroup corresponding to the generator  $\mathcal{L} = \sum_{j=1}^s \mathcal{L}_{\iota_j}$ , is such that

$$\mu_{\mathbf{G}} (f P_t g) = \mu_{\mathbf{G}} (g P_t f)$$

for all  $f, g \in \mathcal{UC}_b^2(E_\alpha)$ , as required.  $\square$

Finally, by standard arguments, we can extend the above result to functions in  $L^p(\mu_{\mathbf{G}})$ .

**Corollary 4.4.** *The semigroup  $(P_t)_{t \geq 0}$  acting on  $\mathcal{UC}_b(E_\alpha)$  can be extended to  $L^p(\mu_{\mathbf{G}})$  for any  $p \geq 1$ . Moreover we have*

$$\mu_{\mathbf{G}} (f P_t g) = \mu_{\mathbf{G}} (g P_t f)$$

for any  $f, g \in L^2(\mu_{\mathbf{G}})$ .



## 5 Weak and strong continuity

In this section we will show that the semigroup  $e^{-\beta t}P_t$ ,  $t \geq 0$ , is weakly continuous for some  $\beta > 0$  in the sense of definition given in [8]. This will allow us to deduce closedness of the generator  $\mathcal{L}$  and strong continuity of  $(P_t)_{t \geq 0}$  in  $L^2(E_\alpha, d\mu_{\mathbf{G}})$ . Another approach to strong continuity of diffusion semigroups and connected questions is discussed in [11].

Let  $\mathcal{E}$  be an arbitrary separable Hilbert space. The following definition is found in [8].

**Definition 5.1.** *A semigroup of bounded linear operators  $(S_t)_{t \geq 0}$  defined on  $\mathcal{UC}_b(\mathcal{E})$  is said to be weakly continuous if there exist  $\mathcal{M}, \omega > 0$  such that*

- (i) *the family of functions  $\{S_t\phi\}_{t \geq 0}$  is equi-uniformly continuous for every  $\phi \in \mathcal{UC}_b(\mathcal{E})$ ;*
- (ii) *for every  $\phi \in \mathcal{UC}_b(\mathcal{E})$  and for every compact set  $K \subset H$*

$$\lim_{t \rightarrow 0} \sup_{x \in K} |S_t\phi(x) - \phi(x)| = 0; \quad (5.1)$$

- (iii) *for every  $\phi \in \mathcal{UC}_b(\mathcal{E})$  and for every sequence  $\{\phi_j\}_{j \in \mathbb{N}} \subset \mathcal{UC}_b(\mathcal{E})$  such that*

$$\left\{ \begin{array}{l} \sup_j |\phi_j|_{L^\infty} < \infty, \\ \lim_{j \rightarrow \infty} \sup_{x \in K} |\phi_j(x) - \phi(x)| = 0, \text{ for all compact } K \subset \mathcal{E}, \end{array} \right.$$

*it holds that*

$$\lim_{j \rightarrow \infty} \sup_{x \in K} |S_t\phi_j(x) - S_t\phi(x)| = 0, \quad (5.2)$$

*for every compact set  $K \subset E$ , and furthermore the limit is uniform in  $t \geq 0$ ;*

- (iv)

$$|S_t f|_{\mathcal{UC}_b(\mathcal{E})} \leq \mathcal{M} e^{-\omega t} |f|_{\mathcal{UC}_b(\mathcal{E})}, \quad t \geq 0 \quad (5.3)$$

*for all  $f \in \mathcal{UC}_b(\mathcal{E})$ .*

Now suppose we are in the situation of Sections 2, 3 and 4 above. Define  $\tilde{H} \subset E_\alpha$  by

$$\tilde{H} := \left\{ x \in \Omega : \sum_{\mathbf{i}} x_{\mathbf{i}}^2 < \infty \right\}.$$

**Theorem 5.2.** *Assume that there exist  $C_1, C_2 > 0$  such that*

$$C_2 V(x) \leq |x|_{\tilde{H}}^2 \leq C_1 V(x), \quad x \in \tilde{H}. \quad (5.4)$$

*Then there exists  $\beta > 0$  such that semigroup  $(\tilde{P}_t)_{t \geq 0} := (e^{-\beta t} P_t)_{t \geq 0}$  is weakly continuous in  $\mathcal{UC}_b(E_\alpha)$ .*

**Remark 5.3.** Assumption (5.4) is satisfied if  $\mathbf{M}$  is strictly positive definite and the coefficients of  $\mathbf{M}$  are uniformly bounded, as in our case, though we state the result in a more general form.

*Proof.* First notice that there exists  $q = q(\alpha) > 0$ , such that

$$P_t |Id|_{E_\alpha}^2(x) \leq |x|_{E_\alpha}^2 e^{qt}, \quad (5.5)$$

for all  $x \in E_\alpha$  and  $t > 0$ . Indeed,  $P_t |Id|_{E_\alpha}^2(x) = \mathbb{E}|Y_t(x)|_{E_\alpha}^2$ , where  $Y_t$  is a solution of equation (3.4). Inequality (5.5) then follows from Itô's formula, the boundedness of linear maps  $A \in L(E_\alpha, E_\alpha)$  and  $B \in L(E_\alpha, L_{HS}(H, E_\alpha))$  and Gronwall's lemma. Put  $\beta = q$ . We check the requirements of Definition 5.1.

- (i) Let  $\phi \in \mathcal{UC}_b(E)$ . Then for any  $\varepsilon > 0$  there exist  $\delta(\varepsilon) > 0$  such that  $|x - y|_{E_\alpha} < \delta(\varepsilon) \Rightarrow |\phi(x) - \phi(y)| < \varepsilon$ . Thus, for any  $x, y \in E_\alpha$ ,

$$\begin{aligned} |\tilde{P}_t \phi(x) - \tilde{P}_t \phi(y)| &\leq e^{-qt} \mathbb{E} \left( \mathbf{1}_{\{|Y_t(x) - Y_t(y)| < \delta(\varepsilon/2)\}} |\phi(Y_t(x)) - \phi(Y_t(y))| \right) \\ &\quad + e^{-qt} \mathbb{E} \left( \mathbf{1}_{\{|Y_t(x) - Y_t(y)| \geq \delta(\varepsilon/2)\}} |\phi(Y_t(x)) - \phi(Y_t(y))| \right) \\ &\leq \frac{\varepsilon}{2} + 2e^{-qt} |\phi|_{L^\infty} \mathbb{P} \{ |Y_t(x) - Y_t(y)|_{E_\alpha} \geq \delta(\varepsilon/2) \} \\ &\leq \frac{\varepsilon}{2} + \frac{2|\phi|_{L^\infty}}{\delta^2(\varepsilon/2)} |x - y|_{E_\alpha}^2, \end{aligned} \quad (5.6)$$

where we have used Chebyshev's inequality. Choose  $\delta_1(\varepsilon)$  such that  $\frac{2|\phi|_{L^\infty}}{\delta^2(\varepsilon/2)} \delta_1(\varepsilon)^2 = \frac{\varepsilon}{2}$ . Then  $|x - y|_{E_\alpha} < \delta_1(\varepsilon) \Rightarrow |\tilde{P}_t \phi(x) - \tilde{P}_t \phi(y)| \leq \varepsilon$ , and so the first requirement holds.

- (ii) Fix compact  $K \subset E_\alpha$  and  $\phi \in \mathcal{UC}_b(E_\alpha)$ . For  $\varepsilon > 0$  again let  $\delta(\varepsilon) > 0$  be such that  $|x - y|_{E_\alpha} \leq \delta(\varepsilon) \Rightarrow |\phi(x) - \phi(y)| \leq \varepsilon$  for any  $x, y \in E_\alpha$ .

Since

$$\begin{aligned} |\tilde{P}_t \phi(x) - \phi(x)| &\leq |P_t \phi(x) - \phi(x)| + (1 - e^{-qt}) |P_t \phi|_{L^\infty} \\ &\leq |P_t \phi(x) - \phi(x)| + (1 - e^{-qt}) |\phi|_{L^\infty}, \end{aligned}$$

it is enough to show that

$$\limsup_{t \rightarrow 0} \sup_{x \in K} |P_t \phi(x) - \phi(x)| = 0. \quad (5.7)$$

A similar calculation to the above yields

$$\begin{aligned} |P_t \phi(x) - \phi(x)| &\leq \mathbb{E} |\phi(Y_t(x)) - \phi(x)| \\ &\leq \mathbb{E} \mathbf{1}_{\{|Y_t(x) - x|_{E_\alpha} \leq \delta(\varepsilon/2)\}} |\phi(Y_t(x)) - \phi(x)| \\ &\quad + \mathbb{E} \mathbf{1}_{\{|Y_t(x) - x|_{E_\alpha} > \delta(\varepsilon/2)\}} |\phi(Y_t(x)) - \phi(x)| \\ &\leq \varepsilon/2 + 2|\phi|_{L^\infty} \mathbb{P} \{ |Y_t(x) - x|_{E_\alpha} > \delta(\varepsilon/2) \} \\ &\leq \varepsilon/2 + \frac{2|\phi|_{L^\infty}}{\delta^2(\varepsilon/2)} \mathbb{E} |Y_t(x) - x|_{E_\alpha}^2. \end{aligned} \quad (5.8)$$

Since  $Y_t$  is a mild solution of equation (3.8), we have

$$Y_t(x) - x = e^{At}x - x + \int_0^t e^{A(t-s)}B(Y_s)dW_s.$$

Therefore using the Itô isometry, for  $x \in E_\alpha$  and  $t \geq 0$ , we see that

$$\begin{aligned} \mathbb{E}|Y_t(x) - x|_{E_\alpha}^2 &\leq 2|e^{At}x - x|_{E_\alpha}^2 + 2\mathbb{E} \int_0^t |e^{A(t-s)}B(Y_s)|_{L_{HS}(H, E_\alpha)}^2 ds \\ &\leq 2|e^{At} - Id|_{L(E_\alpha, E_\alpha)}^2 |x|_{E_\alpha}^2 \\ &\quad + 2 \sup_{\tau \in [0, t]} |e^{A\tau}|_{L(E_\alpha, E_\alpha)}^2 |B|_{L(E_\alpha, L_{HS}(H, E_\alpha))}^2 \int_0^t \mathbb{E}|Y_s|_{E_\alpha}^2 ds \\ &\leq 2|e^{At} - Id|_{L(E_\alpha, E_\alpha)}^2 |x|_{E_\alpha}^2 \\ &\quad + 2e^{2|A|_{L(E_\alpha, E_\alpha)}t} |B|_{L(E_\alpha, L_{HS}(H, E_\alpha))}^2 |x|_{E_\alpha}^2 \frac{e^{qt} - 1}{q}, \end{aligned} \quad (5.9)$$

where the last inequality follows from (5.5). Combining (5.8) and (5.9), we get, for  $t \in [0, 1]$

$$|P_t\phi(x) - \phi(x)| \leq \varepsilon/2 + \frac{4|\phi|_{L^\infty}}{\delta^2(\varepsilon/2)} |x|_{E_\alpha}^2 \left( C(A, B, q)(e^{qt} - 1) + 2|e^{At} - Id|_{L(E_\alpha, E_\alpha)}^2 \right).$$

Choose  $\tau \in (0, 1]$  such that

$$\frac{2|\phi|_{L^\infty}}{\delta^2(\varepsilon/2)} \sup_{x \in K} |x|_{E_\alpha}^2 \left[ C(A, B, q)(e^{q\tau} - 1) + 2 \sup_{t \in [0, \tau]} |e^{At} - Id|_{L(E_\alpha, E_\alpha)}^2 \right] \leq \varepsilon/2.$$

Then for any  $0 \leq t \leq \tau$ ,

$$\sup_{x \in K} |P_t\phi(x) - \phi(x)| \leq \varepsilon,$$

and (5.7) follows.

(iii) Fix compact  $K \subset E_\alpha$ . Denote

$$\tilde{K} = \tilde{K}(\omega) = \overline{\bigcup_{t \geq 0} Y_t(K)}, \quad \omega \in \Omega.$$

We first show that  $\tilde{K}$  is a compact with probability 1. For any  $\varepsilon > 0$  there exist  $x(1), \dots, x(n) \in E_\alpha$  such that

$$K \subset \bigcup_{i=1}^n B_{\varepsilon/2}(x(i)). \quad (5.10)$$

Since  $\tilde{H}$  is dense in  $E_\alpha$  we can always assume that  $x(i) \in \tilde{H}$ . It follows from assumption (5.4) that  $V(x(i)) < \infty$  for  $i = 1, \dots, n$ . Therefore, by Itô's lemma and the identity  $\mathbf{X}_{i,j}V = 0$ , we conclude<sup>1</sup> that  $\mathbb{P}$ -a.s.

$$V(Y_t(x(i))) = V(x(i)), \quad t \geq 0, \quad i = 1, \dots, n. \quad (5.11)$$

Hence, using assumption (5.4) once more, we see that there exists  $C > 0$  such that  $\mathbb{P}$ -a.s.

$$|Y_t(x(i))|_{\tilde{H}} \leq C \sup_l |x(l)|_{\tilde{H}}, \quad t \geq 0, \quad i = 1, \dots, n. \quad (5.12)$$

Since the embedding  $\tilde{H} \subset E_\alpha$  is compact, there exist  $y(1), \dots, y(m) \in E_\alpha$  such that

$$\bigcup_{t,i} Y_t(x(i)) \subset \bigcup_{l=1}^m B_{\varepsilon/2}(y(l)) \quad (5.13)$$

$\mathbb{P}$ -a.s. Combining identities (5.10) and (5.13) we deduce that

$$\bigcup_{t \geq 0} Y_t(K) \subset \bigcup_{l=1}^m B_\varepsilon(y(l)) \quad (5.14)$$

and so  $\tilde{K}$  is compact  $\mathbb{P}$ -a.s. as claimed.

Now let  $\phi \in \mathcal{UC}_b(\mathcal{E})$  and  $\{\phi_j\}_{j \in \mathbb{N}} \subset \mathcal{UC}_b(\mathcal{E})$  be such that  $\sup_j |\phi_j|_{L^\infty} < \infty$  and

$$\lim_{j \rightarrow \infty} \sup_{x \in K} |\phi_j(x) - \phi(x)| = 0$$

for all compact  $K \subset E_\alpha$ . Note that

$$\begin{aligned} \sup_{x \in K} |\tilde{P}_t \phi_j(x) - \tilde{P}_t \phi(x)| &\leq e^{-qt} \sup_{x \in K} \mathbb{E} |\phi_j(Y_t(x)) - \phi(Y_t(x))| \\ &\leq \mathbb{E} \sup_{y \in \tilde{K}} |\phi_j(y) - \phi(y)|, \end{aligned} \quad (5.15)$$

for all  $t \geq 0, j \in \mathbb{N}$ . Since  $\tilde{K}$  is compact with probability 1, we have that  $\mathbb{P}$ -a.s.

$$\sup_{y \in \tilde{K}} |\phi_j(y) - \phi(y)| \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Thus, by the dominated convergence theorem, we conclude that

$$\lim_{j \rightarrow \infty} \sup_{x \in K} |\tilde{P}_t \phi_j(x) - \tilde{P}_t \phi(x)| = 0$$

for all compact  $K \subset E_\alpha$ .

---

<sup>1</sup>We can assume that exceptional set of measure 0 in equality (5.11) is the same for all  $t \geq 0$  because we can choose continuous modification of the process  $Y$ .

(iv) Since  $\tilde{P}_t f = e^{-qt} \mathbb{E} f(Y_t)$ , we have that

$$|\tilde{P}_t f|_{\mathcal{UC}_b(E_\alpha)} \leq e^{-qt} |f|_{\mathcal{UC}_b(E_\alpha)},$$

for all  $f \in \mathcal{UC}_b(E_\alpha)$  and  $t \geq 0$ .

□

**Corollary 5.4.** *The operator  $\mathcal{L}$  is closed and the semigroup  $(P_t)_{t \geq 0}$  is strongly continuous in  $L^2(E_\alpha, d\mu_{\mathbf{G}})$ .*

*Proof.* Operator  $\mathcal{L}$  is closed by Theorem 5.1 of [8]. Strong continuity follows from property (ii) of definition of weak continuity above and a standard approximation procedure. □

## 6 Symmetry of infinitesimal generator in Sobolev spaces

In this section we show that the generator  $\mathcal{L}$  is symmetric and dissipative in some family of infinite dimensional Sobolev spaces. In the next section this result will be useful for the proof of ergodicity of the semigroup generated by  $\mathcal{L}$ . We start by introducing the following Dirichlet operator.

$$(f, Lg) = - \sum_{\mathbf{k}, \mathbf{l} \in \mathbb{Z}^N} \mathbf{G}_{\mathbf{k}, \mathbf{l}} (\partial_{\mathbf{k}} f, \partial_{\mathbf{l}} g)$$

where  $(f, h) \equiv (f, h)_{L^2(E_\alpha, \mu_{\mathbf{G}})}$  and  $\mathbf{G}$  is the covariance matrix associated to the measure, as above. That is, on a dense domain including  $\mathcal{UC}_b^2$ , we have

$$Lg = \sum_{\mathbf{k}, \mathbf{l} \in \mathbb{Z}^N} \mathbf{G}_{\mathbf{k}, \mathbf{l}} \partial_{\mathbf{k}} \partial_{\mathbf{l}} g - Dg \tag{6.1}$$

where

$$Dg \equiv \sum_{\mathbf{k} \in \mathbb{Z}^N} x_{\mathbf{k}} \partial_{\mathbf{k}} g. \tag{6.2}$$

$D$  will play the role of the dilation generator in our setup. We remark that

$$[D, \mathbf{X}_{\mathbf{i}, \mathbf{j}}] = 0$$

i.e. our fields are of order zero. Thus

$$[D, \mathcal{L}] = 0.$$

Note also that by a simple computation, we get

$$\left[ \sum_{\mathbf{k}, \mathbf{l} \in \mathbb{Z}^N} \mathbf{G}_{\mathbf{k}, \mathbf{l}} \partial_{\mathbf{k}} \partial_{\mathbf{l}}, \mathcal{L} \right] = 0.$$

This is because

$$[\partial_{\mathbf{k}}, \mathbf{X}_{\mathbf{i}, \mathbf{j}}] = [\partial_{\mathbf{k}}, \sum_{\mathbf{l} \in \mathbb{Z}^N} \mathbf{M}_{\mathbf{i}, \mathbf{l}} x_{\mathbf{l}} \partial_{\mathbf{j}} - \sum_{\mathbf{l} \in \mathbb{Z}^N} \mathbf{M}_{\mathbf{j}, \mathbf{l}} x_{\mathbf{l}} \partial_{\mathbf{i}}] = \mathbf{M}_{\mathbf{i}, \mathbf{k}} \partial_{\mathbf{j}} - \mathbf{M}_{\mathbf{j}, \mathbf{k}} \partial_{\mathbf{i}}$$

and hence we conclude that

$$\left[ \sum_{\mathbf{k}, \mathbf{l} \in \mathbb{Z}^N} \mathbf{G}_{\mathbf{k}, \mathbf{l}} \partial_{\mathbf{k}} \partial_{\mathbf{l}}, \mathbf{X}_{\mathbf{i}, \mathbf{j}} \right] = 0$$

which implies the claim. Thus we obtain the following result.

**Proposition 6.1.** *On  $\mathcal{UC}_b^4$ , we have*

$$[L, \mathbf{X}_{\mathbf{i}, \mathbf{j}}] = 0 \tag{6.3}$$

and

$$[L, \mathcal{L}] = 0. \tag{6.4}$$

Keeping this in mind, we introduce the following family of Hilbert spaces

$$\tilde{\mathbb{X}}_n = \left\{ f \in L^2(E_\alpha, \mu_{\mathbf{G}}) \cap \mathcal{D}(L^n) : |f|_{\tilde{\mathbb{X}}_n}^2 := |f|_{L^2(d\mu_{\mathbf{G}})}^2 + (f, (-L)^n f)_{L^2(d\mu_{\mathbf{G}})} < \infty \right\}$$

equipped with the corresponding inner product

$$(f, g)_{\tilde{\mathbb{X}}_n} = \mu_{\mathbf{G}}(fg) + (f, (-L)^n f)_{L^2(d\mu_{\mathbf{G}})},$$

for  $f, g \in \tilde{\mathbb{X}}_n$ , where  $n \in \mathbb{N} \cup \{0\}$ . Hence we obtain the following fact, (where besides Proposition 6.1 we also use the anti-symmetry of the fields  $\mathbf{X}_{\mathbf{i}, \mathbf{j}}$  in  $L^2(\mu_{\mathbf{G}})$ ).

**Proposition 6.2.** *On a dense set  $\tilde{\mathcal{D}}_n \subset \tilde{\mathbb{X}}_n$ , we have*

$$(f, \mathcal{L}g)_{\tilde{\mathbb{X}}_n} = (\mathcal{L}f, g)_{\tilde{\mathbb{X}}_n} = -\frac{1}{4} \sum_{\mathbf{i} \in \mathbb{Z}^N} \sum_{\mathbf{j}: |\mathbf{i}-\mathbf{j}|_1=1} (\mathbf{X}_{\mathbf{i}, \mathbf{j}} f, \mathbf{X}_{\mathbf{i}, \mathbf{j}} g)_{\tilde{\mathbb{X}}_n}. \tag{6.5}$$

In the case when  $n = 1$ , we have

$$(f, g)_{\tilde{\mathbb{X}}} = \mu_{\mathbf{G}}(fg) + \sum_{\mathbf{i}, \mathbf{j} \in \mathbb{Z}^N} \mu_{\mathbf{G}}(\mathbf{G}_{\mathbf{i}\mathbf{j}} \partial_{\mathbf{i}} f \partial_{\mathbf{j}} g) = \mu_{\mathbf{G}}(fg) + \mu_{\mathbf{G}}(\mathbf{G}^{\frac{1}{2}} \nabla f \cdot \mathbf{G}^{\frac{1}{2}} \nabla g)$$

where for simplicity here and later we set  $\tilde{\mathbb{X}} \equiv \tilde{\mathbb{X}}_1 (= \mathbb{X})$ . By induction, and using the fact that  $[\mathbf{G}^{\frac{1}{2}} \nabla, L] = \mathbf{G}^{\frac{1}{2}} \nabla$ , one can show that there exist non-negative constants  $a_{m,n}$ ,  $m = 1, \dots, n$  with  $a_{n,n} > 0$ , such that

$$(f, g)_{\tilde{\mathbb{X}}_n} = \mu_{\mathbf{G}}(fg) + \sum_{m=1, \dots, n} a_{m,n} \mu_{\mathbf{G}}((\mathbf{G}^{\frac{1}{2}} \nabla)^{\otimes m} f \cdot (\mathbf{G}^{\frac{1}{2}} \nabla)^{\otimes m} g).$$

This motivates the introduction of the associated family  $\mathbb{X}_n$  of Hilbert spaces with corresponding scalar products

$$(f, g)_{\mathbb{X}_n} \equiv \mu_{\mathbf{G}}(fg) + \mu_{\mathbf{G}}((\mathbf{G}^{\frac{1}{2}} \nabla)^{\otimes m} f \cdot (\mathbf{G}^{\frac{1}{2}} \nabla)^{\otimes m} g).$$

Again by induction we conclude with the following property.

**Proposition 6.3.** *On a dense set  $\mathcal{D}_n \subset \mathbb{X}_n$ , we have*

$$(f, \mathcal{L}g)_{\mathbb{X}_n} = (\mathcal{L}f, g)_{\mathbb{X}_n}. \quad (6.6)$$

We remark that neither of the families are orthogonal, but the tilded one further allows Fock's type stratification, which provides invariant subspaces other than the one given as eigenspaces of the dilation generator  $D$ .

**Remark 6.4.** *The operator  $\mathcal{L}$  is closable in  $\mathbb{X}$  and its closure has bounded from above self-adjoint extension, which we continue to denote by the same symbol  $\mathcal{L}$ . Moreover, the self-adjoint extension  $\mathcal{L}$  generates a strongly continuous semigroup  $T_t = e^{t\mathcal{L}} : \mathbb{X} \rightarrow \mathbb{X}$  such that  $T_t = P_t|_{\mathbb{X}}$ .*

## 7 Ergodicity

Before we get into general estimates, it is interesting to consider a few cases where some explicit bounds can be obtained. First of all consider the linear functions

$$F(x) \equiv \sum_{\mathbf{k} \in \mathbb{Z}^N} \alpha_{\mathbf{k}} x_{\mathbf{k}}.$$

We note that

$$\mathcal{L}F(x) = \frac{1}{4} \sum_{\mathbf{k} \in \mathbb{Z}^N} \left( \sum'_{\mathbf{i}, \mathbf{j}} ((\mathbf{M}_{\mathbf{j}, \mathbf{k}} \mathbf{M}_{\mathbf{j}, \mathbf{i}} - \mathbf{M}_{\mathbf{i}, \mathbf{k}} \mathbf{M}_{\mathbf{j}, \mathbf{j}}) \alpha_{\mathbf{i}} + (\mathbf{M}_{\mathbf{j}, \mathbf{k}} \mathbf{M}_{\mathbf{i}, \mathbf{j}} - \mathbf{M}_{\mathbf{j}, \mathbf{k}} \mathbf{M}_{\mathbf{i}, \mathbf{i}}) \alpha_{\mathbf{j}}) \right) x_{\mathbf{k}},$$

where the sum  $\sum'_{\mathbf{i}, \mathbf{j}}$  indicates that we sum over the pairs of indices as in the definition of  $\mathcal{L}$ . In particular, in the case when  $\mathbf{M} = b\mathbf{Id}$ ,  $b \in (0, \infty)$ , we have

$$\mathcal{L}F = -Nb^2 \sum_{\mathbf{k} \in \mathbb{Z}^N} \alpha_{\mathbf{k}} x_{\mathbf{k}}.$$

Since the semigroup maps the space of linear functions into itself, we conclude that

$$\mu_{\mathbf{G}}|P_t F - \mu_{\mathbf{G}} F|^2 \leq e^{-mt} \mu_{\mathbf{G}}|F - \mu_{\mathbf{G}} F|^2$$

with some  $m \in (0, \infty)$ , i.e. on linear functions we get exponential decay to equilibrium. An inequality of this form on a dense set would imply a Poincaré inequality. One can, however, show that such an inequality cannot hold. To this end consider a sequence of functions of the following form:

$$f_{\Lambda}(x) \equiv \sum_{\mathbf{i} \in \Lambda} x_{\mathbf{i}}^2$$

for a finite set  $\Lambda$ . Then, for the measure with diagonal covariance matrix, we have

$$\mu_{\mathbf{G}}|f_{\Lambda} - \mu_{\mathbf{G}} f_{\Lambda}|^2 \geq |\Lambda| \mu_{\mathbf{G}}|x_{\mathbf{i}}^2 - \mu_{\mathbf{G}} x_{\mathbf{i}}^2|^2 \equiv \text{const} \cdot |\Lambda|$$

with  $|\Lambda|$  denoting cardinality of  $\Lambda$ . Moreover,

$$\mu_{\mathbf{G}}(f_{\Lambda}(-\mathcal{L}f_{\Lambda})) = \frac{1}{4} \sum_{\substack{\mathbf{i} \in \Lambda, \mathbf{j} \in \Lambda^c \\ |\mathbf{i} - \mathbf{j}|_1 = 1}} \mu_{\mathbf{G}}(\mathbf{X}_{\mathbf{ij}} f_{\Lambda})^2 = \text{const} \cdot |\partial \Lambda|.$$

From this we see that for a suitable sequence of subsets  $\Lambda$  invading the lattice, the ratio of  $\mu_{\mathbf{G}}(f_{\Lambda}(-\mathcal{L}f_{\Lambda}))$  to  $\mu_{\mathbf{G}}|f_{\Lambda} - \mu_{\mathbf{G}} f_{\Lambda}|^2$  converges to 0.

In the remainder of this section we develop a strategy to obtain optimal estimates on the decay to equilibrium for more general spaces of functions, for simplicity working in the setup when the matrix  $\mathbf{M}$  is given by  $\mathbf{M} = b\mathbf{Id}$ ,  $b \in (0, \infty)$ . We show that the corresponding semigroup is ergodic with polynomial decay.

Define

$$\mathcal{A}(f) \equiv \left( \sum_{\mathbf{i} \in \mathbb{Z}^N} \mu_{\mathbf{G}}|\partial_{\mathbf{i}} f|^2 \right)^{1/2}.$$

**Lemma 7.1.** *For any  $f \in \mathbb{X}$ ,  $\mathbf{i} \in \mathbb{Z}^N$  and  $t > 0$ ,*

$$\mu_{\mathbf{G}}(|\partial_{\mathbf{i}}(P_t f)|^2) \leq \frac{A^N}{t^{\frac{N}{2}}} \mathcal{A}(f), \quad (7.1)$$

where  $A = \frac{1}{b} \sup_{t>0} \sqrt{t} \int_0^1 e^{-2t(1-\cos(2\pi\beta))} d\beta$ .



*Proof.* It is enough to show (7.1) for  $f \in \mathcal{UC}_b^4(E_\alpha)$ . Indeed,  $\mathcal{UC}_b^4(E_\alpha)$  is dense in  $\mathbb{X}$  and  $(P_t)_{t \geq 0}$  is a contraction on  $\mathbb{X}$ .

Denote  $f_t = P_t f$  for  $t \geq 0$ . For  $\mathbf{i} \in \mathbb{Z}^N$ , we can calculate that

$$\begin{aligned}
|\partial_{\mathbf{i}} f_t|^2 - P_t |\partial_{\mathbf{i}} f|^2 &= \int_0^t \frac{d}{ds} P_{t-s} |\partial_{\mathbf{i}} f_s|^2 ds \\
&= \int_0^t P_{t-s} (-\mathcal{L}(|\partial_{\mathbf{i}} f_s|^2) + 2\partial_{\mathbf{i}} f_s \mathcal{L} \partial_{\mathbf{i}} f_s + 2\partial_{\mathbf{i}} f_s [\partial_{\mathbf{i}}, \mathcal{L}] f_s) ds \\
&= \int_0^t P_{t-s} \left( - \sum_{\mathbf{m}, \mathbf{l} \in \mathbb{Z}^N: |\mathbf{m}-\mathbf{l}|_1=1} |\mathbf{X}_{\mathbf{m}, \mathbf{l}}(\partial_{\mathbf{i}} f_s)|^2 \right. \\
&\quad \left. + 2b^2 \partial_{\mathbf{i}} f_s \sum_{k=1}^N (-\partial_{\mathbf{i}} f_s + \mathbf{X}_{\mathbf{i}, \mathbf{i}^-(k)} \partial_{\mathbf{i}^-(k)} f_s + \mathbf{X}_{\mathbf{i}, \mathbf{i}^+(k)} \partial_{\mathbf{i}^+(k)} f_s) \right) ds.
\end{aligned} \tag{7.2}$$

Integrating (7.2) with respect to the invariant measure  $\mu_{\mathbf{G}}$  yields

$$\begin{aligned}
\mu_{\mathbf{G}} |\partial_{\mathbf{i}} f_t|^2 - \mu_{\mathbf{G}} |\partial_{\mathbf{i}} f|^2 &= \int_0^t \left( - \sum_{\mathbf{m}, \mathbf{l} \in \mathbb{Z}^N: |\mathbf{m}-\mathbf{l}|_1=1} \mu_{\mathbf{G}} |\mathbf{X}_{\mathbf{m}, \mathbf{l}}(\partial_{\mathbf{i}} f_s)|^2 \right. \\
&\quad \left. - 2Nb^2 \mu_{\mathbf{G}} |\partial_{\mathbf{i}} f_s|^2 + 2b^2 \sum_{k=1}^N \mu_{\mathbf{G}} (\partial_{\mathbf{i}} f_s \mathbf{X}_{\mathbf{i}, \mathbf{i}^-(k)} \partial_{\mathbf{i}^-(k)} f_s) \right. \\
&\quad \left. + 2b^2 \sum_{k=1}^N \mu_{\mathbf{G}} (\partial_{\mathbf{i}} f_s \mathbf{X}_{\mathbf{i}, \mathbf{i}^+(k)} \partial_{\mathbf{i}^+(k)} f_s) \right) ds.
\end{aligned} \tag{7.3}$$

Notice that the operators  $\mathbf{X}_{\mathbf{i}, \mathbf{j}}, \mathbf{i}, \mathbf{j} \in \mathbb{Z}^N$ , are anti-symmetric in  $L^2(\mu_{\mathbf{G}})$ . Therefore

$$\begin{aligned}
\mu_{\mathbf{G}} |\partial_{\mathbf{i}} f_t|^2 - \mu_{\mathbf{G}} |\partial_{\mathbf{i}} f|^2 &= \int_0^t \left( - \sum_{\mathbf{m}, \mathbf{l} \in \mathbb{Z}^N: |\mathbf{m}-\mathbf{l}|_1=1} \mu_{\mathbf{G}} |\mathbf{X}_{\mathbf{m}, \mathbf{l}}(\partial_{\mathbf{i}} f_s)|^2 \right. \\
&\quad - 2Nb^2 \mu_{\mathbf{G}} |\partial_{\mathbf{i}} f_s|^2 - 2b^2 \sum_{k=1}^N \mu_{\mathbf{G}} (\partial_{\mathbf{i}^-(k)} f_s \mathbf{X}_{\mathbf{i}, \mathbf{i}^-(k)} \partial_{\mathbf{i}} f_s) \\
&\quad \left. - 2b^2 \sum_{k=1}^N \mu_{\mathbf{G}} (\partial_{\mathbf{i}^+(k)} f_s \mathbf{X}_{\mathbf{i}, \mathbf{i}^+(k)} \partial_{\mathbf{i}} f_s) \right) ds.
\end{aligned} \tag{7.4}$$

Hence, by Young's inequality we deduce that

$$\begin{aligned}
\mu_{\mathbf{G}}|\partial_{\mathbf{i}}f_t|^2 &- \mu_{\mathbf{G}}|\partial_{\mathbf{i}}f|^2 \leq \int_0^t \left( - \sum_{\mathbf{m}, \mathbf{l} \in \mathbb{Z}^N: |\mathbf{m}-\mathbf{l}|_1=1} \mu_{\mathbf{G}}|\mathbf{X}_{\mathbf{m}, \mathbf{l}}(\partial_{\mathbf{i}}f_s)|^2 \right. \\
&- 2Nb^2\mu_{\mathbf{G}}|\partial_{\mathbf{i}}f_s|^2 + \sum_{k=1}^N b^2\mu_{\mathbf{G}}|\partial_{\mathbf{i}-(k)}f_s|^2 + \mu_{\mathbf{G}}|\mathbf{X}_{\mathbf{i}, \mathbf{i}-(k)}\partial_{\mathbf{i}}f_s|^2 \\
&+ \sum_{k=1}^N b^2\mu_{\mathbf{G}}|\partial_{\mathbf{i}+(k)}f_s|^2 + \mu_{\mathbf{G}}|\mathbf{X}_{\mathbf{i}, \mathbf{i}+(k)}\partial_{\mathbf{i}}f_s|^2 \Big) ds \\
&\leq \int_0^t b^2 \sum_{k=1}^N \left( \mu_{\mathbf{G}}|\partial_{\mathbf{i}-(k)}f_s|^2 + \mu_{\mathbf{G}}|\partial_{\mathbf{i}+(k)}f_s|^2 - 2\mu_{\mathbf{G}}|\partial_{\mathbf{i}}f_s|^2 \right) ds. \quad (7.5)
\end{aligned}$$

Let  $\Delta$  denote the Laplacian on the lattice  $\mathbb{Z}^N$  and set  $F(\mathbf{i}, t) = \mu_{\mathbf{G}}|\partial_{\mathbf{i}}(P_t f)|^2$  for  $t \geq 0, \mathbf{i} \in \mathbb{Z}^N$ . Then we can rewrite (7.5) as

$$F(t) \leq F(0) + \int_0^t b^2 \Delta F(s) ds, \quad t \in [0, \infty). \quad (7.6)$$

Hence, by the positivity of the semigroup  $(e^{tb^2\Delta})_{t \geq 0}$ , and Duhamel's principle, we can conclude that

$$F(t) \leq e^{tb^2\Delta} F(0) \quad (7.7)$$

for  $t \in [0, \infty)$ . By taking the Fourier transform, we can see that this is equivalent to

$$\mu_{\mathbf{G}}|\partial_{\mathbf{i}}(P_t f)|^2 \leq \sum_{\mathbf{l} \in \mathbb{Z}^N} \mu_{\mathbf{G}}(|\partial_{\mathbf{l}}f|^2) c_{\mathbf{i}+\mathbf{l}}(\phi^{b^2t}), \quad (7.8)$$

where

$$c_{\mathbf{k}}(\phi^t) = \int_{[0,1]^N} \varphi^t(\alpha) \cos \left( 2\pi \sum_{l=1}^N k_l \alpha_l \right) d\alpha_1 \dots d\alpha_N,$$

is the Fourier coefficient of the function  $\varphi^t(\alpha) = \exp(-2t \sum_{n=1}^N (1 - \cos(2\pi\alpha_n)))$ ,  $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ .  $c_{\mathbf{k}}(\varphi^t), \mathbf{k} \in \mathbb{Z}^N$  can then be bounded above by

$$|c_{\mathbf{k}}(\phi^t)| \leq \int_{[0,1]^N} \varphi^t(\alpha) d\alpha = \left( \int_0^1 e^{-2t(1-\cos(2\pi\beta))} d\beta \right)^N, \quad (7.9)$$

and the result follows.  $\square$

Define also

$$\mathcal{B}(f) \equiv \left( \sum_{\mathbf{i} \in \mathbb{Z}^N} (\mu_{\mathbf{G}} |\partial_{\mathbf{i}} f|^2)^{\frac{1}{2}} \right).$$

**Corollary 7.2.** *For  $f \in \mathbb{X}$  such that  $\mathcal{B}(f) < \infty$ , we have*

$$\sum_{\mathbf{i} \in \mathbb{Z}^N} \mu_{\mathbf{G}} |\partial_{\mathbf{i}} (P_t f)|^2 \leq \frac{A^{\frac{N}{2}}}{t^{\frac{N}{4}}} \mathcal{A}(f) \mathcal{B}(f). \quad (7.10)$$

Furthermore, there exists a constant  $C \in (0, \infty)$  such that

$$\mu_{\mathbf{G}} \left( (P_t f)^2 \log \frac{(P_t f)^2}{\mu_{\mathbf{G}} (P_t f)^2} \right) \leq C \frac{A^{\frac{N}{2}}}{t^{\frac{N}{4}}} \mathcal{A}(f) \mathcal{B}(f), \quad (7.11)$$

and hence

$$\mu_{\mathbf{G}} (P_t f - \mu_{\mathbf{G}}(f))^2 \leq C \frac{A^{\frac{N}{2}}}{t^{\frac{N}{4}}} \mathcal{A}(f) \mathcal{B}(f), \quad (7.12)$$

i. e. our system is ergodic with polynomial rate of convergence.

*Proof.* By symmetry of the semigroup  $P_t$  in  $\mathbb{X}$ , we have

$$\begin{aligned} \sum_{\mathbf{i} \in \mathbb{Z}^N} \mu_{\mathbf{G}} |\partial_{\mathbf{i}} (P_t f)|^2 &= \sum_{\mathbf{i} \in \mathbb{Z}^N} \mu_{\mathbf{G}} (\partial_{\mathbf{i}} f \partial_{\mathbf{i}} P_{2t} f) \\ &\leq \sum_{\mathbf{i} \in \mathbb{Z}^N} (\mu_{\mathbf{G}} |\partial_{\mathbf{i}} f|^2)^{\frac{1}{2}} (\mu_{\mathbf{G}} |\partial_{\mathbf{i}} P_{2t} f|^2)^{\frac{1}{2}} \\ &\leq \left( \sum_{\mathbf{i} \in \mathbb{Z}^N} (\mu_{\mathbf{G}} |\partial_{\mathbf{i}} f|^2)^{\frac{1}{2}} \right) \sup_{\mathbf{j} \in \mathbb{Z}^N} (\mu_{\mathbf{G}} |\partial_{\mathbf{j}} P_{2t} f|^2)^{1/2}. \end{aligned} \quad (7.13)$$

Combining (7.13) with (7.1) we immediately arrive at the estimate (7.10). Now inequalities (7.11) and (7.12) follow from the logarithmic Sobolev and Poincaré inequalities for the Gaussian measure  $\mu_{\mathbf{G}}$ .  $\square$

**Remark 7.3.** *The convergence in Lemma 7.1 cannot be improved, while the rate of convergence in Corollary 7.2 is not far from optimal. Indeed, let  $W(\mathbf{k}, t) = P_t(x_{\mathbf{k}}^2)$  for  $t \geq 0$  and  $\mathbf{k} \in \mathbb{Z}^N$ . Then  $\mathcal{L}x_{\mathbf{k}}^2 = b^2 \sum_{m=1}^N (x_{\mathbf{k}+(m)}^2 + x_{\mathbf{k}-(m)}^2 - 2x_{\mathbf{k}}^2)$ , so that,*

$$\frac{\partial W}{\partial t} = b^2 \Delta W,$$

where as above  $\Delta$  denotes the discrete Laplacian on  $\mathbb{Z}^N$ . Thus

$$W(t) = e^{tb^2\Delta}W(0), t \geq 0 \quad (7.14)$$

so that convergence in the Lemma 7.1 is precise (see the end of the proof of the Lemma 7.1). Furthermore, using (7.14) it is possible to explicitly calculate  $\mu_{\mathbf{G}}(P_t x_k^2 - \mu_{\mathbf{G}}(x_k^2))^2, t \geq 0$  and show that this expression converges to 0 polynomially. Hence the operator  $\mathcal{L}$  does not have a spectral gap.

The following result shows that the class of functions for which system is ergodic is larger than the one considered in Corollary 7.2.

**Proposition 7.4.** *The semigroup  $(P_t)_{t \geq 0}$  is ergodic in the Orlicz space  $L_{\Psi}(\mu_{\mathbf{G}})$ , with  $\Psi(s) \equiv s^2 \log(1 + s^2)$ , in the sense that*

$$\|P_t f - \mu_{\mathbf{G}} f\|_{L_{\Psi}(\mu_{\mathbf{G}})} \rightarrow 0$$

for any  $f \in L_{\Psi}(\mu_{\mathbf{G}})$  as  $t \rightarrow \infty$ . Furthermore, for all  $f \in \mathbb{X}$ ,  $|P_t f - \mu_{\mathbf{G}} f|_{\mathbb{X}} \rightarrow 0$  as  $t \rightarrow \infty$ .

*Proof.* For  $f \in \mathbb{X} \cap \left\{ f \in L_{\Psi}(\mu_{\mathbf{G}}) : \sum_{\mathbf{i} \in \mathbb{Z}^N} \left( \mu_{\mathbf{G}} |\partial_{\mathbf{i}} f|^2 \right)^{\frac{1}{2}} < \infty \right\}$  the result follows from Corollary 7.2. Now, it is enough to notice that such set of functions is dense in  $L_{\Psi}(\mu_{\mathbf{G}})$  (resp. in  $\mathbb{X}$ ) for the natural topology and  $P_t$  is a contraction on  $L_{\Psi}(\mu_{\mathbf{G}})$  (resp. in  $\mathbb{X}$ ).  $\square$

**Remark 7.5.** *It should be noted that the measures  $\mu_{r\mathbf{G}}$ , for  $r > 0$  are all invariant with respect to the semigroup  $P_t, t \geq 0$ . Consequently, the results of the section 7 are also valid for the measures  $\mu_{r\mathbf{G}}, r > 0$ , with the proofs being exactly the same as in the case of  $\mu_{\mathbf{G}}$ .*

## 8 Liggett-Nash type inequalities

In this section we will show how to deduce Liggett-Nash type inequalities from the results of the previous section.

**Theorem 8.1.** *For  $f \in \mathbb{X} \cap \mathcal{D}(\mathcal{L})$  such that  $\sum_{\mathbf{i} \in \mathbb{Z}^N} \left( \mu_{\mathbf{G}} |\partial_{\mathbf{i}} f|^2 \right)^{\frac{1}{2}} < \infty$  we have that there exists a constant  $C \in (0, \infty)$  such that*

$$\mu_{\mathbf{G}}(f - \mu_{\mathbf{G}}(f))^2 \leq C (-\mathcal{L}f, f)_{L^2(\mu_{\mathbf{G}})}^{\frac{N}{N+4}} (\mathcal{A}(f)\mathcal{B}(f))^{\frac{2}{N+4}}. \quad (8.1)$$

Furthermore, for  $f \in \mathcal{D}_{\mathbb{X}}(\mathcal{L})$  such that  $\mathcal{B}(f) < \infty$ , we have

$$[\mathcal{A}(f)]^{2+\frac{4}{N}} \leq C(N) \left[ \sum_{\mathbf{i} \in \mathbb{Z}^N} \int_{E_\alpha} \partial_{\mathbf{i}} f \partial_{\mathbf{i}} (-\mathcal{L}f) d\mu_{\mathbf{G}} \right] \mathcal{B}(f)^{\frac{4}{N}}, \quad (8.2)$$

where  $C(N) = A^{\frac{2N}{N+8}} (1 + \frac{8}{N}) (\frac{N}{8})^{\frac{8}{N+8}}$ , and  $A$  is as in Lemma 7.1.

**Remark 8.2.** Note that inequality (8.2) can be considered as an analog of the Nash inequality in  $\mathbb{R}^N$  (see [19], p.936). Indeed, such an inequality takes the form

$$|u|_{L^2(\mathbb{R}^N)}^{2+\frac{4}{N}} \leq C(-\Delta u, u)_{L^2(\mathbb{R}^N)} |u|_{L^1(\mathbb{R}^N)}^{\frac{4}{N}}, \quad u \in L^1(\mathbb{R}^N) \cap W^{1,2}(\mathbb{R}^N),$$

for some constant  $C > 0$ , and where  $\Delta$  is the standard Laplacian on  $\mathbb{R}^N$ . The main difference is that the natural space for our operator  $\mathcal{L}$  is  $\mathbb{X}$  instead of  $L^2$ .

*Proof.* Inequality (8.1) immediately follows from (7.12), Corollary 4.4 and part (b) of Theorem 2.2 of [16].

Let us show the inequality (8.2). As before we denote  $f_t = P_t f$ . We have by Hölder's inequality and Lemma 7.1 that

$$\begin{aligned} \sum_{\mathbf{i} \in \mathbb{Z}^N} \int_{E_\alpha} \partial_{\mathbf{i}} f \partial_{\mathbf{i}} f_t d\mu_{\mathbf{G}} &\leq \left( \sum_{\mathbf{i} \in \mathbb{Z}^N} |\partial_{\mathbf{i}} f|_{L^2(\mu_{\mathbf{G}})}^2 \right)^{\frac{1}{2}} \left( \sum_{\mathbf{i} \in \mathbb{Z}^N} |\partial_{\mathbf{i}} f_t|_{L^2(\mu_{\mathbf{G}})}^2 \right)^{\frac{1}{2}} \\ &\leq \frac{A^{\frac{N}{4}} \left( \sum_{\mathbf{i} \in \mathbb{Z}^N} \left( \mu_{\mathbf{G}} |\partial_{\mathbf{i}} f|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \sum_{\mathbf{i} \in \mathbb{Z}^N} |\partial_{\mathbf{i}} f|_{L^2(\mu_{\mathbf{G}})}^2 \right)^{\frac{3}{4}}}{t^{\frac{N}{8}}} \end{aligned} \quad (8.3)$$

Furthermore, note that

$$\sum_{\mathbf{i} \in \mathbb{Z}^N} \int_{E_\alpha} \partial_{\mathbf{i}} f \partial_{\mathbf{i}} f_t d\mu_{\mathbf{G}} = \sum_{\mathbf{i} \in \mathbb{Z}^N} |\partial_{\mathbf{i}} f|_{L^2(\mu_{\mathbf{G}})}^2 + \int_0^t \sum_{\mathbf{i} \in \mathbb{Z}^N} \int_{E_\alpha} \partial_{\mathbf{i}} f \partial_{\mathbf{i}} (\mathcal{L}f_s) d\mu_{\mathbf{G}} ds. \quad (8.4)$$

Define  $\phi(s) = \sum_{\mathbf{i} \in \mathbb{Z}^N} \int_{E_\alpha} \partial_{\mathbf{i}} f \partial_{\mathbf{i}} (\mathcal{L}f_s) d\mu_{\mathbf{G}}$  for  $s \geq 0$ .  $\mathcal{L}$  is symmetric in  $\mathbb{X}$ , so

$$\phi(s) = \sum_{\mathbf{i} \in \mathbb{Z}^N} \int_{E_\alpha} \partial_{\mathbf{i}} (\mathcal{L}f) \partial_{\mathbf{i}} f_s d\mu_{\mathbf{G}}, \quad s \geq 0.$$

We can then calculate that

$$\begin{aligned} \phi'(s) &= \sum_{\mathbf{i} \in \mathbb{Z}^N} \int_{E_\alpha} \partial_{\mathbf{i}} (\mathcal{L}f) \partial_{\mathbf{i}} (\mathcal{L}f_s) d\mu_{\mathbf{G}} = \sum_{\mathbf{i} \in \mathbb{Z}^N} \int_{E_\alpha} \partial_{\mathbf{i}} (\mathcal{L}f) \partial_{\mathbf{i}} (P_s \mathcal{L}f) d\mu_{\mathbf{G}} \\ &= \sum_{\mathbf{i} \in \mathbb{Z}^N} \int_{E_\alpha} \partial_{\mathbf{i}} (\mathcal{L}P_{\frac{s}{2}} f) \partial_{\mathbf{i}} (\mathcal{L}P_{\frac{s}{2}} f) d\mu_{\mathbf{G}} \geq 0, \end{aligned}$$

for all  $s \geq 0$ . Consequently,

$$\phi(t) = \sum_{\mathbf{i} \in \mathbb{Z}^N} \int_{E_\alpha} \partial_{\mathbf{i}}(\mathcal{L}f) \partial_{\mathbf{i}} f_t d\mu_{\mathbf{G}} \geq \phi(0) = \sum_{\mathbf{i} \in \mathbb{Z}^N} \int_{E_\alpha} \partial_{\mathbf{i}}(\mathcal{L}f) \partial_{\mathbf{i}} f d\mu_{\mathbf{G}} \quad (8.5)$$

for all  $t \geq 0$ . Using (8.5) in (8.4) yields

$$\sum_{\mathbf{i} \in \mathbb{Z}^N} \int_{E_\alpha} \partial_{\mathbf{i}} f \partial_{\mathbf{i}} f_t d\mu_{\mathbf{G}} \geq \sum_{\mathbf{i} \in \mathbb{Z}^N} |\partial_{\mathbf{i}} f|_{L^2(\mu_{\mathbf{G}})}^2 - t \sum_{\mathbf{i} \in \mathbb{Z}^N} \int_{E_\alpha} \partial_{\mathbf{i}}(-\mathcal{L}f) \partial_{\mathbf{i}} f d\mu_{\mathbf{G}},$$

for  $t \geq 0$ . Therefore, using this in (8.3), we obtain

$$\begin{aligned} \sum_{\mathbf{i} \in \mathbb{Z}^N} |\partial_{\mathbf{i}} f|_{L^2(\mu_{\mathbf{G}})}^2 &\leq t \sum_{\mathbf{i} \in \mathbb{Z}^N} \int_{E_\alpha} \partial_{\mathbf{i}}(-\mathcal{L}f) \partial_{\mathbf{i}} f d\mu_{\mathbf{G}} \\ &\quad + \frac{A^{\frac{N}{4}} \left( \sum_{\mathbf{i} \in \mathbb{Z}^N} \left( \mu_{\mathbf{G}} |\partial_{\mathbf{i}} f|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \sum_{\mathbf{i} \in \mathbb{Z}^N} |\partial_{\mathbf{i}} f|_{L^2(\mu_{\mathbf{G}})}^2 \right)^{\frac{3}{4}}}{t^{\frac{N}{8}}}. \end{aligned} \quad (8.6)$$

Optimization of the right-hand side of (8.6) with respect to  $t$  leads to (8.2).  $\square$

## 9 Phase transition in stochastic dynamics

In this section we consider a family of stochastic dynamics defined by the following generators

$$\mathcal{L} \equiv \sum_{\mathbf{k} \in \mathbb{Z}^N} \mathbf{X}_{\Xi+\mathbf{k}}^2 - \beta D$$

where  $\beta \in [0, \infty)$ , and for a finite subset  $\Xi \subset \mathbb{Z}^N$  we set

$$\mathbf{X}_{\Xi+\mathbf{k}} \equiv \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Xi+\mathbf{k} \\ \mathbf{j} \sim \mathbf{i}}} a_{\mathbf{ij}} \mathbf{X}_{\mathbf{i}, \mathbf{j}}$$

with some constants  $a_{\mathbf{ij}} = a_{\mathbf{i}+\mathbf{k}, \mathbf{j}+\mathbf{k}} \in \mathbb{R}$ . As a special case, we can take  $\Xi$  to be a set of two neighbors and  $\beta = 0$ , which includes the model studied earlier. Now  $P_t \equiv e^{t\mathcal{L}}$ . Then, with  $f_s \equiv P_s f$ , we have the following simple computation.

$$\begin{aligned}
\frac{d}{ds} P_{t-s} |\nabla f_s|^2 &= P_{t-s} (-\mathcal{L} |\nabla f_s|^2 + 2 \nabla f_s \nabla \mathcal{L} f_s) \\
&= P_{t-s} \left( -2 \sum_{\mathbf{l}, \mathbf{k}} |\mathbf{X}_{\Xi+\mathbf{k}} \nabla_{\mathbf{l}} f_s|^2 + 2 \sum_{\mathbf{l}, \mathbf{k}} \nabla_{\mathbf{l}} f_s [\nabla_{\mathbf{l}}, \mathbf{X}_{\Xi+\mathbf{k}}^2] f_s - 2\beta |\nabla f_s|^2 \right) \\
&= P_{t-s} \left( -2 \sum_{\mathbf{l}, \mathbf{k}} |\mathbf{X}_{\Xi+\mathbf{k}} \nabla_{\mathbf{l}} f_s|^2 \right. \\
&\quad \left. + 2 \sum_{\mathbf{l}} \sum_{\mathbf{k}} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Xi+\mathbf{k} \\ \mathbf{j} \sim \mathbf{i}}} a_{\mathbf{i}\mathbf{j}} \nabla_{\mathbf{l}} f_s \{ \mathbf{X}_{\Xi+\mathbf{k}}, [\nabla_{\mathbf{l}}, \mathbf{X}_{\mathbf{i}\mathbf{j}}] \} f_s - 2\beta |\nabla f_s|^2 \right),
\end{aligned} \tag{9.1}$$

where  $\{\cdot, \cdot\}$  denotes the anti-commutator. Now, since

$$\{ \mathbf{X}_{\Xi+\mathbf{k}}, [\nabla_{\mathbf{l}}, \mathbf{X}_{\mathbf{i}\mathbf{j}}] \} f_s = 2 \mathbf{X}_{\Xi+\mathbf{k}} (\delta_{\mathbf{l}\mathbf{j}} \nabla_{\mathbf{i}} - \delta_{\mathbf{l}\mathbf{i}} \nabla_{\mathbf{j}}) + \sum_{\substack{\mathbf{i}', \mathbf{j}' \in \Xi+\mathbf{k} \\ \mathbf{j}' \sim \mathbf{i}'}} a_{\mathbf{i}'\mathbf{j}'} (\delta_{\mathbf{l}\mathbf{j}} \delta_{\mathbf{i}\mathbf{j}'} \nabla_{\mathbf{i}'} - \delta_{\mathbf{l}\mathbf{i}} \delta_{\mathbf{i}\mathbf{j}'} \nabla_{\mathbf{j}'})$$

there are constants  $\varepsilon \in (0, 2)$  and  $\eta \in \mathbb{R}$  such that

$$\begin{aligned}
\frac{d}{ds} P_{t-s} |\nabla f_s|^2 &\leq P_{t-s} \left( -(2 - \varepsilon) \sum_{\mathbf{l}, \mathbf{k}} |\mathbf{X}_{\Xi+\mathbf{k}} \nabla_{\mathbf{l}} f_s|^2 - 2(\beta - \eta) |\nabla f_s|^2 \right) \\
&\leq -2(\beta - \eta) P_{t-s} |\nabla f_s|^2.
\end{aligned} \tag{9.2}$$

Integrating this differential inequality, we obtain

$$|\nabla f_t|^2 \leq e^{-2(\beta-\eta)t} P_t |\nabla f|^2.$$

In the case when  $\Xi$  is a two point set, combining this with our analysis in previous section we conclude with the following result.

**Theorem 9.1.** *A stochastic system described by the family of generators*

$$\mathcal{L}_\beta \equiv \sum_{\mathbf{i} \sim \mathbf{j}} \mathbf{X}_{\mathbf{i}\mathbf{j}}^2 - \beta D$$

*with  $\beta \in [0, \infty)$ , undergoes a phase transition at some  $\beta_c \in [0, \infty)$ . That is, for  $\beta > \beta_c$  it decays to equilibrium exponentially fast, while for  $\beta \in [0, \beta_c)$  the decay to equilibrium (for certain cylinder functions) can only be algebraic.*

## 10 Homogenisation

We have shown in Section 7 that the semigroup  $(P_t)_{t \geq 0}$  with generator  $\mathcal{L}$  given by (3.9) is ergodic. Therefore we can apply Theorem 1.8, of [13] (see also [14]) to conclude that the following functional CLT holds.

**Proposition 10.1.** *Let  $\mathcal{L}$  be given by (3.9), and  $Y_t$ ,  $t \geq 0$  be the corresponding Markov process. Suppose  $F \in \mathcal{D}((-\mathcal{L})^{-\frac{1}{2}})$  is such that  $\mu(F) = 0$ . Let  $\mathbf{P}^\mu$  be a probability measure corresponding to the stationary Markov process with the same transition functions as  $Y_t$ , and  $\mathcal{G}_t = \sigma\{Y_s, s \leq t\}$ ,  $t \geq 0$  be the filtration generated by  $Y_t$ . Then there exists a square integrable martingale  $M_t$ ,  $t \geq 0$ , on the probability space  $(\Omega, (\mathcal{G}_t)_{t \geq 0}, \mathbf{P}^\mu)$  with stationary increments such that  $M_0 = 0$  and*

$$\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \sup_{0 \leq s \leq t} \left| \int_0^s F(Y_s) ds - M_s \right| = 0,$$

in probability with respect to  $\mathbf{P}^\mu$ . Moreover,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}^{\mathbf{P}^\mu} |Y_t - M_t|^2 = 0.$$

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