Global Types with Internal Delegation

To the memory of Maurice Nivat

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Abstract

This paper investigates a new form of delegation for multiparty session calculi. Usually, delegation allows a session participant to appoint a participant in another session to act on her behalf. This means that delegation is inherently an inter-session mechanism, which requires session interleaving. Hence delegation falls outside the descriptive power of global types, which specify single sessions. As a consequence, properties such as deadlock-freedom or lock-freedom are difficult to ensure in the presence of delegation. Here we adopt a different view of delegation, by allowing participants to delegate tasks to each other within the same multiparty session. This way, delegation occurs within a single session (internal delegation) and may be captured by its global type. To increase flexibility in the use of delegation, our calculus uses connecting communications, which allow optional participants in the branches of choices. By these means, we are able to express conditional delegation. We present a session type system based on global types with internal delegation, and show that it ensures the usual safety properties of multiparty sessions, together with a progress property.

Keywords: Communication-centric Systems, Process Calculi, Multiparty Session Types.

1. Introduction

Multiparty session types model the sequence and types of messages exchanged between two or more parties interacting according to a predefined protocol. Session types are typically used to ensure safety properties such as *communication safety* (absence of communication errors) and *session fidelity*

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(conformance to the protocol), which in turn entail (internal) *deadlock-freedom* (absence of mutual blocking between session participants). Often a stronger property than deadlock-freedom is targeted for multiparty sessions, called *progress*, stating that in addition to being able to proceed as a whole, a session should leave no participant behind. Note that for binary sessions, namely sessions involving only two parties, progress reduces to deadlock-freedom. Similarly, in the binary case, the type of the whole session is entirely determined by the type of anyone of its participants. In the multiparty case, on the other hand, a *global type* is required to specify the overall choreography of the session. From this global type, one can determine – via an operation of projection – the contribution of each individual party to the protocol, which is described by a *(local) session type*. Then, if all participants in a session behave according to their session type, we have guarantees about the overall behaviour of the session.

In early work [20], session type theory was enhanced with the ability to *delegate* interactions from one participant in a session to another participant in a different session. Thereby, by using delegation, a participant involved in a session can at any point request that some participant in a different session conducts part of the interaction in the first session on her behalf. By construction, the delegation mechanism proposed in [20], which is also the one generally adopted in later work, cannot take place within a single session.

Delegation is typically modelled by sending a channel, on which the session is conducted, over another channel. For example, suppose two participants — a client and a seller — are initially engaged in a session on a channel, say *cs*. During the session, the seller would like another participant, — a bank —, which does not belong to the current session, to process a payment on her behalf. For a seamless experience for the client, the seller opens a new session with the bank, say on channel *sb*, and sends the initial session channel *cs* over the new channel *sb* to the bank, who is then able to complete the session with the client on behalf of the seller. This avoids the client directly opening a new session with the bank; it also avoids any need for the seller to act as an intermediary, forwarding messages back and forth between the client and the bank. After delegation, all interactions occur directly between the client and the bank. The client does not know that the seller has subcontracted part of its interaction to the bank. Such subcontracting may be useful both for abstraction purposes, to hide useless details from some participant (the client does not need to know how the transaction is implemented) or to provide additional guarantees on sensitive parts of the interaction (the credit card should only be processed by a trusted entity like the bank).

The idea of delegation originates in early research into concurrent object-oriented programming [40, 2]. Hence, delegation via channels, as described above, has been natural to implement in object-oriented languages [15], such as Java [24], F# [11] and Scala [37]. With regards to session calculi, delegation for binary sessions was introduced in [20]. The first paper with global types for multiparty sessions [21] also considers delegation. Delegation increases the expressiveness of a session calculus regardless of whether the sessions are binary or multiparty. In fact, channel-based delegation allows communications

between participants which operate in different sessions and are unknown to each other. This behaviour cannot be obtained in session calculi without delegation.

This work takes a slightly more abstract approach to delegation than the channel-passing approach described above. In our calculus, there are no channels representing sessions, and hence delegation cannot be explicitly modelled by passing channels over channels. Instead, delegation is modelled by enabling one participant in a given multiparty session (the principal) to temporarily "lend" her behaviour to another participant in the same session (the deputy). The approach adopted has two advantages:

- Firstly, by being more abstract, we are less explicit about how delegation is implemented; thereby permitting a wide range of delegation mechanisms to be modelled. For example, delegation to a bank may be implemented as a browser redirection secured by a delegation certificate [1], which is quite different from the object-oriented approach. Thus the global types in this work are largely language-independent.
- Secondly, by "localising" delegation within a single multiparty session, we get around two well-known issues of multiparty sessions types, both of which may prevent progress: session interleaving (which is required to model delegation in the channel-based approach), and the "non-local" character of channel-based delegation, which is inherently shared between two sessions, and hence cannot be captured by standard global types.

Global types provide a view of the behaviour of all participants partaking in a session [22, 32]. They are a required feature of all multiparty session frameworks. Without global types, using standard techniques, deadlock cannot be prevented even within a single multiparty session: one can easily program a 3-philosopher deadlock by assuming three participants in the same session, which are all waiting to receive from their left neighbour before sending to their right neighbour. These participants are all pairwise dual but the session is stuck. The problem can be solved by using global types, and requiring the session types of all participants to be projections of the same global type.

Global types are closely related to message sequence charts [26, 29], widely understood by engineers as a tool for describing the flow of messages in a protocol. This work explores how global types can be enhanced with mechanisms for describing internal delegation (i.e., delegation within the same session), thus going beyond the expressiveness of message sequence charts.

Although early work on multiparty session types [21] features delegation, the progress result in that work holds only for a single session, which therefore excludes delegation since delegation requires multiple sessions in that framework. Existing approaches to progress for global types with delegation include the interleaving approach where several sessions are described separately and their actions may be interleaved [6]. A problem with having several separate global types is that it is not immediately clear how the actions in one session are related to the actions in another session. Worse still, without a view of

how all interactions are related, interleaving between different sessions may cause deadlock, preventing progress of typed sessions. Proposals for ensuring progress in the presence of session interleaving have been put forward in [10, 34], but they require sophisticated additional machinery. The approach presented in the current paper has the advantage that there is a single global type expressing the relationships between the behaviours of all participants in a session.

To increase flexibility in the use of delegation, our calculus accommodates *connecting communications*, a notion introduced in [23] to describe protocols with optional participants. Here we adopt the variant of this notion proposed in [9]. The intuition behind connecting communications is that in some parts of the protocol, delimited by a choice construct, some participants may be optional, namely they are "invited" to join the interaction only in some branches of the choice, by means of connecting communications. As argued in [23] and [9], this feature allows for a more natural description of typical communication protocols. In the present setting, connecting communications also enable us to express conditional delegation: this will be obtained by writing a choice where the delegation appears only in some branches of the choice, following a connecting communication.

Outline. Section 2 provides a motivating example describing a global type with internal delegation and its end-point projections. Section 3 introduces a core process calculus with delegation actions. Section 4 introduces global types and session types reflecting our notion of internal delegation. Section 5 formalises the type system which assures progress for networks with internal delegation and connecting communications. Section 6 establishes the main properties of our typed calculus: subject reduction, session fidelity and progress. Section 7 sketches future work on multiparty compatibility and on various extensions of the present calculus. Section 8 discusses related work, followed by concluding remarks in Section 9. Most of the proofs are included in the body of the paper. The proofs of projection correctness, subject reduction and session fidelity are given in Appendix A, Appendix B and Appendix C, respectively.

2. A Motivating Example of a Global Type with Internal Delegation

Here we revisit a typical example of delegation involving three participants: Client, Seller, and Bank. In the protocol, the client and seller engage in a session in which they present each other with the terms of a purchase (the title and price of a book). At this point the client is faced with a choice, depending on whether the proposed price is within her intended budget or not. If the client decides to purchase, she notifies the seller, who connects with the bank, sending the agreed price, and then delegates the processing of the client's credit card to the bank. Otherwise, the client indicates to the seller that she is not interested in proceeding with the purchase and the session ends without contacting the bank.

A global type for such a client-seller-bank protocol is presented in Figure 1. In this figure, an exchange of a message, say `price`, from a participant sending the message, say `S`, to another participant receiving the message, say `C`, is

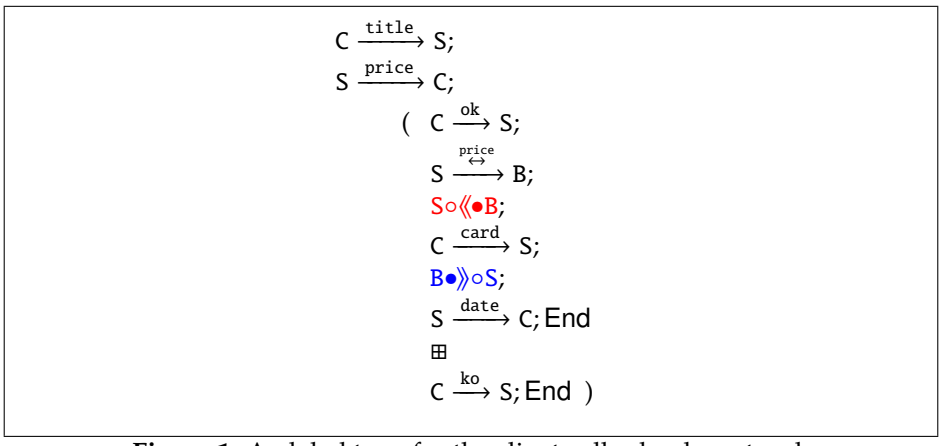

Figure 1: A global type for the client-seller-bank protocol.

denoted $\mathtt{S} \xrightarrow{\mathtt{price}} \mathtt{C}$. Communications are assumed to be synchronous, meaning that any communication on the left of a sequential composition (;) must be fully completed before we begin any communication on the right of the sequential composition, unless the two communications involve disjoint sets of participants. A choice (⊞) indicates that the sending party decides which branch of the choice she takes. Note that sequential composition binds tighter than choice; hence, in this figure, the client named C, at runtime, chooses either the branch beginning with an exchange of message ok or the branch beginning with an exchange of message ko. The new constructs for internal delegation $\mathtt{S}\circ\langle\!\langle\bullet\mathtt{B}$ and $\mathtt{B}\bullet\rangle\!\rangle\circ\mathtt{S}$ are explained at the appropriate point in the following message sequence:

- Firstly, Client sends a title to Seller, indicated by $\mathtt{C} \xrightarrow{\mathtt{title}} \mathtt{S}$;
- Seller responds by sending a price to Client, indicated by $\mathtt{S} \xrightarrow{\mathtt{price}} \mathtt{C}$;
- Depending on the price received at runtime, Client will decide whether the book is within her budget or not. This is an internal runtime decision made by Client. The two options are detailed separately below.
- If the price is within Client's budget, then the following occurs:
  - Client sends a message ok to Seller;
  - Seller sends the price to Bank. The annotation on the message $\overset{\mathtt{price}}{\leftrightarrow}$ indicates that the receiver Bank connects to the session at that point.
  - After the above connecting communication, Seller **delegates** her interaction to Bank, indicated by $\mathtt{S}\circ\langle\!\langle\bullet\mathtt{B}$.
  - Client sends her credit card number apparently to the participant Seller but actually — thanks to delegation — to the participant Bank.
  - Bank delegates back the interaction to Seller, indicated by $\mathtt{B}\bullet\rangle\!\rangle\circ\mathtt{S}$;

  - Seller sends a `date` to Client. Since the delegation has ended, this is a communication between the actual seller and the client;

- Otherwise, Client sends message `ko` to Seller, terminating the session.

Notice, in the above protocol, it is possible that the bank is never contacted, in the case the client terminates the protocol by giving up the purchase. In other words, the bank is an optional participant. For this reason, it receives its first message via a connecting communication. The local view of the bank in this protocol can be described by the following session type.

$$\text{Participant Bank:}\quad \mathtt{S?}\overset{\mathtt{price}}{\leftrightarrow}\ ;\ \mathtt{S}\circ\langle\!\langle\bullet\ ;\ \mathtt{C?\,card}\ ;\ \bullet\rangle\!\rangle\circ\mathtt{S}\ ;\ \mathsf{End}$$

In the above session type for the bank, the first action $\mathtt{S?}\overset{\mathtt{price}}{\leftrightarrow}$ indicates the bank is ready to receive a message containing a `price` from the seller via a connecting communication. The construct $\mathtt{S}\circ\langle\!\langle\bullet$, pronounced *passive forward delegation*, indicates that the bank is delegated to act as a deputy for the seller. On behalf of the seller it receives a card number from the client, indicated by $\mathtt{C?\,card}$. Finally, the bank returns control to the seller, as indicated by the construct $\bullet\rangle\!\rangle\circ\mathtt{S}$, pronounced *active backward delegation*.

The session type of the bank, presented above, is obtained automatically by *projection* from the global type. This projection always succeeds under some mild constraints removing ambiguity and ensuring that delegations begin and end correctly. For the other two participants — Client and Seller — we obtain the following session types:

Participant Client:

```
S!title;
S?price;
     ( S!ok;
       S!card;
       S?date;End
       ∨
       S!ko;End )
```

Participant Seller:

```
C?title;
C!price;
     ( C?ok;
       B!↔^price;
       ○⟪•B;B•⟫○;
       C!date;End
       ∧
       C?ko;End )
```

In the above two types, observe that the client always thinks she is interacting with the seller. Also observe that the seller does not perform any actions between delegating her behaviour to the bank (indicated by $\circ\langle\!\langle\bullet\mathtt{B}$, pronounced *active forward delegation*) and receiving it back (indicated by $\mathtt{B}\bullet\rangle\!\rangle\circ$, pronounced *passive backward delegation*).

As standard for session types, since the client decides whether to purchase or not by sending the message `ok` or `ko` respectively, the branches for the client are composed using the union operator ($\vee$). The union indicates that at compile-time we do not know what decision the client will make. In contrast, we know that the seller will react in response to the decision the client makes — by either receiving `ok` or `ko` — and hence the branches of the seller are composed using intersection ($\wedge$).

2.1. Comparison to the approach with session interleaving

In previous work [21], a quite different approach to global types involving delegation is adopted. To model the scenario defined above, two global types are employed. The first describes the session involving interactions between the client and the seller. The second describes the session between the seller and the bank, in which some actions of the seller from the former session are delegated to the bank. As discussed in Section 1, this delegation is implemented via channel passing and it appears in the global type of the deputy's session as a communication labelled by a channel type. The two separate global types are presented in Figure 2.

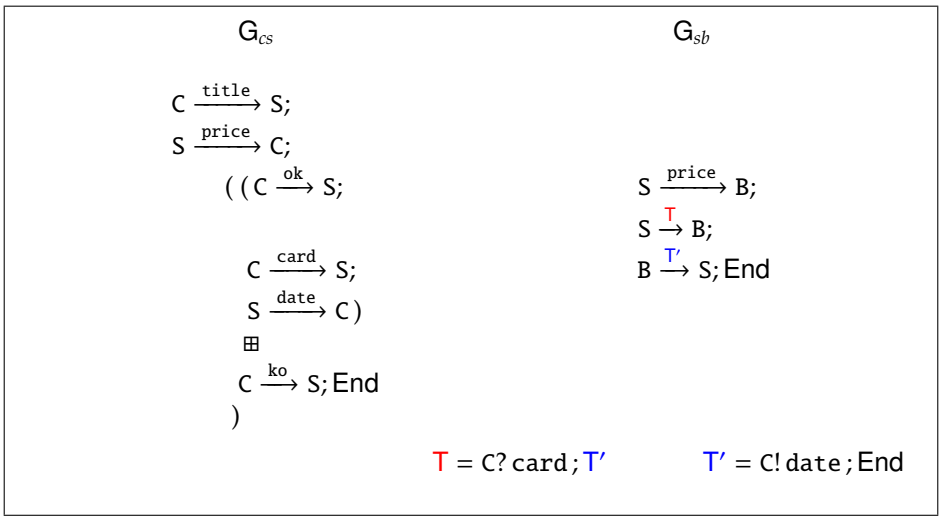

Figure 2: Two global types for the client-seller-bank protocol in the channel-based approach.

All communications in Figure 1 are also accounted for in Figure 2. However, there is information missing in Figure 2, regarding how the communications in the two global types are related to each other. Moreover, it is not clear whether the delegation in the first session is enforced, since G_{cs} describes a binary session that can complete independently of G_{sb}.

To infer the connection between the two global types in Figure 2 it is required to know that the transmitted type C? `card`; C! `date`; End is the type of the seller in G_{cs} after having received the message `ok`. This however is still ambiguous. For example, we may wonder whether the message `price` is sent from the seller to the bank before or after receiving the message `ok`. In order to guarantee that the bank does not get stuck once it engages in the session, the seller must send the `price` after `ok` is received from the client. Unfortunately, this is not provided by the types in Figure 2. In contrast, there is no such ambiguity in interpreting the global type in Figure 1.

3. A Core Calculus Modelling Processes with Internal Delegation

We begin the formal development by introducing untyped networks of processes. Each process in a network is a single, possibly infinite, thread that can communicate by message passing with the other processes. In session calculi, we typically have two types of choice indicating whether, at runtime, the process makes a choice (by sending one among a possible set of messages) or reacts to a choice (by receiving one among a possible set of messages). Beyond these standard choice constructs, our session calculus features primitives for beginning and ending internal delegations and for connecting to a new party that would otherwise not participate in the session.

We assume the following base sets: *simple messages*, ranged over by $\lambda, \lambda', \ldots$ and forming the set Msg; *connecting messages*, ranged over by $\overset{\lambda}{\leftrightarrow}, \overset{\lambda'}{\leftrightarrow}, \ldots$ and forming the set CMsg; and *session participants*, ranged over by $\mathsf{p}, \mathsf{q}, \mathsf{r}$ and forming the set Part. We use Λ to range over both messages and connecting messages.

Let $\pi \in \{\mathsf{p}?\Lambda, \mathsf{p}!\Lambda \mid \mathsf{p} \in \mathsf{Part}, \Lambda \in \mathsf{Msg} \cup \mathsf{CMsg}\}$ denote an *I/O action*, namely a simple input/output action or an input/output action establishing a connection. An input action $\mathsf{p}?\Lambda$ whose message Λ is connecting is called a *connecting input* (and similarly for output actions). The choice between using simple or connecting inputs gives us additional freedom, as in [23, 9]. Using a simple input indicates that we expect the input action is capable of participating in a communication; whereas connecting inputs may never be capable of participating in a communication. For example, in the client-seller-bank protocol described in Figure 1, the seller (which uses only simple inputs) participates in all sessions, but the bank is incapable of performing its connecting input when the client chooses the `ko` branch.

In addition to exchanging messages, processes may perform *delegation actions*. Delegation involves two participants: the *principal*, say p in what follows, and the *deputy*, say q. The principal p suspends execution between two matching delegation actions $\circ\langle\!\langle\bullet\mathsf{q}$ and $\mathsf{q}\bullet\rangle\!\rangle\circ$. Dually, the deputy q acts on behalf of the principal p in the piece of code delimited by two matching delegation actions $\mathsf{p}\circ\langle\!\langle\bullet$ and $\bullet\rangle\!\rangle\circ\mathsf{p}$.

Definition 3.1 (Processes). Processes *are defined by:*

$$\begin{array}{rcl} P & ::= & \Sigma_{i\in I}\pi_i;P_i \;\mid\; \oplus_{i\in I}\,\pi_i;P_i \;\mid\; \mathsf{p}\circ\langle\!\langle\bullet;P \;\mid\; \bullet\rangle\!\rangle\circ\mathsf{p};P \\ & & \mid\; \circ\langle\!\langle\bullet\mathsf{q};P \;\mid\; \mathsf{q}\bullet\rangle\!\rangle\circ;P \;\mid\; \mu X.P \;\mid\; X \;\mid\; \mathbf{0} \end{array}$$

External choice ($\sum$) and internal choice ($\oplus$) are assumed to be associative, commutative, and non-empty. A process prefixed by an I/O action may be either an *input process* or an *output process*. We require recursion to be guarded, and we assume its scope to extend as far right as possible. Processes are treated equirecursively, i.e., they are identified with their generated tree [35] (Chapter 21). Sequential composition (;) is assumed to have higher precedence than both choice constructs ($\sum$ and $\oplus$). We omit choice symbols in one-branch choices, as well as trailing $\mathbf{0}$ processes.

The prefixes $\mathsf{p}\circ\langle\!\langle\bullet, \bullet\rangle\!\rangle\circ\mathsf{p}$, $\circ\langle\!\langle\bullet\mathsf{q}, \mathsf{q}\bullet\rangle\!\rangle\circ$ are used for delegation and should be read as follows: the open/closed parentheses mean forward/backward delegation, while the white/black circle next to the participant indicates whether she is the principal/deputy in the delegation (as a rule of thumb, the colour of the principal is white since she plays first, as in a chess game).

The construct $\circ\langle\!\langle\bullet\mathsf{q}; P$ is *active forward delegation*: it represents the behaviour of a principal who delegates deputy q to act on her behalf, and freezes her continuation P until q returns the delegation. The construct $\mathsf{q}\bullet\rangle\!\rangle\circ; P$ is *passive backward delegation*: it represents the behaviour of a principal who gets back the delegation from the deputy q and resumes executing her continuation P. An active forward delegation followed by a passive backward delegation in the code for some participant p, as in $\circ\langle\!\langle\bullet\mathsf{q}; \mathsf{q}\bullet\rangle\!\rangle\circ; P$, indicates that p remains inactive while her behaviour is being delegated to q (see participant Seller at page 6).

Dually, the construct $\mathsf{p}\circ\langle\!\langle\bullet; P$ is *passive forward delegation*: it represents the behaviour of a deputy who will execute the continuation P as if she were the principal p until the end of the delegation. The construct $\bullet\rangle\!\rangle\circ\mathsf{p}; P$ is *active backward delegation*: it represents the behaviour of a deputy who returns the delegation to the principal p and resumes executing P under her own identity. For example, if a participant q is modelled as process $\mathsf{r}?\lambda_1; \mathsf{p}\circ\langle\!\langle\bullet; \mathsf{r}!\lambda_2; \bullet\rangle\!\rangle\circ\mathsf{p}; \mathsf{r}?\lambda_3$, then messages λ_1 and λ_3 at the beginning and end are received under the participant's own identity q; however, for sending message λ_2, the participant acts as a deputy for p. See for example participant Bank in Section 2.

An *atomic action* is either an I/O action or an active/passive forward/backward delegation action.

In a full-fledged calculus, messages would carry values or data types, namely they would be of the form $\Lambda(v)$. For simplicity, we consider only pure messages.

Networks are comprised of pairs $\mathsf{p}[\![\, P \,]\!]$ or $\overset{*}{\mathsf{p}}[\![\, P \,]\!]$ composed in parallel. The pair $\mathsf{p}[\![\, P \,]\!]$ indicates participant p is *active* and has behaviour P; while in $\overset{*}{\mathsf{p}}[\![\, P \,]\!]$ the participant p is *frozen* and is ready to resume P after unfreezing.

Definition 3.2 (Networks). Networks *are defined by:*

$$\mathbb{N} \quad ::= \quad \mathsf{p}[\![\, P \,]\!] \;\mid\; \overset{*}{\mathsf{p}}[\![\, P \,]\!] \;\mid\; \mathbb{N} \parallel \mathbb{N}$$

The operator $\parallel$ is associative and commutative, with neutral elements $\mathsf{p}[\![\, \mathbf{0} \,]\!]$ for each p. These laws give rise to the structural equivalence on networks.

To keep track of whom is involved in a communication, it is convenient to record the overall behaviour of a communication on labels whenever a sender and receiver interact. For this purpose, we define *communications*, ranged over by α, to be of the form $\alpha = \mathsf{p}\Lambda\mathsf{q}$, where $\mathsf{p}, \mathsf{q} \in \mathsf{Part}$ and $\Lambda \in \mathsf{Msg} \cup \mathsf{CMsg}$. A communication $\mathsf{p}\lambda\mathsf{q}$ is *simple*, while a communication $\mathsf{p} \overset{\lambda}{\leftrightarrow} \mathsf{q}$ is *connecting*. Given a communication $\alpha = \mathsf{p}\Lambda\mathsf{q}$, we define the sender and receiver of α as $\mathsf{sender}(\alpha) = \mathsf{p}$ and $\mathsf{rec}(\alpha) = \mathsf{q}$. Moreover, we denote by $\mathsf{part}(\alpha) = \{\mathsf{rec}(\alpha), \mathsf{sender}(\alpha)\}$ the participants of the communication α.

We also wish to record when a principal and a deputy are engaged in a delegation. We let *atomic interactions*, ranged over by ϕ, be defined by

$$\phi \in \{\mathsf{p}\Lambda\mathsf{q}, \mathsf{p}\circ\langle\!\langle\bullet\mathsf{q}, \mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p} \mid \mathsf{p},\mathsf{q} \in \mathsf{Part}, \mathsf{p} \neq \mathsf{q}, \Lambda \in \mathsf{Msg} \cup \mathsf{CMsg}\}$$

An atomic interaction is either a communication from a participant p to a participant q, or the start of a delegation from a principal p to a deputy q, or the end of a delegation from a deputy q to a principal p. In the delegation interactions, the participant to the left is the one who triggers the interaction, while the white and black colours denote respectively the principal and the deputy.

The operational semantics is given by two labelled transition systems, one for processes and one for networks. The transitions for processes are labelled by I/O actions, while those for networks are labelled by atomic interactions. As a convention, when we want to discuss sequences of multiple interactions, we write $\mathbb{N} \xrightarrow{\phi_1\cdots\phi_n} \mathbb{N}'$ for the transitive closure of $\xrightarrow{\phi}$. Formally, for $n \geq 1$ we let $\mathbb{N} \xrightarrow{\phi_1\cdots\phi_n} \mathbb{N}'$ if there are $\mathbb{N}_i$, $0 \leq i \leq n$, such that $\mathbb{N}_0 = \mathbb{N}$ and $\mathbb{N}_n = \mathbb{N}'$ and $\mathbb{N}_i \xrightarrow{\phi_{i+1}} \mathbb{N}_{i+1}$ for $0 \leq i < n$.

$$\Sigma_{i\in I}\pi_i; P_i \xrightarrow{\pi_j} P_j \quad j \in I \quad [\textsc{ExtCh}] \qquad \oplus_{i\in I}\pi_i; P_i \xrightarrow{\pi_j} P_j \quad j \in I \quad [\textsc{IntCh}]$$

$$\frac{P \xrightarrow{\mathsf{q}!\Lambda} P' \qquad Q \xrightarrow{\mathsf{p}?\Lambda} Q'}{\mathsf{p}[\![\, P \,]\!] \parallel \mathsf{q}[\![\, Q \,]\!] \xrightarrow{\mathsf{p}\Lambda\mathsf{q}} \mathsf{p}[\![\, P' \,]\!] \parallel \mathsf{q}[\![\, Q' \,]\!]} \; [\textsc{Com}]$$

$$\mathsf{p}[\![\, \circ\langle\!\langle\bullet\mathsf{q}; P \,]\!] \parallel \mathsf{q}[\![\, \mathsf{p}\circ\langle\!\langle\bullet; Q \,]\!] \xrightarrow{\mathsf{p}\circ\langle\bullet\mathsf{q}} \overset{\ast}{\mathsf{p}}[\![\, P \,]\!] \parallel \mathsf{p}[\![\, Q \,]\!] \quad [\textsc{BDel}]$$

$$\overset{\ast}{\mathsf{p}}[\![\, \mathsf{q}\bullet\rangle\!\rangle\circ; P \,]\!] \parallel \mathsf{p}[\![\, \bullet\rangle\!\rangle\circ\mathsf{p}; Q \,]\!] \xrightarrow{\mathsf{q}\bullet\rangle\circ\mathsf{p}} \mathsf{p}[\![\, P \,]\!] \parallel \mathsf{q}[\![\, Q \,]\!] \quad [\textsc{EDel}]$$

$$\frac{\mathbb{N} \xrightarrow{\phi} \mathbb{N}'}{\mathbb{N} \parallel \mathbb{N}'' \xrightarrow{\phi} \mathbb{N}' \parallel \mathbb{N}''} \; [\textsc{Ct}]$$

Figure 3: LTS for processes and networks.

The LTSs for processes and networks are given in Figure 3. Rules [ExtCh] and [IntCh] allow an I/O action to be extracted from one of the summands, as usual. Rule [BDel] describes delegation start: when the delegation from p to q begins, p becomes frozen and q becomes p. Rule [EDel] describes delegation end: when the delegation from p to q ends, the active p becomes q again and the frozen p becomes active. Notice that the deputy "forgets" her identity when acting on behalf of the principal; her identity is only remembered by the principal. Hence it is crucial that the principal does not perform any other delegation while being

personified by the deputy: otherwise the principal could mix up the identities of her different deputies, who would then resume executing with the wrong continuation (see Example 4.3 in Section 4). Our type system will make sure that this does not happen. Rule [Ct] is the standard contextual rule for networks.

4. Global Types and Session Types

The untyped networks in the previous section offer few guarantees regarding their behaviours. It is trivial to construct networks where progress is not guaranteed, or which exhibit wild delegation behaviours, where we lose track of which participants are the principals and which are the deputies. In this section we present such examples.

Example 4.1 (Deadlock due to delegation to another principal's deputy). *The use of nested delegations may lead to deadlock, at least when they are not well nested. Consider the following protocol, described using the same notation as in Figure 1:*

$$\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q};\mathsf{r}\circ\langle\!\langle\bullet\mathsf{p};\mathsf{r}\xrightarrow{\lambda_1}\mathsf{s};\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p};\mathsf{r}\xrightarrow{\lambda_2}\mathsf{s};\mathsf{p}\bullet\rangle\!\rangle\circ\mathsf{r};\mathsf{End}$$

Note that the delegation from r *to* p *is actually a delegation from* r *to* q *acting as* p*, namely it is a delegation to someone else's deputy. Note also that the two delegations are not well nested (the corresponding blocks are not contained one in the other), since the first delegation is closed before the second.*

Any network implementing this protocol will get stuck when trying to close the delegation from r *to* p*, as shown by the following maximal computation:*

$$\begin{array}{c}\mathsf{p}[\![\,\circ\langle\!\langle\bullet\mathsf{q};\mathsf{q}\bullet\rangle\!\rangle\circ;\mathsf{s}!\lambda_2;\bullet\rangle\!\rangle\circ\mathsf{r}\,]\!]\parallel\mathsf{q}[\![\,\mathsf{p}\circ\langle\!\langle\bullet;\mathsf{r}\circ\langle\!\langle\bullet;\mathsf{s}!\lambda_1;\bullet\rangle\!\rangle\circ\mathsf{p}\,]\!]\parallel\mathsf{r}[\![\,\circ\langle\!\langle\bullet\mathsf{p};\mathsf{p}\bullet\rangle\!\rangle\circ\,]\!]\parallel\mathsf{s}[\![\,\mathsf{r}?\lambda_1;\mathsf{r}?\lambda_2\,]\!]\xrightarrow{\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q}}\\ \overset{*}{\mathsf{p}}[\![\,\mathsf{q}\bullet\rangle\!\rangle\circ;\mathsf{s}!\lambda_2;\bullet\rangle\!\rangle\circ\mathsf{r}\,]\!]\parallel\mathsf{p}[\![\,\mathsf{r}\circ\langle\!\langle\bullet;\mathsf{s}!\lambda_1;\bullet\rangle\!\rangle\circ\mathsf{p}\,]\!]\parallel\mathsf{r}[\![\,\circ\langle\!\langle\bullet\mathsf{p};\mathsf{p}\bullet\rangle\!\rangle\circ\,]\!]\parallel\mathsf{s}[\![\,\mathsf{r}?\lambda_1;\mathsf{r}?\lambda_2\,]\!]\xrightarrow{\mathsf{r}\circ\langle\!\langle\bullet\mathsf{p}}\\ \overset{*}{\mathsf{p}}[\![\,\mathsf{q}\bullet\rangle\!\rangle\circ;\mathsf{s}!\lambda_2;\bullet\rangle\!\rangle\circ\mathsf{r}\,]\!]\parallel\mathsf{r}[\![\,\mathsf{s}!\lambda_1;\bullet\rangle\!\rangle\circ\mathsf{p}\,]\!]\parallel\overset{*}{\mathsf{r}}[\![\,\mathsf{p}\bullet\rangle\!\rangle\circ\,]\!]\parallel\mathsf{s}[\![\,\mathsf{r}?\lambda_1;\mathsf{r}?\lambda_2\,]\!]\xrightarrow{\mathsf{r}\lambda_1\mathsf{s}}\\ \overset{*}{\mathsf{p}}[\![\,\mathsf{q}\bullet\rangle\!\rangle\circ;\mathsf{s}!\lambda_2;\bullet\rangle\!\rangle\circ\mathsf{r}\,]\!]\parallel\mathsf{r}[\![\,\bullet\rangle\!\rangle\circ\mathsf{p}\,]\!]\parallel\overset{*}{\mathsf{r}}[\![\,\mathsf{p}\bullet\rangle\!\rangle\circ\,]\!]\parallel\mathsf{s}[\![\,\mathsf{r}?\lambda_2\,]\!]\end{array}$$

The last network is stuck because the code of the deputy r *does not start with* $\bullet\rangle\!\rangle\circ\mathsf{r}$ *and hence it is not of the required form for Rule* [EDel] *to be applied.*

The deadlock of Example 4.1 could be avoided by requiring delegations to be well nested.

Example 4.2 (Variant of Example 4.1 with well nested delegation). *Consider now a variant of the protocol of Example 4.1 where the two backward delegations are swapped:*

$$\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q};\mathsf{r}\circ\langle\!\langle\bullet\mathsf{p};\mathsf{r}\xrightarrow{\lambda_1}\mathsf{s};\mathsf{p}\bullet\rangle\!\rangle\circ\mathsf{r};\mathsf{r}\xrightarrow{\lambda_2}\mathsf{s};\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p};\mathsf{End}$$

In a network implementing the protocol, the two backward delegations will be similarly swapped in the code of participant q*. Moreover, the backward delegation to* r *is now in*

participant q *rather than in participant* p*. This network has the transition sequence:*

$$\begin{array}{c}
\mathsf{p}[\![\ \circ\langle\!\langle \bullet \mathsf{q}; \mathsf{q} \bullet \rangle\!\rangle \circ\ ]\!] \parallel \mathsf{q}[\![\ \mathsf{p} \circ \langle\!\langle \bullet; \mathsf{r} \circ \langle\!\langle \bullet; \mathsf{s}!\lambda_1; \bullet \rangle\!\rangle \circ \mathsf{r}; \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{r}[\![\ \circ\langle\!\langle \bullet \mathsf{p}; \mathsf{p} \bullet \rangle\!\rangle \circ; \mathsf{s}!\lambda_2\ ]\!] \parallel \mathsf{s}[\![\ \mathsf{r}?\lambda_1; \mathsf{r}?\lambda_2\ ]\!] \xrightarrow{\mathsf{p} \circ \langle\!\langle \bullet \mathsf{q}} \\
\overset{*}{\mathsf{p}}[\![\ \mathsf{q} \bullet \rangle\!\rangle \circ\ ]\!] \parallel \mathsf{p}[\![\ \mathsf{r} \circ \langle\!\langle \bullet; \mathsf{s}!\lambda_1; \bullet \rangle\!\rangle \circ \mathsf{r}; \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{r}[\![\ \circ\langle\!\langle \bullet \mathsf{p}; \mathsf{p} \bullet \rangle\!\rangle \circ; \mathsf{s}!\lambda_2\ ]\!] \parallel \mathsf{s}[\![\ \mathsf{r}?\lambda_1; \mathsf{r}?\lambda_2\ ]\!] \xrightarrow{\mathsf{r} \circ \langle\!\langle \bullet \mathsf{p}} \\
\overset{*}{\mathsf{p}}[\![\ \mathsf{q} \bullet \rangle\!\rangle \circ\ ]\!] \parallel \mathsf{r}[\![\ \mathsf{s}!\lambda_1; \bullet \rangle\!\rangle \circ \mathsf{r}; \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \overset{*}{\mathsf{r}}[\![\ \mathsf{p} \bullet \rangle\!\rangle \circ; \mathsf{s}!\lambda_2\ ]\!] \parallel \mathsf{s}[\![\ \mathsf{r}?\lambda_1; \mathsf{r}?\lambda_2\ ]\!] \xrightarrow{\mathsf{r}\lambda_1 \mathsf{s}} \\
\overset{*}{\mathsf{p}}[\![\ \mathsf{q} \bullet \rangle\!\rangle \circ\ ]\!] \parallel \mathsf{r}[\![\ \bullet \rangle\!\rangle \circ \mathsf{r}; \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \overset{*}{\mathsf{r}}[\![\ \mathsf{p} \bullet \rangle\!\rangle \circ; \mathsf{s}!\lambda_2\ ]\!] \parallel \mathsf{s}[\![\ \mathsf{r}?\lambda_2\ ]\!] \xrightarrow{\mathsf{p} \bullet \rangle\!\rangle \circ \mathsf{r}} \\
\overset{*}{\mathsf{p}}[\![\ \mathsf{q} \bullet \rangle\!\rangle \circ\ ]\!] \parallel \mathsf{p}[\![\ \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{r}[\![\ \mathsf{s}!\lambda_2\ ]\!] \parallel \mathsf{s}[\![\ \mathsf{r}?\lambda_2\ ]\!] \xrightarrow{\mathsf{r}\lambda_2 \mathsf{s}} \\
\overset{*}{\mathsf{p}}[\![\ \mathsf{q} \bullet \rangle\!\rangle \circ\ ]\!] \parallel \mathsf{p}[\![\ \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{r}[\![\ \mathbf{0}\ ]\!] \parallel \mathsf{s}[\![\ \mathbf{0}\ ]\!] \xrightarrow{\mathsf{q} \bullet \rangle\!\rangle \circ \mathsf{p}} \\
\mathsf{p}[\![\ \mathbf{0}\ ]\!] \parallel \mathsf{q}[\![\ \mathbf{0}\ ]\!] \parallel \mathsf{r}[\![\ \mathbf{0}\ ]\!] \parallel \mathsf{s}[\![\ \mathbf{0}\ ]\!]
\end{array}$$

In this case, delegation is well nested, hence the nested delegations are closed in the right order and the session terminates successfully.

However, well-nestedness does not suffice to prevent deadlock in case of a chain of delegations originating from the same principal.

Example 4.3 (Deadlock due to delegation from a deputy). *Consider now a protocol where a principal* p *delegates to a participant* q *who in turn delegates to a participant* r *(while acting on behalf of* p*):*

$$\mathsf{p} \circ \langle\!\langle \bullet \mathsf{q}; \mathsf{p} \circ \langle\!\langle \bullet \mathsf{r}; \mathsf{p} \xrightarrow{\lambda} \mathsf{s}; \mathsf{r} \bullet \rangle\!\rangle \circ \mathsf{p}; \mathsf{q} \bullet \rangle\!\rangle \circ \mathsf{p}; \mathsf{End}$$

A network implementing this protocol is truly nondeterministic: it has two possible non-confluent maximal computations, one of which leads to deadlock, while the other one terminates successfully. The deadlocked computation is the following:

$$\begin{array}{c}
\mathsf{p}[\![\ \circ\langle\!\langle \bullet \mathsf{q}; \mathsf{q} \bullet \rangle\!\rangle \circ\ ]\!] \parallel \mathsf{q}[\![\ \mathsf{p} \circ \langle\!\langle \bullet; \circ \langle\!\langle \bullet \mathsf{r}; \mathsf{r} \bullet \rangle\!\rangle \circ; \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{r}[\![\ \mathsf{p} \circ \langle\!\langle \bullet; \mathsf{s}!\lambda; \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{s}[\![\ \mathsf{p}?\lambda\ ]\!] \xrightarrow{\mathsf{p} \circ \langle\!\langle \bullet \mathsf{q}} \\
\overset{*}{\mathsf{p}}[\![\ \mathsf{q} \bullet \rangle\!\rangle \circ\ ]\!] \parallel \mathsf{p}[\![\ \circ\langle\!\langle \bullet \mathsf{r}; \mathsf{r} \bullet \rangle\!\rangle \circ; \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{r}[\![\ \mathsf{p} \circ \langle\!\langle \bullet; \mathsf{s}!\lambda; \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{s}[\![\ \mathsf{p}?\lambda\ ]\!] \xrightarrow{\mathsf{p} \circ \langle\!\langle \bullet \mathsf{r}} \\
\overset{*}{\mathsf{p}}[\![\ \mathsf{q} \bullet \rangle\!\rangle \circ\ ]\!] \parallel \overset{*}{\mathsf{p}}[\![\ \mathsf{r} \bullet \rangle\!\rangle \circ; \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{p}[\![\ \mathsf{s}!\lambda; \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{s}[\![\ \mathsf{p}?\lambda\ ]\!] \xrightarrow{\mathsf{p} \longrightarrow \mathsf{s}} \\
\overset{*}{\mathsf{p}}[\![\ \mathsf{q} \bullet \rangle\!\rangle \circ\ ]\!] \parallel \overset{*}{\mathsf{p}}[\![\ \mathsf{r} \bullet \rangle\!\rangle \circ; \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{p}[\![\ \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{s}[\![\ \mathbf{0}\ ]\!] \xrightarrow{\mathsf{q} \bullet \rangle\!\rangle \circ \mathsf{p}} \\
\mathsf{p}[\![\ \mathbf{0}\ ]\!] \parallel \overset{*}{\mathsf{p}}[\![\ \mathsf{r} \bullet \rangle\!\rangle \circ; \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{q}[\![\ \mathbf{0}\ ]\!] \parallel \mathsf{s}[\![\ \mathbf{0}\ ]\!]
\end{array}$$

The last network is stuck, although delegation is well nested in this case. The problem arises from the fact that the second forward delegation creates a second avatar of $\overset{*}{\mathsf{p}}[\![\ P\ ]\!]$ *in the network, and this introduces an ambiguity for backward delegation. In the above computation, the backward delegation swaps the identities of* q *and* r*, assigning the wrong continuation to* q*. In this way, the second delegation from* p *to* r *cannot be closed anymore, since the code of* p *does not have the right form. So this network is deadlocked.*

If instead we had closed first the delegation r•⟫∘p*, the computation from the fourth line would have been:*

$$\begin{array}{c}
\overset{*}{\mathsf{p}}[\![\ \mathsf{q} \bullet \rangle\!\rangle \circ\ ]\!] \parallel \overset{*}{\mathsf{p}}[\![\ \mathsf{r} \bullet \rangle\!\rangle \circ; \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{p}[\![\ \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{s}[\![\ \mathbf{0}\ ]\!] \xrightarrow{\mathsf{r} \bullet \rangle\!\rangle \circ \mathsf{p}} \\
\overset{*}{\mathsf{p}}[\![\ \mathsf{q} \bullet \rangle\!\rangle \circ\ ]\!] \parallel \mathsf{p}[\![\ \bullet \rangle\!\rangle \circ \mathsf{p}\ ]\!] \parallel \mathsf{r}[\![\ \mathbf{0}\ ]\!] \parallel \mathsf{s}[\![\ \mathbf{0}\ ]\!] \xrightarrow{\mathsf{q} \bullet \rangle\!\rangle \circ \mathsf{p}} \\
\mathsf{p}[\![\ \mathbf{0}\ ]\!] \parallel \mathsf{q}[\![\ \mathbf{0}\ ]\!] \parallel \mathsf{r}[\![\ \mathbf{0}\ ]\!] \parallel \mathsf{s}[\![\ \mathbf{0}\ ]\!]
\end{array}$$

leading to a successful termination. Note that we could get around the ambiguity problem by numbering the different avatars of $\overset{*}{\mathsf{p}}[\![\, P \,]\!]$ *according to their order of creation, but this would make our calculus considerably more complex.*

Even in situations where deadlock is avoided, it can be desirable to enforce the guarantee that a principal knows what her deputy will do, e.g., she should not make choices on behalf of the principal. Such problems with deadlock can be avoided and trust assumptions enforced by the types in this section, which constrain networks.

4.1. Unambiguous global types and session pre-types

A *multiparty session* is a series of atomic interactions among participants [21, 22], which follows a predefined protocol specified by a *global type*. Global types are built from choices among communications with the same sender (*the choice leader*) and from forward/backward delegations, possibly using recursion. We write α^{p} when $\mathsf{sender}(\alpha) = \mathsf{p}$: this is useful to denote the choice leader in global types.

Definition 4.4 (Global types). Global types G *are defined by:*

$$\mathsf{G} \quad ::= \quad \boxplus_{i\in I}\, \alpha_i^{\mathsf{p}}; \mathsf{G}_i \;\mid\; \mathsf{p}\circ\langle\!\langle \bullet\mathsf{q}; \mathsf{G} \;\mid\; \mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p}; \mathsf{G} \;\mid\; \mu\mathbf{t}.\mathsf{G} \;\mid\; \mathbf{t} \;\mid\; \mathsf{End}$$

Sequential composition (;) has higher precedence than choice ($\boxplus$*). Recursion must be guarded and it is treated equi-recursively.*

In the figures of Section 2, as well as in the examples given earlier in this section, we used a more verbose syntax for global types, rendering $\mathsf{p}\Lambda\mathsf{q}$ as $\mathsf{p} \xrightarrow{\Lambda} \mathsf{q}$, as this seemed easier to understand in an informal description of the protocol.

Note that our syntax for global types is more liberal than that typically used in the literature, since we permit global choices among communications with distinct receiving participants.

To avoid non-determinism, we require all atomic communications in choices to be different, a condition formalised by the following definition.

Definition 4.5 (Non-ambiguous global type). *A choice* $\boxplus_{i\in I}\, \alpha_i^{\mathsf{p}}; \mathsf{G}_i$ *is ambiguous if* $\alpha_h^{\mathsf{p}} = \alpha_k^{\mathsf{p}}$ *for some* $h, k \in I$*. A global type is* non-ambiguous *if it does not contain ambiguous choices.*

In standard global types [22], where choices are among communications with the same receiver, non-ambiguity amounts to requiring that all labels be distinct.

Session types (also called *local types*, as opposed to global types) describe conversations from the viewpoint of session participants. They are used to type the processes implementing session participants [22]. We start by defining *session pre-types*, which are a superset of session types. Session types will be session pre-types which are projections of non-ambiguous global types. Session

pre-types are obtained from the syntax of processes by replacing external and internal choices with intersections and unions, X with $\mathbf{t}$ and $\mathbf{0}$ with End, with similar conventions.

Definition 4.6 (Session pre-types). Session pre-types *are defined by:*

$$\begin{array}{rcl} \mathsf{T} & ::= & \bigwedge_{i\in I} \pi_i;\mathsf{T}_i \mid \bigvee_{i\in I} \pi_i;\mathsf{T}_i \mid \mathsf{p}\circ\langle\!\langle\bullet;\mathsf{T} \mid \bullet\rangle\!\rangle\circ\mathsf{p};\mathsf{T} \\ & & \mid \circ\langle\!\langle\bullet\mathsf{q};\mathsf{T} \mid \mathsf{q}\bullet\rangle\!\rangle\circ;\mathsf{T} \mid \mu\mathbf{t}.\mathsf{T} \mid \mathbf{t} \mid \mathsf{End} \end{array}$$

As with global types and processes, session type recursion must be guarded and treated equi-recursively.

4.2. Projection of a global type onto participants

$$\begin{array}{l} \mathsf{part}(\boxplus_{i\in I}\, \alpha_i;\mathsf{G}_i) = \bigcup_{i\in I}(\mathsf{part}(\alpha_i) \cup \mathsf{part}(\mathsf{G}_i)) \\ \mathsf{part}(\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q};\mathsf{G}) = \mathsf{part}(\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p};\mathsf{G}) = \mathsf{part}(\mathsf{G}) \cup \{\mathsf{p},\mathsf{q}\} \\ \mathsf{part}(\mu\mathbf{t}.\mathsf{G}) = \mathsf{part}(\mathsf{G}) \\ \mathsf{part}(\mathbf{t}) = \mathsf{part}(\mathsf{End}) = \emptyset \end{array}$$

Figure 4: Participants of global types.

Session types are defined as projections of unambiguous global types onto their participants (when such a projection is well defined), where *the set of participants of* G, denoted $\mathsf{part}(\mathsf{G})$, is defined in Figure 4. Each session type represents the contribution of an individual participant to the session. The projection of a choice yields a union for the choice leader and intersections for all other participants.

The projection of global types is defined in terms of two projection operators, the *direct projection* "$\upharpoonright$", which projects a global type on participants when no delegation is underway, and the *delegation projection* "$\upharpoonright_i$" with $i = 1, 2$, which projects a global type after a delegation has started. The latter is parameterised by a pair of participants: the pair identifies the principal and the deputy of the delegation, while the subscript 1 or 2 identifies the participant on which the projection is performed. The delegation projection can only be performed on the two participants involved in the delegation. Both projection operators are partial and they have higher precedence than "$;$".

Projection employs a partial operator $\bigsqcap$, ensuring that, in a projection from a global choice, the choice leader makes the decision and all the other participants act accordingly. This is achieved by requiring that, for any participant except for the choice leader, the set of projections of the choice branches on that participant, say $\{\mathsf{T}_i \mid i \in I\}$, be *consistent* in the sense that the following partial operator, *the meet of a set of types* $\bigsqcap_{i\in I} \mathsf{T}_i$, is defined.

Definition 4.7 (Non-ambiguous input intersection). *An intersection of inputs* $\bigwedge_{i\in I} \mathsf{p}_i?\Lambda_i;\mathsf{T}_i$ *is* non-ambiguous *if its initial inputs are either all simple inputs or all connecting inputs, and moreover all such inputs are distinct, namely* $h, k \in I$ *and* $h \neq k$ *imply either* $\mathsf{p}_h \neq \mathsf{p}_k$ *or* $\Lambda_h \neq \Lambda_k$.

Definition 4.8 (Meet of types). *To define the meet of intersection types, we firstly define the meet between a non-ambiguous input intersection and an input type or an* End *type, as well as the meet between two* End *types:*

1. $(\bigwedge_{i\in I} \mathsf{p}_i?\Lambda_i;\mathsf{T}_i) \sqcap \mathsf{p}?\Lambda;\mathsf{T} = \bigwedge_{i\in I} \mathsf{p}_i?\Lambda_i;\mathsf{T}_i \ \wedge\ \mathsf{p}?\Lambda;\mathsf{T}$
   if the resulting intersection is not ambiguous
2. $(\bigwedge_{i\in I} \mathsf{p}_i?\Lambda_i;\mathsf{T}_i) \sqcap \mathsf{p}?\Lambda;\mathsf{T} = \bigwedge_{i\in I} \mathsf{p}_i?\Lambda_i;\mathsf{T}_i$
   if $\mathsf{p} = \mathsf{p}_j$ *and* $\Lambda = \Lambda_j$ *and* $\mathsf{T} = \mathsf{T}_j$ *for some* $j \in I$
3. $(\bigwedge_{i\in I} \mathsf{p}_i? \overset{\lambda_i}{\leftrightarrow};\mathsf{T}_i) \sqcap \mathsf{End} = \bigwedge_{i\in I} \mathsf{p}_i? \overset{\lambda_i}{\leftrightarrow};\mathsf{T}_i$
4. $\mathsf{End} \sqcap \mathsf{End} = \mathsf{End}$

To define the meet of two intersection types, we simply iterate the above definition on the members of one of the intersections. Thereby, meet can be extended to a set of types.

The above meet $\bigsqcap_{i\in I} \mathsf{T}_i$ checks that the T_i's are consistent behaviours and then combines them into a single session type. Intuitively, $\bigsqcap_{i\in I} \mathsf{T}_i$ is defined if the concerned participant receives a message that "notifies" her about the chosen T_i. If one of the T_i's is an intersection of simple inputs, then so must be all the other T_i's. Intersections of connecting inputs are also allowed to be combined with End. In this way, branches of a choice where a participant never acts can be merged with branches in which the participant performs a connecting input.

Note that we do not need to define the meet when the arguments are recursive types, since we consider recursion equi-recursively; hence we can unfold a recursion one step in order to expose actions guarded by the recursion. In other cases, meet is not defined, i.e., when unions or delegations are involved.

In the literature there are many definitions of the meet operator [22, 12, 8]. For example, in [12, 8] Clause 2 of Definition 4.8 is replaced by:

$$(\bigwedge_{i\in I} \mathsf{p}_i?\Lambda_i;\mathsf{T}_i) \sqcap \mathsf{p}?\Lambda;\mathsf{T} = \bigwedge_{i\in I\setminus\{j\}} \mathsf{p}_i?\Lambda_i;\mathsf{T}_i \ \wedge\ \mathsf{p}?\Lambda;(\mathsf{T}_j \sqcap \mathsf{T})$$
$$\text{if } \mathsf{p} = \mathsf{p}_j \text{ and } \Lambda = \Lambda_j \text{ and } \mathsf{T}_j \sqcap \mathsf{T} \text{ is defined for some } j \in I$$

Our definition of meet is a trade-off between expressiveness and simplicity. We believe that intra-session delegation could be easily adapted to other definitions of meet.

The direct projection, given in Figure 5, where $|I|$ denotes the cardinality of I, can be defined for all kinds of global types. The projection of a choice (with more than one branch) is a union of outputs for the choice leader, and it is otherwise computed as the meet of the projections of the branches. From the definition of meet we can see that the branches of a choice can differ for the presence of participants whose first communication is a connecting input.

The projection of a forward delegation from principal p to deputy q is an active forward delegation on p and a passive forward delegation on q, followed in both cases by the delegation projection of the remaining global type until the corresponding backward delegation is reached (the forward delegation is where the direct projection turns into the delegation projection). In the case that the projected participant is neither the principal nor the deputy, the forward delegation is ignored. Similarly, the backward delegation from deputy q to

$$(\mathsf{p}\Lambda\mathsf{q};\mathsf{G})\upharpoonright\mathsf{r} = \begin{cases}\mathsf{q}!\Lambda;\mathsf{G}\upharpoonright\mathsf{p} & \text{if } \mathsf{r}=\mathsf{p}\\ \mathsf{p}?\Lambda;\mathsf{G}\upharpoonright\mathsf{q} & \text{if } \mathsf{r}=\mathsf{q}\\ \mathsf{G}\upharpoonright\mathsf{r} & \text{if } \mathsf{r}\notin\{\mathsf{p},\mathsf{q}\}\end{cases}$$

$$(\boxplus_{i\in I}\,\alpha_i^{\mathsf{p}};\mathsf{G}_i)\upharpoonright\mathsf{r} = \begin{cases}\bigvee_{i\in I}(\alpha_i^{\mathsf{p}};\mathsf{G}_i)\upharpoonright\mathsf{r} & \text{if } \mathsf{r}=\mathsf{p}\\ \bigsqcap_{i\in I}(\alpha_i^{\mathsf{p}};\mathsf{G}_i)\upharpoonright\mathsf{r} & \text{otherwise}\end{cases} \qquad \text{where } |I|>1$$

$$(\mu\mathbf{t}.\mathsf{G})\upharpoonright\mathsf{r} = \begin{cases}\mathsf{G}\upharpoonright\mathsf{r} & \text{if } \mathbf{t} \text{ does not occur in } \mathsf{G}\\ \mu\mathbf{t}.\mathsf{G}\upharpoonright\mathsf{r} & \text{if } \mathsf{r}\in\mathsf{part}(\mathsf{G})\\ \mathsf{End} & \text{otherwise}\end{cases} \qquad \mathbf{t}\upharpoonright\mathsf{r}=\mathbf{t} \qquad \mathsf{End}\upharpoonright\mathsf{r}=\mathsf{End}$$

$$(\mathsf{p}\circ\langle\!\langle\bullet\,\mathsf{q};\mathsf{G})\upharpoonright\mathsf{r} = \begin{cases}\circ\langle\!\langle\bullet\,\mathsf{q};\mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q}) & \text{if } \mathsf{r}=\mathsf{p}\\ \mathsf{p}\circ\langle\!\langle\bullet;\mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{q}) & \text{if } \mathsf{r}=\mathsf{q}\\ \mathsf{G}\upharpoonright\mathsf{r} & \text{otherwise}\end{cases}$$

$$(\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p};\mathsf{G})\upharpoonright\mathsf{r} = \mathsf{G}\upharpoonright\mathsf{r} \qquad \text{if } \mathsf{r}\notin\{\mathsf{p},\mathsf{q}\}$$

Figure 5: Direct projection of global types onto participants.

principal p is ignored when projected on participants different from p and q. Notice that the direct projection of backward delegation is not defined for p and q, since this case is handled by the delegation projection.

The delegation projection, given in Figure 6, comes into play when a delegation is encountered. It is not defined for (proper) choices, nor for End, $\mu\mathbf{t}.\mathsf{G}$ and $\mathbf{t}$. Therefore, the portion of computation between a forward delegation from p to q and a backward delegation from q to p must consist of a sequence of atomic interactions. Moreover, these interactions should not involve q, since q is supposed to perform exclusively p's actions in this portion of computation. Furthermore, in the scope of her delegation the principal p should only be involved in communications, and not in further delegations. As hinted above, the delegation projections $\mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q})$ and $\mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{q})$ represent the projections of G on p and on q respectively, within the scope of a delegation from p to q.

The first two clauses of Figure 6 concern the delegation projection of communications. A communication involving the principal p (and not involving the deputy q, as just argued) projects to no action for the principal p and to an I/O action for the deputy q. A communication not involving the principal p nor the deputy q projects to no action for either p or q. The clauses on the third line in Figure 6 define the projection of backward delegation on one of the two involved participants. Notice that, as soon as the delegation ends, one gets back to the direct projection. The last line says that the projections on p and q ignore any forward/backward delegation between other participants. Notice that $(\mathsf{r}\circ\langle\!\langle\bullet\mathsf{s};\mathsf{G})\upharpoonright_i(\mathsf{p},\mathsf{q})$ and $(\mathsf{r}\bullet\rangle\!\rangle\circ\mathsf{s};\mathsf{G})\upharpoonright_i(\mathsf{p},\mathsf{q})$ are undefined for $i=1,2$ when

$$(\ r\Lambda s;\mathsf{G}\ )\upharpoonright_1(p,q) = \mathsf{G}\upharpoonright_1(p,q) \quad \text{if } r \neq q \text{ and } s \neq q$$

$$(\ r\Lambda s;\mathsf{G}\ )\upharpoonright_2(p,q) = \begin{cases} s!\Lambda;\mathsf{G}\upharpoonright_2(p,q) & \text{if } r = p \text{ and } s \neq q \\ r?\Lambda;\mathsf{G}\upharpoonright_2(p,q) & \text{if } s = p \text{ and } r \neq q \\ \mathsf{G}\upharpoonright_2(p,q) & \text{if } \{r,s\}\cap\{p,q\} = \emptyset \end{cases}$$

$$(q\bullet\!\!\gg\!\circ p;\mathsf{G})\upharpoonright_1(p,q) = q\bullet\!\!\gg\!\circ;\mathsf{G}\upharpoonright p \qquad (q\bullet\!\!\gg\!\circ p;\mathsf{G})\upharpoonright_2(p,q) = \bullet\!\!\gg\!\circ p;\mathsf{G}\upharpoonright q$$

$$(r\circ\!\!\ll\!\bullet s;\mathsf{G})\upharpoonright_i(p,q) = (r\bullet\!\!\gg\!\circ s;\mathsf{G})\upharpoonright_i(p,q) = \mathsf{G}\upharpoonright_i(p,q) \quad i = 1,2 \text{ if } \{r,s\}\cap\{p,q\} = \emptyset$$

Figure 6: Delegation projection of global types onto participants.

$\{r,s\}\cap\{p,q\} \neq \emptyset$, since we want to forbid a participant to be involved in more than one delegation at a time.

Example 4.9. *Let us see our notion of projection at work on a few examples.*

1. *Let* $\mathsf{G} = p\lambda_1 q \boxplus p \overset{\lambda_2}{\leftrightarrow} q$. *Then* $\mathsf{G}\upharpoonright q$ *is not defined, since the intersection* $p?\lambda_1 \wedge p? \overset{\lambda_2}{\leftrightarrow}$ *between a simple and a connecting input is ambiguous.*

2. *Let* $\mathsf{G} = \mu\mathbf{t}.(p\lambda_1 q;\mathbf{t} \boxplus p\lambda_2 q)$ *and* $\mathsf{G}' = p\lambda_1 q;\mathsf{G}\boxplus p\lambda_2 q$. *Then* $\mathsf{G}\upharpoonright q = \mu\mathbf{t}.(p?\lambda_1;\mathbf{t}\wedge p?\lambda_2)$ *and* $\mathsf{G}'\upharpoonright q = p?\lambda_1;\mathsf{G}\upharpoonright q \wedge p?\lambda_2$. *Notice that* G' *is the unfolding of* G.

3. *Let* $\mathsf{G} = \mu\mathbf{t}.p\lambda q;\mathbf{t}$ *and* $\mathsf{G}' = \mu\mathbf{t}.p\lambda q;p\lambda q;\mathbf{t}$. *Then* $\mathsf{G}\upharpoonright q = \mu\mathbf{t}.p?\lambda;\mathbf{t}$ *and* $\mathsf{G}'\upharpoonright q = \mu\mathbf{t}.p?\lambda;p?\lambda;\mathbf{t}$. *Notice that* G *and* G' *generate the same tree.*

4. *Let* $\mathsf{G} = p\circ\!\!\ll\!\bullet q;p\lambda_1 r;s\lambda_2 p;q\bullet\!\!\gg\!\circ p;q\lambda_3 p$. *Then*

   - $\mathsf{G}\upharpoonright q = p\circ\!\!\ll\!\bullet;r!\lambda_1;s?\lambda_2;\bullet\!\!\gg\!\circ p;p!\lambda_3$, *where* q *communicates with* r,s *as deputy of* p,
   - $\mathsf{G}\upharpoonright p = \circ\!\!\ll\!\bullet q;q\bullet\!\!\gg\!\circ;q?\lambda_3$, *where* p *is inactive during delegation,*
   - $\mathsf{G}\upharpoonright r = p?\lambda_1$ *and* $\mathsf{G}\upharpoonright s = p!\lambda_2$, *where* r *and* s *believe they are communicating with* p.

The soundness of the direct projection requires that global types generating the same tree are projected into session types generating the same tree. Notice that the syntactic differences between global types generating the same tree can be:

- fold/unfold of recursion, like in Example 4.9(2);
- different occurrences of μ bindings and bound variables, like in Example 4.9(3).

The projection of global types with the same tree yields session types with the same tree, since it preserves the occurrences of μ bindings and bound variables. The formal proof is given in Appendix A.

Definition 4.10 (Projectable global type). *A global type* G *is* projectable *if for all* $\mathsf{r} \in \mathsf{part}(\mathsf{G})$ *either the direct projection* $\mathsf{G} \upharpoonright \mathsf{r}$ *is defined, or there exists* $\mathsf{s} \in \mathsf{part}(\mathsf{G})$ *such that either* $\mathsf{G} \upharpoonright_1(\mathsf{r},\mathsf{s})$ *or* $\mathsf{G} \upharpoonright_2(\mathsf{s},\mathsf{r})$ *is defined.*

In the following we will consider only *non-ambiguous and projectable global types.* Projectability assures the property of well-delegation, defined as follows.

Definition 4.11 (Well-delegated global type). *A global type is* well delegated *if:*

1. *each occurrence of* $\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q}$ *is followed by an occurrence of* $\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p}$*;*
2. *no atomic interaction involving* q *occurs between* $\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q}$ *and* $\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p}$*;*
3. *no choice occurs between* $\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q}$ *and* $\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p}$*;*
4. *no delegation involving* p *occurs between* $\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q}$ *and* $\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p}$*.*

Let us briefly comment on the above definition. Condition 1 simply requires that every forward delegation be matched by the corresponding backward delegation. Conditions 2–4 impose some restrictions on the behaviour that can be delegated: while it should be clear why this behaviour should not involve the deputy q (Condition 2), the fact that the principal p should not be involved in further delegations while being personificated by q, whether actively or passively (Condition 4), is required for deadlock-freedom, as shown by Example 4.1 and Example 4.3. Finally, the absence of choices in the delegated behaviour (Condition 3) simplifies the technical treatment and may be justified by the intention of the principal to retain some control on the deputy.

Lemma 4.12. *Each projectable global type is well delegated.*

Proof Let G be a projectable global type. We show that G satisfies conditions $1 \div 4$ of well-delegation. Let G' follow an open delegation $\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q}$ in G, i.e., $\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q};\mathsf{G}'$ is a subtype of G.

(1). By definition of delegation projection, $\mathsf{End} \upharpoonright_i (\mathsf{p},\mathsf{q})$ and $\mathbf{t} \upharpoonright_i (\mathsf{p},\mathsf{q})$ are undefined with $i = 1,2$. Note that the rightmost operands of a sequential composition can only be $\mathbf{t}$ and End. If the delegation projections $\mathsf{G}' \upharpoonright_i (\mathsf{p},\mathsf{q})$ with $i = 1,2$ are defined, then before reaching the rightmost operands of sequential compositions in G' we have to find a backward delegation $\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p}$ so that we go back to the direct projections on p and q, which are defined for End and $\mathbf{t}$.

(2). Since the delegation projections $\upharpoonright_i (\mathsf{p},\mathsf{q})$ with $i = 1,2$ are not defined for atomic interactions involving q (see side conditions of the clauses in Figure 6), atomic interactions involving q in G' cannot be before the backward delegation $\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p}$.

(3). Delegation projections $\upharpoonright_i (\mathsf{p},\mathsf{q})$ with $i = 1,2$ are not defined for choices. So, as for the previous cases, a (non trivial) choice in G' cannot be before the backward delegation $\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p}$.

(4). We conclude as in the previous cases, noticing that the delegation projections $\upharpoonright_i (\mathsf{p},\mathsf{q})$ with $i = 1,2$ are not defined for p or q (see last line of Figure 6).

Notice that well-delegation only concerns delegation projection, so it does not assure projectability as shown by Example 4.9(1). Projectability does not assure non-ambiguity, since for example $\mathsf{p}\Lambda\mathsf{q} \boxplus \mathsf{p}\Lambda\mathsf{q}$ is ambiguous and projectable.

We end this section by defining session types.

Definition 4.13 (Session types). *A session pre-type* T *is a* session type *if* $\mathsf{T} = \mathsf{G} \upharpoonright \mathsf{p}$ *or* $\mathsf{T} = \mathsf{G} \upharpoonright_1(\mathsf{p},\mathsf{q})$ *or* $\mathsf{T} = \mathsf{G} \upharpoonright_2(\mathsf{p},\mathsf{q})$ *for some non-ambiguous and projectable global type* G *and some participants* $\mathsf{p},\mathsf{q} \in \mathsf{part}(\mathsf{G})$.

When typing networks we consider only non-ambiguous and projectable global types.

5. Guaranteeing Progress for Networks with Internal Delegation

In this section, we define our type system and state the main strong progress result that the type system guarantees. As usual, we assume an environment Γ that associates process variables with session types: $\Gamma ::= \emptyset \mid \Gamma, X : \mathsf{T}$. *Typing judgements* assign session types to processes and global types to networks, respectively. Then they have the form $\Gamma \vdash P : \mathsf{T}$ and $\vdash \mathbb{N} : \mathsf{G}$. The empty environment in the typing of networks assures that all processes are closed.

In order to define our type system, we require the subtyping rules in Figure 7, where the double line indicates that the rules are interpreted *coinductively* [35] (Chapter 21). Subtyping takes into account the rules for intersection and union. Rule [Sub-In-Skip] reflects the fact that unused connecting inputs can be present in the processes without causing problems. In rule [Sub-Del], we denote by δ a delegation action.

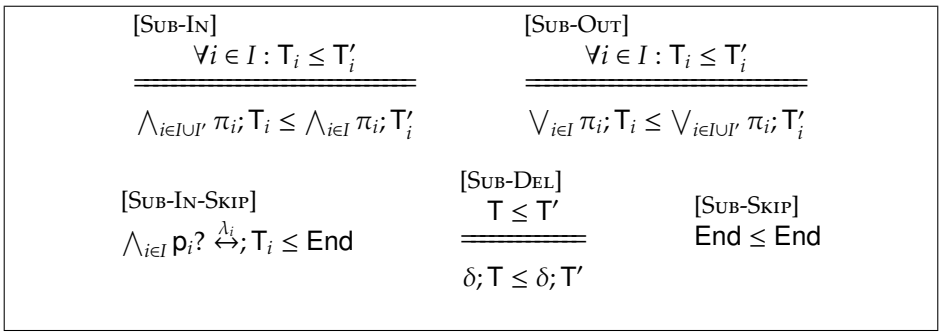

Figure 7: Subtyping rules.

Process typing exploits the correspondence between external choices and intersections, internal choices and unions. The typing rules for processes and networks are given in Figure 8. Since $\pi_h \neq \pi_k$ with $h,k \in I$ for intersections/unions in session types, this is true also for choices in processes. The typing rules for external choice and internal choice, together with the subtyping rules for intersections and unions, assure that choice leaders can freely select both receivers and messages.

Rule [T-NET] is the only rule for typing networks: it requires that the types of all processes be subtypes of the projections of a unique global type. The condition $\mathsf{part}(\mathsf{G}) \subseteq \{\mathsf{p}_i \mid i \in I\} \cup \{\mathsf{q}_i \mid i \in I\} \cup \{\mathsf{r}_j \mid j \in J\}$ ensures all participants of G appear in the network. Note that this condition permits additional participants that do not appear in the global type, allowing the typing of sessions containing $\mathsf{p}[\![\, \mathbf{0} \,]\!]$ for fresh p, which may appear when a participant has already completed her actions in a session. This property is required to guarantee invariance of types under structural equivalence of networks. Furthermore, we require all participants in the network to be distinct, to avoid multiple processes assuming the same role in a network (this is essential for linearity of sessions). The rule [T-NET] assures that:

- the principals partaking in a delegation (i.e. the participants p_i with $i \in I$) are frozen and their first interaction is the end of the delegation;
- the deputies partaking in a delegation (i.e. the participants q_i with $i \in I$) have the behaviours required by the global type for the corresponding principals (recall that, in the labelled transition system, the deputy takes on the name of the principal);
- all other participants (i.e. the participants r_j with $j \in J$) follow the prescription of the global type.

Notably, in the conclusion of rule [T-NET] the deputies' original names do not appear, but they must appear in the initial backward delegation action of the corresponding frozen principals. This is assured by the condition $\vdash P_i : \mathsf{q}_i \bullet \rangle\!\rangle \circ ; \mathsf{T}_i$, where P_i is the process of the frozen principal p_i with deputy q_i for $i \in I$.

$$\frac{\Gamma \vdash P_i : \mathsf{T}_i \ (i \in I)}{\Gamma \vdash \Sigma_{i\in I}\pi_i; P_i : \bigwedge_{i\in I} \pi_i; \mathsf{T}_i}\ [\text{T-ExtCh}] \qquad \frac{\Gamma \vdash P_i : \mathsf{T}_i \ (i \in I)}{\Gamma \vdash \oplus_{i\in I}\pi_i; P_i : \bigvee_{i\in I} \pi_i; \mathsf{T}_i}\ [\text{T-IntCh}]$$

$$\Gamma, X : \mathbf{t} \vdash X : \mathbf{t}\ \ [\text{T-Var}] \qquad \frac{\Gamma, X : \mathbf{t} \vdash P : \mathsf{T}}{\Gamma \vdash \mu X.P : \mu \mathbf{t}.\mathsf{T}}\ [\text{T-Rec}]$$

$$\frac{\Gamma \vdash P : \mathsf{T}}{\Gamma \vdash \delta; P : \delta; \mathsf{T}}\ [\text{T-Del}] \qquad \Gamma \vdash \mathbf{0} : \mathsf{End}\ \ [\text{T-}\mathbf{0}]$$

$$\frac{\begin{array}{lll} \vdash P_i : \mathsf{q}_i \bullet \rangle\!\rangle \circ ; \mathsf{T}_i & \mathsf{q}_i \bullet \rangle\!\rangle \circ ; \mathsf{T}_i \leq \mathsf{G} \upharpoonright_1 (\mathsf{p}_i, \mathsf{q}_i) & (i \in I) \\ \vdash Q_i : \mathsf{S}_i & \mathsf{S}_i \leq \mathsf{G} \upharpoonright_2 (\mathsf{p}_i, \mathsf{q}_i) & (i \in I) \\ \vdash R_j : \mathsf{U}_j & \mathsf{U}_j \leq \mathsf{G} \upharpoonright \mathsf{r}_j & (j \in J) \end{array} \quad \mathsf{part}(\mathsf{G}) \subseteq \{\mathsf{p}_i \mid i \in I\} \cup \{\mathsf{q}_i \mid i \in I\} \cup \{\mathsf{r}_j \mid j \in J\} \quad \text{all participants distinct}}{\vdash \Pi_{i\in I} \overset{*}{\mathsf{p}_i}[\![\, P_i \,]\!] \parallel \Pi_{i\in I}\, \mathsf{p}_i[\![\, Q_i \,]\!] \parallel \Pi_{j\in J}\, \mathsf{r}_j[\![\, R_j \,]\!] : \mathsf{G}}\ [\text{T-Net}]$$

Figure 8: Typing rules for processes and networks.

Example 5.1. *The following are examples of typed networks with different delegation patterns. The first does not require connecting inputs but the third does. The only difference between these examples is to whom a message is sent during the delegated part of the session. Recall that sequential composition has precedence over choices.*

1. *Global type* $\mathsf{G} = \mathsf{p}\lambda_1\mathsf{q};\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q};\mathsf{p}\lambda_2\mathsf{r};\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p} \boxplus \mathsf{p}\lambda_1\mathsf{r};\mathsf{p}\circ\langle\!\langle\bullet\mathsf{r};\mathsf{p}\lambda_2\mathsf{q};\mathsf{r}\bullet\rangle\!\rangle\circ\mathsf{p}$ *can be used to type the following network.*

$$\begin{array}{ll} & \mathsf{p}[\![\ \mathsf{q}!\lambda_1;\circ\langle\!\langle\bullet\mathsf{q};\mathsf{q}\bullet\rangle\!\rangle\circ \oplus \mathsf{r}!\lambda_1;\circ\langle\!\langle\bullet\mathsf{r};\mathsf{r}\bullet\rangle\!\rangle\circ\ ]\!] \\ \| & \mathsf{q}[\![\ \mathsf{p}?\lambda_1;\mathsf{p}\circ\langle\!\langle\bullet;\mathsf{r}!\lambda_2;\bullet\rangle\!\rangle\circ\mathsf{p} + \mathsf{p}?\lambda_2\ ]\!] \\ \| & \mathsf{r}[\![\ \mathsf{p}?\lambda_1;\mathsf{p}\circ\langle\!\langle\bullet;\mathsf{q}!\lambda_2;\bullet\rangle\!\rangle\circ\mathsf{p} + \mathsf{p}?\lambda_2\ ]\!] \end{array}$$

   Notice in the above network, one of the branches of each participant will trigger in every execution.

2. *The type* G *in the example above can be used to type also the above network modified such that participant* p *has process* $P = \mathsf{q}!\lambda_1;\circ\langle\!\langle\bullet\mathsf{q};\mathsf{q}\bullet\rangle\!\rangle\circ$ *since* $\vdash P : \mathsf{T}$, *where* $\mathsf{T} = \mathsf{q}!\lambda_1;\circ\langle\!\langle\bullet\mathsf{q};\mathsf{q}\bullet\rangle\!\rangle\circ \leq \mathsf{q}!\lambda_1;\circ\langle\!\langle\bullet\mathsf{q};\mathsf{q}\bullet\rangle\!\rangle\circ \vee \mathsf{r}!\lambda_1;\circ\langle\!\langle\bullet\mathsf{r};\mathsf{r}\bullet\rangle\!\rangle\circ = \mathsf{G}\upharpoonright\mathsf{p}$.

3. *In contrast to the above, connecting inputs are necessary for the existence of a network with type* $\mathsf{G} = \mathsf{p} \overset{\lambda_1}{\leftrightarrow} \mathsf{q};\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q};\mathsf{p}\lambda_2\mathsf{s};\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p} \boxplus \mathsf{p} \overset{\lambda_1}{\leftrightarrow} \mathsf{r};\mathsf{p}\circ\langle\!\langle\bullet\mathsf{r};\mathsf{p}\lambda_2\mathsf{s};\mathsf{r}\bullet\rangle\!\rangle\circ\mathsf{p}$. *Observe the use of connecting inputs, which allows one of the roles* q *or* r *to hang forever, without causing deadlock. Thus the following network is typable.*

$$\begin{array}{c} \mathsf{p}[\![\ \mathsf{q}! \overset{\lambda_1}{\leftrightarrow};\circ\langle\!\langle\bullet\mathsf{q};\mathsf{q}\bullet\rangle\!\rangle\circ \oplus \mathsf{r}! \overset{\lambda_1}{\leftrightarrow};\circ\langle\!\langle\bullet\mathsf{r};\mathsf{r}\bullet\rangle\!\rangle\circ\ ]\!] \| \mathsf{q}[\![\ \mathsf{p}? \overset{\lambda_1}{\leftrightarrow};\mathsf{p}\circ\langle\!\langle\bullet;\mathsf{s}!\lambda_2;\bullet\rangle\!\rangle\circ\mathsf{p}\ ]\!] \\ \| \mathsf{r}[\![\ \mathsf{p}? \overset{\lambda_1}{\leftrightarrow};\mathsf{p}\circ\langle\!\langle\bullet;\mathsf{s}!\lambda_2;\bullet\rangle\!\rangle\circ\mathsf{p}\ ]\!] \| \mathsf{s}[\![\ \mathsf{p}?\lambda_2\ ]\!] \end{array}$$

   When typing the above observe $\mathsf{G}\upharpoonright\mathsf{q} = \mathsf{p}? \overset{\lambda_1}{\leftrightarrow};\mathsf{p}\circ\langle\!\langle\bullet\mathsf{s}!\lambda_2;\bullet\rangle\!\rangle\circ\mathsf{p} \sqcap \mathsf{End}$, *which, by Definition 4.8 Clause 3, is* $\mathsf{p}? \overset{\lambda_1}{\leftrightarrow};\mathsf{p}\circ\langle\!\langle\bullet\mathsf{s}!\lambda_2;\bullet\rangle\!\rangle\circ\mathsf{p}$.

4. *A principal may delegate to a deputy infinitely often, as long as each delegation is opened and closed within a different unfolding of a recursion. For example* $\mathsf{G} = \mu\mathbf{t}.\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q};\mathsf{r}\lambda\mathsf{p};\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p};\mathbf{t}$ *is a type for the following network.*

$$\mathsf{p}[\![\ \mu X.\circ\langle\!\langle\bullet\mathsf{q};\mathsf{q}\bullet\rangle\!\rangle\circ;X\ ]\!] \| \mathsf{q}[\![\ \mu Y.\mathsf{p}\circ\langle\!\langle\bullet;\mathsf{r}?\lambda;\bullet\rangle\!\rangle\circ\mathsf{p};Y\ ]\!] \| \mathsf{r}[\![\ \mu Z.\mathsf{p}!\lambda;Z\ ]\!]$$

5.1. Main result: progress

We are now in a position to state the main result of the paper. Well-typed networks are guaranteed to satisfy a strong notion of progress, where there are no stuck actions — inputs, outputs, or delegations — except when an input is connecting. Specifically we have the following:

- Every participant offering a choice of outputs may be able to choose to perform **any** of those outputs;
- Every participant offering a choice of simple inputs may be asked to perform **one** of those inputs;

- Every principal may find the deputy and get back the delegation;
- Every deputy may find the principal and return back the delegation.

The progress theorem can be stated as follows.

Theorem 5.2 (Progress). *The following hold for well-typed networks* $\mathbb{N}$*:*

1. *If* $\mathbb{N} = \mathsf{p}[\![\oplus_{i\in I} \mathsf{q}_i ! \Lambda_i ; P_i]\!] \parallel \mathbb{N}_0$*, then* $\mathbb{N} \xrightarrow{\overrightarrow{\phi}\, \mathsf{p}\Lambda_i\mathsf{q}_i} \mathbb{N}'$ *for some* $\overrightarrow{\phi}$ *and for all* $i \in I$*.*
2. *If* $\mathbb{N} = \mathsf{p}[\![\Sigma_{i\in I} \mathsf{q}_i ? \lambda_i ; P_i]\!] \parallel \mathbb{N}_0$*, then* $\mathbb{N} \xrightarrow{\overrightarrow{\phi}\, \mathsf{q}_i\lambda_i\mathsf{p}} \mathbb{N}'$ *for some* $\overrightarrow{\phi}$ *and for some* $i \in I$*.*
3. *If* $\mathbb{N} = \mathsf{p}[\![\circ\langle\!\langle \bullet \mathsf{q}; P]\!] \parallel \mathbb{N}_0$*, then* $\mathbb{N} \xrightarrow{\overrightarrow{\phi}\, \mathsf{p}\circ\langle\!\langle \bullet \mathsf{q}\, \overrightarrow{\phi'}\, \mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p}} \mathbb{N}'$ *for some* $\overrightarrow{\phi}$ *and* $\overrightarrow{\phi'}$*.*
4. *If* $\mathbb{N} = \mathsf{q}[\![\mathsf{p}\circ\langle\!\langle \bullet ; Q]\!] \parallel \mathbb{N}_0$*, then* $\mathbb{N} \xrightarrow{\overrightarrow{\phi}\, \mathsf{p}\circ\langle\!\langle \bullet \mathsf{q}\, \overrightarrow{\phi'}\, \mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p}} \mathbb{N}'$ *for some* $\overrightarrow{\phi}$ *and* $\overrightarrow{\phi'}$*.*

Notice that, in the above theorem, the labels are sequences of atomic interactions, each of which results either from a pair of matching I/O actions, or from a pair of matching delegation actions. The first two clauses in the theorem above state that any participant in a typed network with a choice of actions available may perform one of those actions as a communication with the corresponding party. Notice that in the internal choice of outputs in Clause 1 above, the choice leader can freely choose any of its branches; as formalised by the universal quantification over branches. In contrast, in the external choice of inputs in Clause 2 above, the receiver responds to the decision made by the choice leader, hence cannot freely choose which branch is taken; as formalised by the existential quantification over branches. Observe that these guarantees, which distinguish internal and external choice, are enforced by the type system, rather than by the operational rules for external and internal choice.

Clause 3 of the progress theorem above states that, if a principal in a typed network is ready to send a forward delegation, then the forward delegation may happen and subsequently the corresponding backward delegation may also happen. Similarly, Clause 4 states that, if a deputy in a typed network is ready to receive a forward delegation, then the forward delegation may happen and subsequently the corresponding backward delegation may also happen.

Observe that the analogous of Clause 2 does not hold for connecting inputs, since there are well-typed networks $\mathbb{N} = \mathsf{p}[\![\Sigma_{i\in I} \mathsf{q}_i ? \overset{\lambda_i}{\leftrightarrow} ; P_i]\!] \parallel \mathbb{N}_0$ such that $\mathbb{N} \xrightarrow{\overrightarrow{\phi}} \mathbb{N}'$ implies $\mathbb{N}' = \mathsf{p}[\![\Sigma_{i\in I} \mathsf{q}_i ? \overset{\lambda_i}{\leftrightarrow} ; P_i]\!] \parallel \mathbb{N}'_0$. Indeed, if no q_i for $i \in I$ sends messages to p in $\mathbb{N}$, then $\mathbb{N}$ has the same global type as $\mathsf{p}[\![\mathbf{0}]\!] \parallel \mathbb{N}_0$ using the subtyping rule [Sub-In-Skip]. Take for example the network of Example 5.1.3. After the connecting communication $\mathsf{p} \overset{\lambda_1}{\leftrightarrow} \mathsf{q}$ we would have

$$\mathsf{p}[\![\circ\langle\!\langle \bullet \mathsf{q}; \mathsf{q}\bullet\rangle\!\rangle\circ]\!] \parallel \mathsf{q}[\![\mathsf{p}\circ\langle\!\langle \bullet ; \mathsf{s}!\lambda_2 ; \bullet\rangle\!\rangle\circ\mathsf{p}]\!] \parallel \mathsf{r}[\![\mathsf{p}? \overset{\lambda_1}{\leftrightarrow} ; \mathsf{p}\circ\langle\!\langle \bullet ; \mathsf{s}!\lambda_2 ; \bullet\rangle\!\rangle\circ\mathsf{p}]\!] \parallel \mathsf{s}[\![\mathsf{p}?\lambda_2]\!]$$

where neither p nor her deputy q can send the message $\overset{\lambda_1}{\leftrightarrow}$ to r.

The notion of progress established in Theorem 5.2 is closely related to the formulation of lock freedom in [34], which does not rely on additional fairness assumptions. To get a stronger progress property in which the communications/delegations *must* happen rather than *may* happen, we would require additional assumptions. For example the network

$$\mathbb{N} = \mathsf{p}[\![\, \mathsf{q}!\Lambda \,]\!] \parallel \mathsf{q}[\![\, \mathsf{p}?\Lambda \,]\!] \parallel \mathsf{r}[\![\, \mu X.\mathsf{s}!\Lambda'; X \,]\!] \parallel \mathsf{s}[\![\, \mu Y.\mathsf{r}?\Lambda'; Y \,]\!]$$

typed by the global type $\mathsf{p}\Lambda\mathsf{q}; \mu \mathbf{t}.\mathsf{r}\Lambda'\mathsf{s}; \mathbf{t}$ has the infinite computation

$$\mathbb{N} \xrightarrow{\mathsf{r}\Lambda'\mathsf{s}} \mathbb{N} \xrightarrow{\mathsf{r}\Lambda'\mathsf{s}} \mathbb{N} \xrightarrow{\mathsf{r}\Lambda'\mathsf{s}} \cdots$$

in which p and q never communicate. In contrast, if we assume fairness of interactions in networks, then p and q must communicate in all infinite computations.

The next section provides the key lemmas for establishing Theorem 5.2.

6. Technical Results Required to Establish Progress

We present intermediate results required to establish progress. We employ lemmas for inferring the structure of a type from the structure of a well-typed network (Lemma 6.1), for inferring the structure of a network from its type (Lemma 6.2), and the structure of a network from labelled transitions (Lemma 6.3). Such lemmas are employed to discover the forms in each case of the key theorems *subject reduction* (Theorem 6.6) and *session fidelity* (Theorem 6.7).

As usual, subject reduction ensures that a well-typed network always reduces to a well-typed network. The novel cases in this work are those that involve interactions where at least one participant is a deputy acting on behalf of a principal, in which case Lemma 6.4 ensures the form of types reflects the restrictions on what actions a deputy can perform. Session fidelity refines subject reduction, by allowing us to be more precise about the form of the global types before and after a specific atomic interaction. The proof of progress employs session fidelity. A novelty of the proof of progress is that, in the case of simple inputs, we must check that the receiving participant appears on every control flow path and the first action encountered for that participant is a simple input (Lemma 6.8). Note this subtlety arises due to the style of definition of session projection using the meet operator for combining consistent inputs (Definition 4.8).

6.1. Standard technical lemmas about the type system

We start by proving the classical lemmas for inversion and canonical forms.

Lemma 6.1 (Inversion Lemma). *The following hold:*

1. *If* $\Gamma \vdash \Sigma_{i\in I}\pi_i; P_i : \mathsf{T}$, *then* $\mathsf{T} = \bigwedge_{i\in I} \pi_i; \mathsf{T}_i$ *and* $\Gamma \vdash P_i : \mathsf{T}_i$ *for* $i \in I$.
2. *If* $\Gamma \vdash \oplus_{i\in I}\pi_i; P_i : \mathsf{T}$, *then* $\mathsf{T} = \bigvee_{i\in I} \pi_i; \mathsf{T}_i$ *and* $\Gamma \vdash P_i : \mathsf{T}_i$ *for* $i \in I$.

3. *If* $\Gamma \vdash \delta; P : \mathsf{T}$, *then* $\mathsf{T} = \delta; \mathsf{T}'$ *and* $\Gamma \vdash P : \mathsf{T}'$.

4. *If* $\Gamma \vdash \mu X.P : \mathsf{T}$, *then* $\mathsf{T} = \mu \mathbf{t}.\mathsf{T}'$ *and* $\Gamma, X : \mathbf{t} \vdash P : \mathsf{T}'$.

5. *If* $\Gamma \vdash X : \mathsf{T}$, *then* $\mathsf{T} = \mathbf{t}$ *and* $\Gamma = \Gamma', X : \mathbf{t}$.

6. *If* $\Gamma \vdash \mathbf{0} : \mathsf{T}$, *then* $\mathsf{T} = \mathsf{End}$.

7. *If* $\vdash \Pi_{i \in I}^{\ast} \mathsf{p}_i[\![P_i]\!] \parallel \Pi_{i \in I} \mathsf{p}_i[\![Q_i]\!] \parallel \Pi_{j \in J} \mathsf{r}_j[\![R_j]\!] : \mathsf{G}$, *then*

$$\mathsf{part}(\mathsf{G}) \subseteq \{\mathsf{p}_i \mid i \in I\} \cup \{\mathsf{q}_i \mid i \in I\} \cup \{\mathsf{r}_j \mid j \in J\} \text{ and}$$

   (a) $\vdash P_i : \mathsf{q}_i \bullet \rangle\!\rangle \circ; \mathsf{T}_i$ *where* $\mathsf{q}_i \bullet \rangle\!\rangle \circ; \mathsf{T}_i \leq \mathsf{G} \upharpoonright_1 (\mathsf{p}_i, \mathsf{q}_i)$ *for* $i \in I$;
   (b) $\vdash Q_i : \mathsf{S}_i$ *where* $\mathsf{S}_i \leq \mathsf{G} \upharpoonright_2 (\mathsf{p}_i, \mathsf{q}_i)$ *for* $i \in I$;
   (c) $\vdash R_j : \mathsf{U}_j$ *where* $\mathsf{U}_j \leq \mathsf{G} \upharpoonright \mathsf{r}_j$ *for* $j \in J$.

Lemma 6.2 (Canonical Form Lemma). *The following hold:*

1. *If* $\Gamma \vdash P : \bigwedge_{i \in I} \pi_i; \mathsf{T}_i$ *then* $P = \Sigma_{i \in I} \pi_i; P_i$ *with* $\Gamma \vdash P_i : \mathsf{T}_i$ *for* $i \in I$.

2. *If* $\Gamma \vdash P : \bigvee_{i \in I} \pi_i; \mathsf{T}_i$ *then* $P = \oplus_{i \in I} \pi_i; P_i$ *with* $\Gamma \vdash P_i : \mathsf{T}_i$ *for* $i \in I$.

3. *If* $\Gamma \vdash P : \delta; \mathsf{T}$ *then* $P = \delta; Q$ *with* $\Gamma \vdash Q : \mathsf{T}$.

4. *If* $\Gamma \vdash P : \mu \mathbf{t}.\mathsf{T}$, *then* $P = \mu X.Q$ *and* $\Gamma, X : \mathbf{t} \vdash Q : \mathsf{T}$.

5. *If* $\Gamma \vdash P : \mathbf{t}$, *then* $P = X$ *and* $\Gamma = \Gamma', X : \mathbf{t}$.

6. *If* $\Gamma \vdash P : \mathsf{End}$, *then* $P = \mathbf{0}$.

7. *If* $\vdash \mathbb{N} : \mathsf{G}$ *and* $\mathsf{part}(\mathsf{G}) = \{\mathsf{p}_i \mid i \in I\} \cup \{\mathsf{q}_i \mid i \in I\} \cup \{\mathsf{r}_j \mid j \in J\}$ *and* $\mathsf{G} \upharpoonright_1 (\mathsf{p}_i, \mathsf{q}_i)$, $\mathsf{G} \upharpoonright_2 (\mathsf{p}_i, \mathsf{q}_i)$, $\mathsf{G} \upharpoonright \mathsf{r}_j$ *are defined for* $i \in I$ *and* $j \in J$, *then*

$$\mathbb{N} = \Pi_{i \in I}^{\ast} \mathsf{p}_i[\![P_i]\!] \parallel \Pi_{i \in I} \mathsf{p}_i[\![Q_i]\!] \parallel \Pi_{j \in J} \mathsf{r}_j[\![R_j]\!] \text{ and}$$

   (a) $\vdash P_i : \mathsf{q}_i \bullet \rangle\!\rangle \circ; \mathsf{T}_i$ *where* $\mathsf{q}_i \bullet \rangle\!\rangle \circ; \mathsf{T}_i \leq \mathsf{G} \upharpoonright_1 (\mathsf{p}_i, \mathsf{q}_i)$ *for* $i \in I$;
   (b) $\vdash Q_i : \mathsf{S}_i$ *where* $\mathsf{S}_i \leq \mathsf{G} \upharpoonright_2 (\mathsf{p}_i, \mathsf{q}_i)$ *for* $i \in I$;
   (c) $\vdash R_j : \mathsf{U}_j$ *where* $\mathsf{U}_j \leq \mathsf{G} \upharpoonright \mathsf{r}_j$ *for* $j \in J$.

The following lemma allows us to recover the shapes of processes and networks from their labelled transitions.

Lemma 6.3 (Synthesis Lemma). *The following hold:*

1. *If* $P \xrightarrow{\mathsf{p}?\Lambda} P'$, *then* $P = \Sigma_{i \in I} \pi_i; P_i$, *where* $\pi_j = \mathsf{p}?\Lambda$ *and* $P_j = P'$ *for some* $j \in I$.

2. *If* $P \xrightarrow{\mathsf{p}!\Lambda} P'$, *then* $P = \oplus_{i \in I} \pi_i; P_i$, *where* $\pi_j = \mathsf{p}!\Lambda$ *and* $P_j = P'$ *for some* $j \in I$.

3. *If* $\mathbb{N} \xrightarrow{\mathsf{p}\Lambda\mathsf{q}} \mathbb{N}'$, *then* $\mathbb{N} = \mathsf{p}[\![\, P \,]\!] \parallel \mathsf{q}[\![\, Q \,]\!] \parallel \mathbb{N}_0$ *and* $P \xrightarrow{\mathsf{q}!\Lambda} P'$ *and* $Q \xrightarrow{\mathsf{p}?\Lambda} Q'$ *and* $\mathbb{N}' = \mathsf{p}[\![\, P' \,]\!] \parallel \mathsf{q}[\![\, Q' \,]\!] \parallel \mathbb{N}_0$.

4. *If* $\mathbb{N} \xrightarrow{\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q}} \mathbb{N}'$, *then* $\mathbb{N} = \mathsf{p}[\![\, \circ\langle\!\langle\bullet\mathsf{q}; P \,]\!] \parallel \mathsf{q}[\![\, \mathsf{p}\circ\langle\!\langle\bullet; Q \,]\!] \parallel \mathbb{N}_0$ *and*

$$\mathbb{N}' = \overset{\ast}{\mathsf{p}}[\![\, P \,]\!] \parallel \mathsf{p}[\![\, Q \,]\!] \parallel \mathbb{N}_0$$

5. *If* $\mathbb{N} \xrightarrow{\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p}} \mathbb{N}'$, *then* $\mathbb{N} = \overset{\ast}{\mathsf{p}}[\![\, \mathsf{q}\bullet\rangle\!\rangle\circ; P \,]\!] \parallel \mathsf{p}[\![\, \bullet\rangle\!\rangle\circ\mathsf{p}; Q \,]\!] \parallel \mathbb{N}_0$ *and*

$$\mathbb{N}' = \mathsf{p}[\![\, P \,]\!] \parallel \mathsf{q}[\![\, Q \,]\!] \parallel \mathbb{N}_0$$

We end this section with a lemma which allows us to guess the shape of a global type from the shapes of its projections. In the proof of this lemma we use global type contexts $\mathcal{G}$ defined by:

$$\mathcal{G} \quad ::= \quad \boxplus_{i\in I}\, \alpha_i^{\mathsf{p}}; \mathsf{G}_i \boxplus \alpha^{\mathsf{p}}; \mathcal{G} \;\mid\; \mathsf{p}\circ\langle\!\langle\bullet\mathsf{q}; \mathcal{G} \;\mid\; \mathsf{p}\bullet\rangle\!\rangle\circ\mathsf{q}; \mathcal{G} \;\mid\; [\,]$$

where I could be empty.

A context $\mathcal{G}$ that does not contain any proper choices (i.e., choices with more than one branch) will be called *flat global type context*, since in this case $\mathcal{G}[\mathsf{G}]$ can be flattened to a sequence of atomic interactions followed by G, i.e., $\mathcal{G}[\mathsf{G}] = \phi_1; \ldots; \phi_n; \mathsf{G}$. We convene that the sequence is empty if $n = 0$. Such contexts arise in the scope of a delegation.

Notice that global types describe the whole interaction scenario, while projections of global types, i.e., session types, see the interactions only from the viewpoints of single participants. For this reason, some immediate actions of processes may not correspond to immediate interactions in the global type. A simple example is the network

$$\mathsf{p}[\![\, \mathsf{q}!\Lambda \,]\!] \parallel \mathsf{q}[\![\, \mathsf{p}?\Lambda \,]\!] \parallel \mathsf{r}[\![\, \mathsf{s}!\Lambda' \,]\!] \parallel \mathsf{s}[\![\, \mathsf{r}?\Lambda' \,]\!]$$

which can be typed by the global type $\mathsf{p}\Lambda\mathsf{q}; \mathsf{r}\Lambda'\mathsf{s}$.

The forthcoming lemma consists of four statements which may be summarised as follows:

- Statement (1) says that, if the direct projection on p is a union type, then the global type is a (possibly empty) sequence of atomic interactions not involving p followed by a choice with leader p;
- Statement (2) says that, if the direct projection on the principal or on the deputy is a forward delegation action, then the global type is a (possibly empty) sequence of atomic interactions not involving that participant followed by the forward delegation;
- Statement (3) says that if the delegation projection on the principal is defined, then the global type is a (possibly empty) sequence of atomic interactions not involving the deputy, followed by the backward delegation

from the deputy to the principal. In this sequence the atomic interactions involving the principal must be all communications. Namely, when a well-typed process sends a delegation, then it cannot do anything else than wait for the delegation to return.

- Statement (4) says that, if the delegation projection on the deputy is defined, then the global type is a (possibly empty) sequence of atomic interactions not involving the deputy, followed by the backward delegation from the deputy to the principal. In this sequence the atomic interactions involving the principal must be all communications.

Lemma 6.4 (Key Lemma). *The following hold:*

1. *If* $\mathsf{G} \upharpoonright \mathsf{p} = \bigvee_{i \in I} \pi_i; \mathsf{T}_i$, *then*

$$\mathsf{G} = \phi_1; \ldots; \phi_n; \boxplus_{i \in I} \alpha_i^{\mathsf{p}}; \mathsf{G}_i$$

   where ϕ_j *for* $1 \leq j \leq n$ *is an atomic interaction not involving* p *and* $(\alpha_i^{\mathsf{p}}; \mathsf{G}_i) \upharpoonright \mathsf{p} = \pi_i; \mathsf{T}_i$ *for* $i \in I$.

2. *If either* $\mathsf{G} \upharpoonright \mathsf{p} = \circ\langle\!\langle \bullet \mathsf{q}; \mathsf{T}$ *or* $\mathsf{G} \upharpoonright \mathsf{q} = \mathsf{p} \circ \langle\!\langle \bullet; \mathsf{S}$, *then* $\mathsf{G} = \phi_1; \ldots; \phi_n; \mathsf{p} \circ \langle\!\langle \bullet \mathsf{q}; \mathsf{G}'$, *where* ϕ_i *for* $1 \leq i \leq n$ *is an atomic interaction not involving* p *in the first case and* q *in the second case.*

3. *If* $\mathsf{G} \upharpoonright_1 (\mathsf{p}, \mathsf{q})$ *is defined, then* $\mathsf{G} = \phi_1; \ldots; \phi_n; \mathsf{q} \bullet \rangle\!\rangle \circ \mathsf{p}; \mathsf{G}'$, *where, for* $1 \leq i \leq n$, *each* ϕ_i *is an atomic interaction that never involves* q, *and may only involve* p *if it is a communication. Moreover*

$$\mathsf{G} \upharpoonright_1 (\mathsf{p}, \mathsf{q}) = \mathsf{q} \bullet \rangle\!\rangle \circ; \mathsf{G}' \upharpoonright \mathsf{p}$$

4. *If* $\mathsf{G} \upharpoonright_2 (\mathsf{p}, \mathsf{q})$ *is defined, then* $\mathsf{G} = \phi_1; \ldots; \phi_n; \mathsf{q} \bullet \rangle\!\rangle \circ \mathsf{p}; \mathsf{G}'$, *where, for* $1 \leq i \leq n$, *each* ϕ_i *is an atomic interaction that never involves* q, *and may only involve* p *if it is a communication. Moreover*

$$\mathsf{G} \upharpoonright_2 (\mathsf{p}, \mathsf{q}) = \pi_1; \ldots; \pi_m; \bullet \rangle\!\rangle \circ \mathsf{p}; \mathsf{G}' \upharpoonright \mathsf{q}$$

   with $m \leq n$.

Proof (1). By definition of direct projection, see Figure 5, $\mathsf{G} = \mathcal{G}[\boxplus_{i \in I} \alpha_i^{\mathsf{p}}; \mathsf{G}_i]$ and $\mathcal{G}$ is a flat global context, since otherwise the projection on p would start with an intersection of inputs. Moreover, $\mathcal{G}$ cannot contain interactions involving p, since otherwise the projection on p would start with an atomic action different from $\bigvee_{i \in I} \pi_i; \mathsf{T}_i$.

(2). By definition of direct projection, $\mathsf{G} = \mathcal{G}[\mathsf{p} \circ \langle\!\langle \bullet \mathsf{q}]$ where $\mathcal{G}$ is a flat context, since the direct projection of choices must be either a union of outputs or an intersection of inputs. If $\mathsf{G} \upharpoonright \mathsf{p} = \circ\langle\!\langle \bullet \mathsf{q}; \mathsf{T}$, then p cannot occur in $\phi_1; \ldots; \phi_n$, since $\phi_1; \ldots; \phi_n$ are skipped in the projection. For the same reason q cannot occur in $\phi_1; \ldots; \phi_n$ when $\mathsf{G} \upharpoonright \mathsf{q} = \mathsf{p} \circ \langle\!\langle \bullet; \mathsf{S}$.

(3). By definition of delegation projection, see Figure 6, $\mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q})$ skips all communications (including those involving p), as well as all delegations not involving p or q. Moreover, by Definition 4.11(2), and Lemma 4.12 we know that no delegation involving p may occur before the backward delegation. Notice that the presence of a backward delegation is ensured by the condition of well-delegation, which in turn follows from projectability (Lemma 4.12). Hence G and $\mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q})$ must have the forms shown.

(4). By definition of delegation projection, see Figure 6, if $\mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{q})$ is defined, then before $\bullet\!\!\gg\!\!\circ\mathsf{p}$ only I/O actions generated by the projections of communications involving p (first two lines of the figure) can occur. Hence G and $\mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{q})$ must have the forms shown, where the π_i, for $i \in \{1,\ldots,m\}$, correspond to those among the communications ϕ_j, for $j \in \{1,\ldots,n\}$, that involve p.

6.2. Subject reduction, with respect to global types

The subject reduction proof requires some ingenuity, since the definition of projection is quite tricky. The following lemma connects the projections of a global type plugged into a flat context to its projections after a communication.

Lemma 6.5.

1. *Let* $\mathsf{G} = \phi_1;\ldots;\phi_n;\phi;\mathsf{G}''$ *and let* $\mathsf{G}' = \phi_1;\ldots;\phi_n;\mathsf{G}''$.
   - *If* p *and* q *are such that* $\{\mathsf{p},\mathsf{q}\}\cap\mathsf{part}(\phi)=\emptyset$ *and* $\mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q})$ *is defined, then* $\mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q}) = \mathsf{G}'\upharpoonright_1(\mathsf{p},\mathsf{q})$. *Similarly for* $\mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{q})$.
   - *If* p *is such that* $\mathsf{p}\notin\mathsf{part}(\phi)$ *and* $\mathsf{G}\upharpoonright\mathsf{p}$ *is defined, then* $\mathsf{G}\upharpoonright\mathsf{p} = \mathsf{G}'\upharpoonright\mathsf{p}$.
2. *Let* $\mathsf{G} = \phi_1;\ldots;\phi_n;\boxplus_{i\in I}\alpha_i;\mathsf{G}_i$ *where* $|I|>1$ *and let* $\mathsf{G}' = \phi_1;\ldots;\phi_n;\mathsf{G}_j$, $j\in I$.
   - *If* p *and* q *are such that* $\{\mathsf{p},\mathsf{q}\}\cap\mathsf{part}(\alpha_j)=\emptyset$ *and* $\mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q})$ *is defined, then* $\mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q}) \leq \mathsf{G}'\upharpoonright_1(\mathsf{p},\mathsf{q})$. *Similarly for* $\mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{q})$.
   - *If* p *is such that* $\mathsf{p}\notin\mathsf{part}(\alpha_j)$ *and* $\mathsf{G}\upharpoonright\mathsf{p}$ *is defined, then* $\mathsf{G}\upharpoonright\mathsf{p} \leq \mathsf{G}'\upharpoonright\mathsf{p}$.

Proof (1). Let $\mathsf{G} = \phi_1;\ldots;\phi_n;\phi;\mathsf{G}''$ and p and q be such that $\{\mathsf{p},\mathsf{q}\}\cap\mathsf{part}(\phi)=\emptyset$. From the definition of delegation projection, Figure 6, we can see that ϕ is ignored both in $\mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q})$ and $\mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{q})$. From the definition of direct projection, Figure 5, we can see that ϕ is ignored in $\mathsf{G}\upharpoonright\mathsf{p}$. Therefore $\mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q}) = \mathsf{G}'\upharpoonright_1(\mathsf{p},\mathsf{q})$, $\mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{q}) = \mathsf{G}'\upharpoonright_2(\mathsf{p},\mathsf{q})$ and $\mathsf{G}\upharpoonright\mathsf{p} = \mathsf{G}'\upharpoonright\mathsf{p}$.

(2). Let $\mathsf{G} = \phi_1;\ldots;\phi_n;\boxplus_{i\in I}\alpha_i;\mathsf{G}_i$ where $|I|>1$ and p and q be such that $\{\mathsf{p},\mathsf{q}\}\cap\mathsf{part}(\alpha_j)=\emptyset$. If $\mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q})$ is defined, by Lemma 6.4(3) we get $\phi_m = \mathsf{q}\bullet\!\!\gg\!\!\circ\mathsf{p}$ for some m ($1\leq m\leq n$). This implies $\mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q}) = \mathsf{q}\bullet\!\!\gg\!\!\circ;\gamma_1;\ldots;\gamma_h;(\boxplus_{i\in I}\alpha_i;\mathsf{G}_i)\upharpoonright\mathsf{p}$, where $0\leq h\leq n-m$ (by convention $\gamma_1;\ldots;\gamma_h$ is the empty sequence if $h=0$) and the γ_k with $1\leq k\leq h$ are either I/O actions or delegation actions. Since $\{\mathsf{p},\mathsf{q}\}\cap\mathsf{part}(\alpha_j)=\emptyset$, then $(\boxplus_{i\in I}\alpha_i;\mathsf{G}_i)\upharpoonright\mathsf{p} = (\mathsf{G}_j\upharpoonright\mathsf{p})\wedge\mathsf{T}$ for some T. Therefore by definition of $\leq$, see Figure 7, we get $(\boxplus_{i\in I}\alpha_i;\mathsf{G}_i)\upharpoonright\mathsf{p} \leq \mathsf{G}_j\upharpoonright\mathsf{p}$ and thus also

$$\mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q}) \leq \mathsf{q}\bullet\!\!\gg\!\!\circ;\gamma_1;\ldots;\gamma_h;\mathsf{G}_j\upharpoonright\mathsf{p} = \mathsf{G}'\upharpoonright_1(\mathsf{p},\mathsf{q})$$

If $\mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{q})$ is defined, by Lemma 6.4(4) with a similar argument we get

$$\mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{q}) = \pi_1;\ldots;\pi_{h_1};\bullet\!\!\rangle\!\!\circ\mathsf{p};\gamma'_1;\ldots;\gamma'_{h_2};(\boxplus_{i\in I}\,\alpha_i;\mathsf{G}_i)\upharpoonright\mathsf{q}$$

where $0 \le h_1 + h_2 < n$ (here the first sequence is empty if $h_1 = 0$ and the second sequence is empty if $h_2 = 0$) and the γ'_k are either I/O actions or delegation actions. We can then conclude as in the previous case.
The proof for the case where $\mathsf{G}\upharpoonright\mathsf{p}$ is defined is easier.

As standard, as a step towards establishing progress, we need to prove a subject reduction result. Here we show only the most interesting case.

Theorem 6.6 (Subject Reduction). *If* $\vdash \mathbb{N} : \mathsf{G}$ *and* $\mathbb{N} \xrightarrow{\phi} \mathbb{N}'$, *then* $\vdash \mathbb{N}' : \mathsf{G}'$ *for some* G'.

Proof (One case only) For rule [Com], we consider here only the case where the sending participant p is not involved in any delegation and the receiving participant q is the principal in an ongoing delegation. Further cases, completing the proof, appear in Appendix B. Since q is a principal in a delegation, she must have a deputy r, who has now taken the name q, and there must be a frozen participant q; hence, in this case, the network is of the form

$$\mathbb{N} = \mathsf{p}[\![\, P \,]\!] \parallel \mathsf{q}[\![\, R \,]\!] \parallel \overset{*}{\mathsf{q}}[\![\, Q \,]\!] \parallel \mathbb{N}_0$$

and

$$\frac{P \xrightarrow{\mathsf{q}!\Lambda} P' \qquad R \xrightarrow{\mathsf{p}?\Lambda} R'}{\mathsf{p}[\![\, P \,]\!] \parallel \mathsf{q}[\![\, R \,]\!] \xrightarrow{\mathsf{p}\Lambda\mathsf{q}} \mathsf{p}[\![\, P' \,]\!] \parallel \mathsf{q}[\![\, R' \,]\!]}$$

and $\mathbb{N}' = \mathsf{p}[\![\, P' \,]\!] \parallel \mathsf{q}[\![\, R' \,]\!] \parallel \overset{*}{\mathsf{q}}[\![\, Q \,]\!] \parallel \mathbb{N}_0$.
By Lemma 6.3(2) $P = \oplus_{i\in I}\pi_i; P_i$, where $\pi_j = \mathsf{q}!\Lambda$ and $P_j = P'$ for some $j \in I$. By Lemma 6.3(1) $R = \Sigma_{h\in H}\pi'_h; R_h$, where $\pi'_k = \mathsf{p}?\Lambda$ and $R_k = R'$ for some $k \in H$. Since $\vdash \mathbb{N} : \mathsf{G}$, by Lemma 6.1(7) we have $\vdash P : \mathsf{T}$ and $\vdash R : \mathsf{S}$ and $\vdash Q : \mathsf{U}$ for some T, S and U such that $\mathsf{T} \le \mathsf{G}\upharpoonright\mathsf{p}$ and $\mathsf{S} \le \mathsf{G}\upharpoonright_2(\mathsf{q},\mathsf{r})$ and $\mathsf{U} \le \mathsf{G}\upharpoonright_1(\mathsf{q},\mathsf{r})$. By Lemma 6.1(2) $\mathsf{T} = \bigvee_{i\in I}\pi_i;\mathsf{T}_i$ and $\vdash P_i : \mathsf{T}_i$ for $i \in I$ and by Lemma 6.1(1) $\mathsf{S} = \bigwedge_{h\in H}\pi'_h;\mathsf{S}_h$ and $\vdash R_h : \mathsf{S}_h$ for $h \in H$.

By the definition of $\le$ and Lemma 6.4(1) and (4), $\mathsf{T} \le \mathsf{G}\upharpoonright\mathsf{p}$ and $\mathsf{S} \le \mathsf{G}\upharpoonright_2(\mathsf{q},\mathsf{r})$ imply that the first interaction in G involving p and q must be $\mathsf{p}\Lambda\mathsf{q}$. Again by Lemma 6.4(4), $\mathsf{G} = \phi_1;\ldots;\phi_n;\mathsf{p}\Lambda\mathsf{q};\mathsf{G}_0$, where ϕ_l for $1 \le l \le n$ is an atomic interaction not involving p,q and r. Then $\mathsf{G}\upharpoonright\mathsf{p} = \mathsf{q}!\Lambda;\mathsf{T}'$ and $\mathsf{T} = \mathsf{q}!\Lambda;\mathsf{T}''$ with $\mathsf{T}'' \le \mathsf{T}'$. Moreover $\mathsf{G}\upharpoonright_2(\mathsf{q},\mathsf{r}) = \mathsf{p}?\Lambda;\mathsf{S}'$ and $\mathsf{S} = \mathsf{p}?\Lambda;\mathsf{S}'' \wedge \mathsf{S}_0$ for some S'' with $\mathsf{S}'' \le \mathsf{S}'$. We can then choose $\mathsf{G}' = \phi_1;\ldots;\phi_n;\mathsf{G}_0$. By construction $\vdash P' : \mathsf{T}''$ and $\vdash R' : \mathsf{S}''$. By definition of projection $\mathsf{G}'\upharpoonright\mathsf{p} = \mathsf{T}'$ and $\mathsf{G}\upharpoonright_1(\mathsf{q},\mathsf{r}) = \mathsf{G}'\upharpoonright_1(\mathsf{q},\mathsf{r})$ and $\mathsf{G}'\upharpoonright_2(\mathsf{q},\mathsf{r}) = \mathsf{S}'$. Then $\mathsf{T}'' \le \mathsf{G}'\upharpoonright\mathsf{p}$, $\mathsf{U} = \mathsf{G}'\upharpoonright_1(\mathsf{q},\mathsf{r})$ and $\mathsf{S}'' \le \mathsf{G}'\upharpoonright_2(\mathsf{q},\mathsf{r})$. Moreover by Lemma 6.5(1) $\mathsf{G}\upharpoonright\mathsf{s} = \mathsf{G}'\upharpoonright\mathsf{s}$, $\mathsf{G}\upharpoonright_1(\mathsf{s},\mathsf{s}') = \mathsf{G}'\upharpoonright_1(\mathsf{s},\mathsf{s}')$, $\mathsf{G}\upharpoonright_2(\mathsf{s},\mathsf{s}') = \mathsf{G}'\upharpoonright_2(\mathsf{s},\mathsf{s}')$ for all s,s' such that $\{\mathsf{p},\mathsf{q},\mathsf{r}\} \cap \{\mathsf{s},\mathsf{s}'\} = \emptyset$. Therefore applying rule [T-Net] we can derive $\vdash \mathsf{p}[\![\, P' \,]\!] \parallel \mathsf{q}[\![\, R' \,]\!] \parallel \overset{*}{\mathsf{q}}[\![\, Q \,]\!] \parallel \mathbb{N}_0 : \mathsf{G}'$.

6.3. Session fidelity with internal delegation

In our setting, session fidelity means that the communications and the delegations between participants are performed as prescribed by the global type. This can be stated by deriving the shapes of global types from the transitions of the networks and vice versa by deriving the transitions of the networks from the shapes of global types.

Theorem 6.7 (Session Fidelity). *The following hold:*

1. *If* $\vdash \mathbb{N} : \mathsf{G}$ *and* $\mathbb{N} \xrightarrow{\mathsf{p}\Lambda\mathsf{q}} \mathbb{N}'$, *then* $\mathsf{G} = \phi_1; \ldots; \phi_n; \boxplus_{i\in I}\, \alpha_i^{\mathsf{p}}; \mathsf{G}_i$, *where* $\alpha_j^{\mathsf{p}} = \mathsf{p}\Lambda\mathsf{q}$ *for some* $j \in I$ *and* ϕ_h *for* $1 \le h \le n$ *is an atomic interaction not involving* p *and* q. *Moreover* $\vdash \mathbb{N}' : \phi_1; \ldots; \phi_n; \mathsf{G}_j$.

2. *If* $\vdash \mathbb{N} : \mathsf{G}$ *and* $\mathbb{N} \xrightarrow{\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q}} \mathbb{N}'$, *then* $\mathsf{G} = \phi_1; \ldots; \phi_n; \mathsf{p}\circ\langle\!\langle\bullet\mathsf{q}; \mathsf{G}'$, *where* ϕ_i *for* $1 \le i \le n$ *is an atomic interaction not involving* p *and* q. *Moreover* $\vdash \mathbb{N}' : \phi_1; \ldots; \phi_n; \mathsf{G}'$.

3. *If* $\vdash \mathbb{N} : \mathsf{G}$ *and* $\mathbb{N} \xrightarrow{\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p}} \mathbb{N}'$, *then* $\mathsf{G} = \phi_1; \ldots; \phi_n; \mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p}; \mathsf{G}'$, *where* ϕ_i *for* $1 \le i \le n$ *is an atomic interaction not involving* p *and* q. *Moreover* $\vdash \mathbb{N}' : \phi_1; \ldots; \phi_n; \mathsf{G}'$.

4. *If* $\vdash \mathbb{N} : \boxplus_{i\in I}\mathsf{p}\Lambda_i\mathsf{q}_i; \mathsf{G}_i$, *then* $\mathbb{N} = \mathsf{p}[\![\oplus_{i\in I'}\mathsf{q}_i!\Lambda_i; P_i]\!] \parallel \mathbb{N}_0$ *with* $I' \subseteq I$ *and* $\mathbb{N} \xrightarrow{\mathsf{p}\Lambda_i\mathsf{q}_i} \mathbb{N}_i$ *and* $\vdash \mathbb{N}_i : \mathsf{G}_i$ *for all* $i \in I'$.

5. *If* $\vdash \mathbb{N} : \phi; \mathsf{G}$, *then* $\mathbb{N} \xrightarrow{\phi} \mathbb{N}'$ *and* $\vdash \mathbb{N}' : \mathsf{G}$.

The proof of this theorem, which expands on the proof of Theorem 6.6, is in Appendix C.

6.4. Proof of progress

To prove the progress property it is handy to represent global types as trees. In these trees the internal nodes are decorated by $\boxplus$ or nothing, the branches are decorated by atomic interactions, and the leaves are decorated by End. In case the global type has some recursive subtype, the tree is an infinite (regular) tree.

Lemma 6.8. *If* $\mathsf{G} \upharpoonright \mathsf{p} = \bigwedge_{i\in I} \mathsf{q}_i?\lambda_i; \mathsf{T}_i$, *then* p *occurs on each path from the root of* G *and the first atomic interaction involving* p *is* $\mathsf{q}_i\lambda_i\mathsf{p}$ *for some* $i \in I$.

Proof First we show that p occurs on each path from the root of G. Assume there is a path without occurrences of p, then the projection of the corresponding global type on p would be either End or $\mathbf{t}$. This is impossible, since the meet between an intersection of simple inputs and End or $\mathbf{t}$ is undefined.

We show now that on each path from the root of G the first atomic interaction involving p is $\mathsf{q}_i\lambda_i\mathsf{p}$ for some $i \in I$. The proof is by induction on the maximal length of the paths from the root of G to the first atomic interaction involving p. Let $\mathsf{G} = \boxplus_{j\in J}\, \alpha_j^{\mathsf{r}}; \mathsf{G}_j$: by definition of projection $\mathsf{p} \neq \mathsf{r}$ and $\mathsf{G} \upharpoonright \mathsf{p} = \bigsqcap_{j\in J}(\, \alpha_j^{\mathsf{r}}; \mathsf{G}_j\,) \upharpoonright \mathsf{p}$. Consider $h \in J$, then either $\alpha_h^{\mathsf{r}} = \mathsf{r}\lambda_h\mathsf{p}$ or $\mathsf{p} \notin \mathsf{part}(\alpha_h^{\mathsf{r}})$. In the first case $\mathsf{r}?\lambda_h = \mathsf{q}_i?\lambda_i$ for some $i \in I$. In the second case from the definition of meet $\mathsf{G}_h \upharpoonright \mathsf{p} =$

$\bigwedge_{k\in K} \mathsf{q}'_k?\lambda'_k;\mathsf{T}'_k$ and for all $k \in K$ we have $\mathsf{q}'_k?\lambda'_k = \mathsf{q}_i?\lambda_i$ for some $i \in I$. By induction hypothesis on each path from the root of G_h the first atomic interaction involving p is $\mathsf{q}'_k\lambda'_k\mathsf{p}$ for some $k \in K$.
Let $\mathsf{G} = \delta;\mathsf{G}_0$. By definition of projection δ cannot involve p. By induction hypothesis on each path from the root of G_0 the first atomic interaction involving p is $\mathsf{q}_i\lambda_i\mathsf{p}$ for some $i \in I$.

Note that previous lemma does not hold if simple inputs are replaced by connecting inputs, since the meet between connecting inputs and End is defined.

By employing the above lemma and Theorem 6.7 we can establish the main result of this paper: progress — Theorem 5.2. To this end, we prove the following four lemmas, whose conjunction amounts to Theorem 5.2.

Lemma 6.9 (Progress scenario 1). *If* $\mathbb{N} = \mathsf{p}[\![\oplus_{i\in I}\mathsf{q}_i!\Lambda_i;P_i]\!] \parallel \mathbb{N}_0$ *is a well typed network, then* $\mathbb{N} \xrightarrow{\vec{\phi}\,\mathsf{p}\Lambda_i\mathsf{q}_i} \mathbb{N}'$ *for some* $\vec{\phi}$ *and for all* $i \in I$.

Proof By Lemma 6.1(7b) or (7c) and the definition of projection $\vdash \oplus_{i\in I}\mathsf{q}_i!\Lambda_i;P_i : \mathsf{T}$ with either $\mathsf{T} \leq \mathsf{G}\upharpoonright\mathsf{p}$ or $\mathsf{T} \leq \mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{r})$ when $|I| = 1$. By Lemma 6.1(2) $\mathsf{T} = \bigvee_{i\in I}\mathsf{q}_i!\Lambda_i;\mathsf{T}_i$. Then either $\mathsf{G}\upharpoonright\mathsf{p} = \bigvee_{i\in I'}\mathsf{q}_i!\Lambda_i;\mathsf{T}_i$ with $I \subseteq I'$ or $\mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{r}) = \mathsf{q}!\Lambda;\mathsf{T}'$ by definition of $\leq$ and projection. In the first case Lemma 6.4(1) gives $\mathsf{G} = \phi_1;\ldots;\phi_n;\boxplus_{i\in I'}\mathsf{p}\Lambda_i\mathsf{q}_i;\mathsf{G}_i$, where ϕ_j for $1 \leq j \leq n$ is an atomic interaction not involving p. In the second case Lemma 6.4(4) gives $\mathsf{G} = \phi_1;\ldots;\phi_n;\mathsf{p}\Lambda\mathsf{q};\mathsf{G}'$, where ϕ_j for $1 \leq j \leq n$ is an atomic interaction not involving p. In both cases we obtain the desired computation by repeated applications of Theorem 6.7(5) and in the first case by one application of Theorem 6.7(4) for the last transition.

Lemma 6.10 (Progress scenario 2). *If* $\mathbb{N} = \mathsf{p}[\![\Sigma_{i\in I}\mathsf{q}_i?\lambda_i;P_i]\!] \parallel \mathbb{N}_0$ *is a well typed network, then* $\mathbb{N} \xrightarrow{\vec{\phi}\,\mathsf{q}_i\lambda_i\mathsf{p}} \mathbb{N}'$ *for some* $\vec{\phi}$ *and for some* $i \in I$.

Proof By Lemma 6.1(7b) or (7c) and the definition of projection $\vdash \Sigma_{i\in I}\mathsf{q}_i?\lambda_i;P_i : \mathsf{T}$ with either $\mathsf{T} \leq \mathsf{G}\upharpoonright\mathsf{p}$ or $\mathsf{T} \leq \mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{r})$. By Lemma 6.1(2) $\mathsf{T} = \bigwedge_{i\in I}\mathsf{q}_i?\lambda_i;\mathsf{T}_i$. By definition of $\leq$ and projection either $\mathsf{G}\upharpoonright\mathsf{p} = \bigwedge_{i\in I'}\mathsf{q}_i?\lambda_i;\mathsf{T}_i$ with $I \supseteq I'$ or $\mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{r}) = \mathsf{q}?\lambda;\mathsf{T}'$. In the first case Lemma 6.8 assures that p occurs on each path from the root of G and the first atomic interaction involving p is $\mathsf{q}_i\lambda_i\mathsf{p}$ for some $i \in I$. In the second case Lemma 6.4(4) gives $\mathsf{G} = \phi_1;\ldots;\phi_n;\mathsf{q}\lambda\mathsf{p};\mathsf{G}'$, where ϕ_j for $1 \leq j \leq n$ is an atomic interaction not involving p. In both cases we obtain the desired computation by repeated applications of Theorem 6.7(4) and (5).

Lemma 6.11 (Progress scenario 3). *If* $\mathbb{N} = \mathsf{p}[\![\circ\langle\!\langle\bullet\mathsf{q};P]\!] \parallel \mathbb{N}_0$ *is a well typed network, then* $\mathbb{N} \xrightarrow{\vec{\phi}\,\mathsf{p}\circ\langle\bullet\mathsf{q}\vec{\phi}'\mathsf{q}\bullet\rangle\circ\mathsf{p}} \mathbb{N}'$ *for some* $\vec{\phi}$ *and* $\vec{\phi}'$.

Proof By Lemma 6.1(7b) or (7c) $\vdash \circ\langle\!\langle\bullet\mathsf{q};P : \mathsf{T}$ with either $\mathsf{T} \leq \mathsf{G}\upharpoonright\mathsf{p}$ or $\mathsf{T} \leq \mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{r})$ for some r. By Lemma 6.1(3) $\mathsf{T} = \circ\langle\!\langle\bullet\mathsf{q};\mathsf{T}'$. From definition of $\leq$ and Lemma 6.4(4) we have that $\mathsf{T} \leq \mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{r})$ is not possible. Therefore

$\mathsf{T} \leq \mathsf{G} \upharpoonright \mathsf{p}$. The definition of $\leq$ gives $\mathsf{G} \upharpoonright \mathsf{p} = \circ\langle\!\langle \bullet \mathsf{q}; \mathsf{S}$. By Lemma 6.4(2) $\mathsf{G} = \phi_1; \ldots; \phi_n; \mathsf{p} \circ \langle\!\langle \bullet \mathsf{q}; \mathsf{G}'$, where ϕ_i for $1 \leq i \leq n$ is an atomic interaction not involving p. The definition of projection implies $\mathsf{S} = \mathsf{G}' \upharpoonright_1 (\mathsf{p}, \mathsf{q})$. By Lemma 6.4(3) $\mathsf{G}' = \phi'_1; \ldots; \phi'_m; \mathsf{q} \bullet \rangle\!\rangle \circ \mathsf{p}; \mathsf{G}''$ and $\mathsf{G}' \upharpoonright_1 (\mathsf{p}, \mathsf{q}) = \mathsf{q} \bullet \rangle\!\rangle \circ; \mathsf{G}'' \upharpoonright \mathsf{p}$, where ϕ'_j for $1 \leq j \leq m$ is an atomic interaction not involving q. We obtain the desired computation by repeated applications of Theorem 6.7(5).

Lemma 6.12 (Progress scenario 4). *If* $\mathbb{N} = \mathsf{q}[\![\mathsf{p} \circ \langle\!\langle \bullet; Q]\!] \parallel \mathbb{N}_0$, *is a well typed network, then* $\mathbb{N} \xrightarrow{\overrightarrow{\phi} \mathsf{p} \circ \langle\!\langle \bullet \mathsf{q} \overrightarrow{\phi'} \mathsf{q} \bullet \rangle\!\rangle \circ \mathsf{p}} \mathbb{N}'$ *for some* $\overrightarrow{\phi}$ *and* $\overrightarrow{\phi'}$.

Proof The proof is similar to that of the previous lemma.

The above four lemmas cover all cases in Theorem 5.2.

7. Future Work: Multiparty Compatibility and Extensions

Multiparty compatibility. The standard notion of duality, ensuring two session types are compatible, does not guarantee progress or the existence of a global type in the multiparty setting. An example mentioned in the introduction is the classic 3-philosopher problem, where each party is waiting for an input from another party. The 3 philosophers are pairwise dual but the network is deadlocked. Global types, as investigated throughout this work, offer one approach to addressing this problem. Here we briefly discuss a notion of multiparty compatibility [13, 14] as an alternative to global types where networks are typed based on the (local) session types of each participant. Selecting an appropriate notion of multiparty compatibility avoids the above mentioned problem with pair-wise compatibility.

In the following definition, we propose an extended notion of multiparty compatibility that is sensitive to delegation. In the presence of fixed points, there are additional subtleties if we wish to capture a notion of progress guaranteeing more than just deadlock freedom, such as the notion of progress captured in Theorem 5.2. Hence we push infinite sessions to future work and restrict ourselves to finite processes in what follows.

Located types are defined similarly to networks, by the syntax:

$$\mathcal{T} ::= \mathsf{p}[\![\mathsf{T}]\!] \mid \overset{*}{\mathsf{p}}[\![\mathsf{T}]\!] \mid \mathcal{T} \,,\, \mathcal{T}$$

where the comma is an associative, commutative operator. Multiparty compatibility of located types is defined according to the rules and axiom in Fig. 9. Observe that only the rule for union types has multiple premises.

We say that a global type is *initial* if all backward delegations have corresponding forward delegations. For finite and initial global types the following holds by induction over the structure of the global type.

Proposition 7.1. *Let* G *be a finite and initial global type and* $\{\mathsf{p}_1, \ldots, \mathsf{p}_n\} = \mathsf{part}(\mathsf{G})$, *then* $\mathsf{p}_1[\![\mathsf{G} \upharpoonright \mathsf{p}_1]\!], \ldots, \mathsf{p}_n[\![\mathsf{G} \upharpoonright \mathsf{p}_n]\!]$ *are multiparty compatible.*

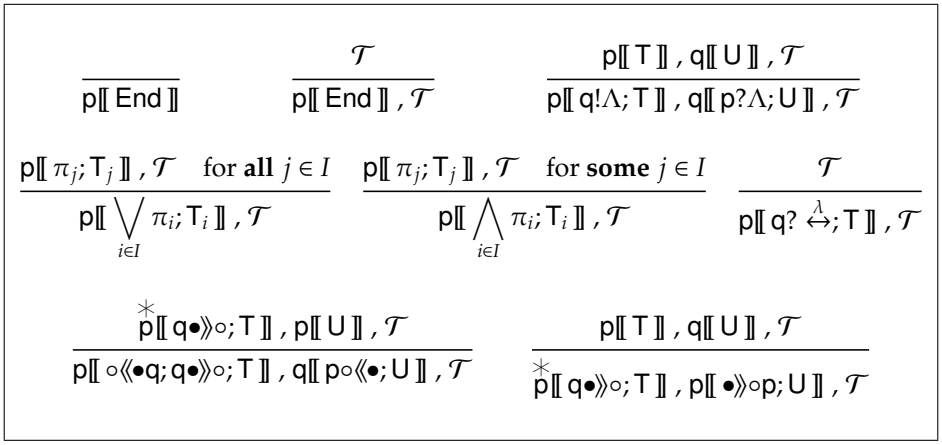

Figure 9: Rules determining whether located types are multiparty compatible.

The condition of being initial is needed in the previous proposition, since after some delegation started we need to use the delegation projection.

The converse direction would establish that, for multiparty-compatible located types, there exists a global type such that the projection for each participant of the global type is a super type of the session type of each participant. This property is sometimes referred to as checking whether types are *coherent* [32, 28] with respect to a global type. However, care must be taken to establish the assumptions under which this holds; for example, we must ensure that deputies neither make choices nor engage in further (nested) delegation actions before returning back the delegation to the principal. A detailed study of such conditions is pushed to future work. It is not clear if global type synthesis can be satisfactory, since there exist realistic multiparty compatible networks with no global type, some examples of which are given in [38]. Hence future work would likely focus on what conditions are required to ensure that multiparty-compatible networks directly guarantee progress.

Reversible sessions. A separate line of future work is to study the hypothesis that internal delegation facilitates a theory of reversible sessions. In the literature, there are various proposals for formalising reversibility in sessions, but few [39, 30] consider session interleaving and delegation, since these features highly complicate the matter. Thanks to its simplicity, internal delegation could be more easily amenable to a smooth integration with reversibility, leading to a flexible and expressive calculus of reversible multiparty sessions.

Choice under recursion. Type systems are not expected to be complete for properties they guarantee. However, a major limitation to address in future work is that we cannot type networks where a choice with branches involving distinct sets of participants occurs inside a recursion. For example, we cannot type an extension of our motivating example in Section 2, where the client and seller negotiate forever, perhaps sometimes contacting the bank to make payments.

For a minimal example illustrating the source of this challenge, consider the following two projections involving connecting inputs.

$$\left(\mathsf{p} \overset{\lambda_1}{\leftrightarrow} \mathsf{q}; \mathsf{End} \boxplus \mathsf{p} \overset{\lambda_2}{\leftrightarrow} \mathsf{r}; \mathsf{End}\right) \upharpoonright \mathsf{q} \qquad \left(\mu \mathbf{t}.\mathsf{p} \overset{\lambda_1}{\leftrightarrow} \mathsf{q}; \mathbf{t} \boxplus \mathsf{p} \overset{\lambda_2}{\leftrightarrow} \mathsf{r}; \mathbf{t}\right) \upharpoonright \mathsf{q}$$

Observe the first projection is defined since a connecting input will be merged with End. The second is undefined since we would attempt to merge a connecting input with $\mathbf{t}$ (or end up in an infinite loop unfolding the recursion).

The fact the latter projection is undefined may be considered to be restrictive with respect to the intuition that a connecting input can be merged with branches in which the participant performing the input never acts. The above problem, which limits the use of choice under recursion, may be addressed by considering a more general merge operator (or perhaps otherwise modifying the notion of projection), allowing connecting inputs to be merged with an infinite branch in which the participant awaiting the connecting inputs never acts.

Alternative notations for internal delegation. Our proposal for internal delegation is by no means the only possible approach. Our solution uses block constructs for both principals and deputies, and enables delegation from a principal to a unique deputy at a time, while suspending the principal for the whole duration of the delegation. An interesting suggestion for an alternative - and arguably more powerful - internal delegation mechanism was put forward by our reviewer: the idea was to refine the message exchange construct by introducing appropriate role annotations in the messages. For example, we could stipulate that the seller delegates the reception of the credit card to the bank by using a construct of the form $\mathtt{C} \xrightarrow{\mathtt{card}} \mathtt{S}(\mathtt{B})$, meaning that $\mathtt{C}$ asks $\mathtt{B}$ to handle the reception of the card on behalf of $\mathtt{S}$. This construct would allow the principal to remain active and engage in other interactions during the delegation (since messages may have distinguishing sender/receiver). Note that this solution would require a substantial change in the operational semantics, since the freezing/unfreezing mechanism is suitable only when delegation is disciplined by blocks. Another issue is that delegation would no longer be transparent for participants different from the principal and the deputy. The rightfulness of this transparency principle is debatable, however. Almost all papers in the literature claim that delegation should be transparent. To the best of our knowledge, explicit delegation was only used in [5] to send sensitive data to trusted participants. We plan to further investigate this question.

Alternative process syntax. Clearly, a process implementing a participant who is ready to delegate starts with two matching delegation actions (an active forward delegation followed by a passive backward delegation with the same participant). As observed by our reviewer, the second action could be initially omitted and then replaced for the first action with rule [BDEL]. Accordingly, the projection of forward delegations on principals should contain both matching delegation actions, while the projection of backward delegations on principals should be empty. A feature of the current design choice is the isomorphism between

processes and session types which results in very simple typing rules. This design decision has a small impact on the semantics, hence can be accommodated mainly by exposing an alternative concrete syntax for delegation in processes.

8. Related Work

Session types to govern communication protocols have been first proposed for dyadic communications [19]. Delegation for binary sessions was introduced as early as the end of the 90's, in [20], as a means of appointing part of the conversation in a session to a participant acting in another session. In a binary session framework, delegation is also the only way to allow more than two participants to communicate in order to achieve a common task.

The first papers on multiparty sessions are [4, 21]. In their foundational work [21], Honda et al. introduced the keystone notion of global type, from which the session types of participants could be retrieved by means of a projection operation. The calculus of [21] included delegation, which was a simple generalisation of that of [20] to the multiparty case. In the last ten years the literature on this subject has grown with very high speed. In the majority of papers session calculi are channel based, as in [21] or in the simplified setting of [3], and delegation is implemented by means of channel passing, thus exploiting an essential feature of the pi-calculus. Here instead we consider multiparty conversations confined to a single session, and delegation takes place within the session itself (delegation is an intra-session rather than a inter-session mechanism), hence we can omit channels, as done for example in [33, 16].

The paper [25] is a recent overview on the genesis and evolution of session types, produced by the IC Cost Action Betty (`http://www.dcs.gla.ac.uk/research/betty/www.behavioural-types.eu/index.html`). An account of several implementations of session types, both as languages with native session types or as extensions of mainstream languages, may be found at `http://groups.inf.ed.ac.uk/abcd/session-implementations.html`.

The papers that are most closely related to our work are [12, 23, 9, 36]. Our notion of progress coincides with that of [12], when restricted to the intersection of the two calculi. The same paper also uses a notion of meet for defining projection, which however is incomparable with ours since it is based on a different syntax for global types.

Connecting communications have been first discussed in [23]. Here we take the simplified and more flexible version proposed in [9]. The introductory example of [36] was our inspiration for defining delegation within a single global type. Notably, delegation in [36] is realised by sending channels as usual, while we avoid channels. The drawback is that our delegation is less expressive than usual, because some behaviours cannot be delegated (as specified by our well-delegation conditions). For example in our intra-session delegation the deputy cannot make choices or behave recursively, while in inter-session delegation the transmitted channel has an arbitrary session type [22]. On the positive side, we are able to achieve both deadlock-freedom and lock-freedom without additional machinery.

Deadlock-freedom ensures there are no states where there are hanging actions and yet no atomic interactions can be performed. Lock-freedom [34] strengthens deadlock-freedom by guaranteeing that certain atomic interactions may succeed. Some definitions of lock-freedom [27] are subject to assumptions guaranteeing fairness of schedulers. In the current work, the type system is sufficiently restrictive for recursive processes with choices that we are not required to make fairness assumptions. However, it would be interesting to investigate in future work whether, by introducing explicit fairness assumptions, the type system can be liberated to permit more processes to be typed while maintaining lock-freedom. Recent work [18, 17] suggests that weak notions of fairness are sufficient assumptions to ensure lock-freedom. In particular, it appears that *justness* is sufficient; where justness assumes that one party cannot prevent another party from eventually acting; thereby avoiding scenarios where one party schedules her actions in such a way that an action critical for another party to progress cannot occur.

Another approach to multiparty sessions that yields deadlock-freedom "for free" in the presence of session interleaving and delegation is that of *Choreographic Programming* [7, 31]. A choreography is a global description of a network engaged in one or more sessions, possibly interleaved with each other. In fact, a choreography may be viewed as the "abstract skeleton" of a concrete network execution. Another way to put it is that a choreography is an interleaving of global types. Therefore, although it adopts channel-based delegation which requires session interleaving, the choreographic approach guarantees progress because it provides complete information about the interleaving of different sessions and about the points where delegation actions are performed.

9. Conclusion

In this paper we have provided an approach to delegation that alleviates the challenging problem of finding conditions on session types guaranteeing progress in the presence of delegation. The approach we adopt does guarantee progress for (non-ambiguous and projectable) global types (Theorem 5.2), ensuring participants connected to the session do not have hanging actions that are never triggered. Progress is established from *session fidelity* (Theorem 6.7), which ensures the processes adhere to their global type. In turn, session fidelity follows from *subject reduction* (Theorem 6.6), which requires some ingenuity in cases where at least one of the principals involved in an atomic interaction is acting as a deputy in an internal delegation.

As expected of a type system, not all networks exhibiting strong progress can be typed. However, we argue that we cover a sufficiently large class of networks. Notably, by including connecting inputs, we are able to increase the class of networks we can type that feature internal delegation. For example, without connecting inputs there would be no type for the network in Example 5.1(3) and it would not be possible to express the motivating example in Section 2. We note that with additional effort the expressive power could still be increased, for example, to allow the deputy in a delegation to make choices or to permit

nested delegations. However, we argue that the current constraints on the considered fragment are desirable, since they ensure that all choices are made by the principal, including the choice of whom is the current deputy.

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A. Projection respects Equirecursion

$$\Theta \vdash^g \mathsf{End} \sim \mathsf{End} \qquad \Theta, (\mathsf{G}, \mathsf{G}') \vdash^g \mathsf{G} \sim \mathsf{G}' \qquad \frac{\Theta \vdash^g \mathsf{G} \sim \mathsf{G}'}{\Theta \vdash^g \mathsf{G}' \sim \mathsf{G}}$$

$$\frac{\Theta \vdash^g \mathsf{G} \sim \mathsf{G}'}{\Theta \vdash^g \phi; \mathsf{G} \sim \phi; \mathsf{G}'} \qquad \frac{\Theta \vdash^g \mathsf{G}_i \sim \mathsf{G}'_i \quad \forall i \in I}{\Theta \vdash^g \boxplus_{i \in I} \alpha_i; \mathsf{G}_i \sim \boxplus_{i \in I} \alpha_i; \mathsf{G}'_i}$$

$$\frac{\Theta, (\mu \mathbf{t}.\mathsf{G}, \mathsf{G}') \vdash^g \mathsf{G}\{\mu \mathbf{t}.\mathsf{G}/\mathbf{t}\} \sim \mathsf{G}'}{\Theta \vdash^g \mu \mathbf{t}.\mathsf{G} \sim \mathsf{G}'}$$

Figure A.10: Equirecursive global types.

$$\Delta \vdash^s \mathsf{End} \sim \mathsf{End} \qquad \Delta, (\mathsf{T}, \mathsf{T}') \vdash^s \mathsf{T} \sim \mathsf{T}' \qquad \frac{\Delta \vdash^s \mathsf{T} \sim \mathsf{T}'}{\Delta \vdash^s \mathsf{T}' \sim \mathsf{T}}$$

$$\frac{\Delta \vdash^s \mathsf{T} \sim \mathsf{T}'}{\Delta \vdash^s \delta; \mathsf{T} \sim \delta; \mathsf{T}'} \qquad \frac{\Delta \vdash^s \mathsf{T}_i \sim \mathsf{T}'_i \quad \forall i \in I}{\Delta \vdash^s \bigwedge_{i \in I} \pi_i; \mathsf{T}_i \sim \bigwedge_{i \in I} \pi_i; \mathsf{T}'_i}$$

$$\frac{\Delta \vdash^s \mathsf{T}_i \sim \mathsf{T}'_i \quad \forall i \in I}{\Delta \vdash^s \bigvee_{i \in I} \pi_i; \mathsf{T}_i \sim \bigvee_{i \in I} \pi_i; \mathsf{T}'_i} \qquad \frac{\Delta, (\mu \mathbf{t}.\mathsf{T}, \mathsf{T}') \vdash^s \mathsf{T}\{\mu \mathbf{t}.\mathsf{T}/\mathbf{t}\} \sim \mathsf{T}'}{\Delta \vdash^s \mu \mathbf{t}.\mathsf{T} \sim \mathsf{T}'}$$

Figure A.11: Equirecursive session types.

Let Θ denote a set of closed global type pairs. Figure A.10 gives a proof system for showing that two global types generate the same tree. Let Δ denote a set of closed session type pairs. Figure A.11 gives a proof system for showing that two session types generate the same tree.

Lemma A.1 (Inversion Lemma). *1. If $\Delta \vdash^s \bigwedge_{i \in I} \pi_i; \mathsf{T}_i \sim \bigwedge_{i \in I'} \pi'_i; \mathsf{T}'_i$, then $I = I'$, $\pi_i = \pi'_i$ and $\Delta \vdash^s \mathsf{T}_i \sim \mathsf{T}'_i$ for all $i \in I$.*

2. If $\Delta \vdash^s \bigvee_{i \in I} \pi_i; \mathsf{T}_i \sim \bigvee_{i \in I'} \pi'_i; \mathsf{T}'_i$, then $I = I'$, $\pi_i = \pi'_i$ and $\Delta \vdash^s \mathsf{T}_i \sim \mathsf{T}'_i$ for all $i \in I$.

Lemma A.2. *If $\mathsf{G}\{\mathsf{G}'/\mathbf{t}\}$ is a projectable global type and $\mathbf{t}'$ occurs in G, then* $\mathsf{part}(\mathsf{G}) = \mathsf{part}(\mathsf{G}\{\mathsf{G}'/\mathbf{t}\})$.

Proof If $\mathbf{t}$ does not occur in G this is obvious. So we can assume that both $\mathbf{t}$ and $\mathbf{t}'$ occur in G. Since all type variables can only occur as rightmost operands of sequential compositions, there must be a subtype G'' of G, such that all the following conditions hold:

- $\mathsf{G}'' = \boxplus_{i \in I} \alpha_i; \mathsf{G}_i$,

- $\mathbf{t}$ occurs in G_j and $\mathbf{t}'$ does not occur in G_j for some $j \in I$,
- $\mathbf{t}'$ occurs in G_k and $\mathbf{t}$ does not occur in G_k for some $k \in I$.

Assume ad absurdum that there is $\mathsf{q} \in \mathsf{part}(\mathsf{G}\{\mathsf{G}'/\mathbf{t}\})$ such that $\mathsf{q} \notin \mathsf{part}(\mathsf{G})$. Then by construction $\mathsf{q} \in \mathsf{part}(\mathsf{G}_j\{\mathsf{G}'/\mathbf{t}\})$ and $\mathsf{q} \notin \mathsf{part}(\mathsf{G}_k\{\mathsf{G}'/\mathbf{t}\})$, which gives $\mathsf{G}_k\{\mathsf{G}'/\mathbf{t}\} \upharpoonright \mathsf{q} = \mathbf{t}'$. This implies that $(\mathsf{G}\{\mathsf{G}'/\mathbf{t}\}) \upharpoonright \mathsf{q}$ is undefined, since the meet between a session type and $\mathbf{t}'$ is undefined. This contradicts our hypothesis.

Lemma A.3 (Projection commutes with substitution). *If* $\mathsf{G}\{\mathsf{G}'/\mathbf{t}\}$ *is a projectable global type, then* $(\mathsf{G}\{\mathsf{G}'/\mathbf{t}\}) \upharpoonright \mathsf{p} = \mathsf{G} \upharpoonright \mathsf{p}\{\mathsf{G}' \upharpoonright \mathsf{p}/\mathbf{t}\}$.

Proof By structural induction on G.

Let $\mathsf{G} = \mathbf{t}'$, by definition of projection $\mathsf{G} \upharpoonright \mathsf{p} = \mathbf{t}'$. The result is immediate in both cases $\mathbf{t}' = \mathbf{t}$ or $\mathbf{t}' \neq \mathbf{t}$.

Let $\mathsf{G} = \boxplus_{i\in I}\, \alpha_i; \mathsf{G}_i$. We get $\mathsf{G}\{\mathsf{G}'/\mathbf{t}\} = \boxplus_{i\in I}\, \alpha_i; \mathsf{G}_i\{\mathsf{G}'/\mathbf{t}\}$. By induction hypothesis

$$(\mathsf{G}_i\{\mathsf{G}'/\mathbf{t}\}) \upharpoonright \mathsf{p} = \mathsf{G}_i \upharpoonright \mathsf{p}\{\mathsf{G}' \upharpoonright \mathsf{p}/\mathbf{t}\} \text{ for all } i \in I$$

If $(\boxplus_{i\in I}\, \alpha_i; \mathsf{G}_i) \upharpoonright \mathsf{p} = \bigsqcap_{i\in I}(\, \alpha_i; \mathsf{G}_i\,) \upharpoonright \mathsf{p}$ let q be the choice leader of $\boxplus_{i\in I}\, \alpha_i; \mathsf{G}_i$ and $J = \{i \in I \mid \alpha_i = \mathsf{q}\Lambda_i\mathsf{p}\}$, then

$$\begin{array}{rcl} \bigsqcap_{i\in I}(\, \alpha_i; \mathsf{G}_i\{\mathsf{G}'/\mathbf{t}\}\,) \upharpoonright \mathsf{p} & = & (\bigsqcap_{i\in J} \mathsf{q}?\Lambda_i; \mathsf{G}_i\{\mathsf{G}'/\mathbf{t}\} \upharpoonright \mathsf{p}\,) \bigsqcap (\bigsqcap_{i\in I\setminus J} \mathsf{G}_i\{\mathsf{G}'/\mathbf{t}\} \upharpoonright \mathsf{p}\,) \\ & = & (\bigsqcap_{i\in J} \mathsf{q}?\Lambda_i; \mathsf{G}_i \upharpoonright \mathsf{p}\{\mathsf{G}' \upharpoonright \mathsf{p}/\mathbf{t}\}) \bigsqcap (\bigsqcap_{i\in I\setminus J} \mathsf{G}_i \upharpoonright \mathsf{p}\{\mathsf{G}' \upharpoonright \mathsf{p}/\mathbf{t}\}) \\ & = & \bigsqcap_{i\in I}(\alpha_i; \mathsf{G}_i) \upharpoonright \mathsf{p}\{\mathsf{G}' \upharpoonright \mathsf{p}/\mathbf{t}\} \end{array}$$

The proof in the case $\boxplus_{i\in I}\, \alpha_i; \mathsf{G}_i \upharpoonright \mathsf{p} = \bigvee_{i\in I}(\, \alpha_i; \mathsf{G}_i\,) \upharpoonright \mathsf{p}$ is similar and simpler.

Let $\mathsf{G} = \phi; \mathsf{G}_1$. If ϕ is a communication, then it is a particular case of choice with $|I|$=1. If $\phi = \mathsf{r}\circ\langle\!\langle\bullet\mathsf{s}$ and neither $\mathsf{r} = \mathsf{p}$ nor $\mathsf{s} = \mathsf{p}$ the result follows easily by induction on G_1.

If $\phi = \mathsf{p}\circ\langle\!\langle\bullet\mathsf{s}$, since by Lemma 4.12, G is well delegated, we have that $\mathsf{G} = \mathsf{p}\circ\langle\!\langle\bullet\mathsf{s}; \phi_1; \ldots; \phi_n; \mathsf{s}\bullet\rangle\!\rangle\circ\mathsf{p}; \mathsf{G}_2$ for some G_2 and s does not occur in $\phi_1; \ldots; \phi_n$. By definition of projection $(\mathsf{G}\{\mathsf{G}'/\mathbf{t}\}) \upharpoonright \mathsf{p} = \circ\langle\!\langle\bullet\mathsf{s}; \mathsf{s}\bullet\rangle\!\rangle\circ; (\mathsf{G}_2\{\mathsf{G}'/\mathbf{t}\}) \upharpoonright \mathsf{p}$. By induction hypothesis $\circ\langle\!\langle\bullet\mathsf{s}; \mathsf{s}\bullet\rangle\!\rangle\circ; (\mathsf{G}_2\{\mathsf{G}'/\mathbf{t}\}) \upharpoonright \mathsf{p} = \circ\langle\!\langle\bullet\mathsf{s}; \mathsf{s}\bullet\rangle\!\rangle\circ; (\mathsf{G}_2 \upharpoonright \mathsf{p})\{\mathsf{G}' \upharpoonright \mathsf{p}/\mathbf{t}\}$ and $\circ\langle\!\langle\bullet\mathsf{s}; \mathsf{s}\bullet\rangle\!\rangle\circ; (\mathsf{G}_2 \upharpoonright \mathsf{p})\{\mathsf{G}' \upharpoonright \mathsf{p}/\mathbf{t}\} = \mathsf{G} \upharpoonright \mathsf{p}\{\mathsf{G}' \upharpoonright \mathsf{p}/\mathbf{t}\}$.

If $\phi = \mathsf{r}\circ\langle\!\langle\bullet\mathsf{p}$, as before $\mathsf{G} = \mathsf{r}\circ\langle\!\langle\bullet\mathsf{p}; \phi_1; \ldots; \phi_n; \mathsf{p}\bullet\rangle\!\rangle\circ\mathsf{r}; \mathsf{G}_2$ for some G_2. By definition of projection $(\mathsf{G}\{\mathsf{G}'/\mathbf{t}\}) \upharpoonright \mathsf{p} = \mathsf{r}\circ\langle\!\langle\bullet; \pi_1; \ldots; \pi_m; \bullet\rangle\!\rangle\circ\mathsf{r}; (\mathsf{G}_2\{\mathsf{G}'/\mathbf{t}\}) \upharpoonright \mathsf{p}$. As before the result follows by induction hypothesis on G_2.

Let $\mathsf{G} = \mu\mathbf{t}'.\mathsf{G}_1$. We may assume that $\mathbf{t} \neq \mathbf{t}'$ and $\mathbf{t}'$ does not occur in G'.

$$(\mu\mathbf{t}'.\mathsf{G}_1\{\mathsf{G}'/\mathbf{t}\}) \upharpoonright \mathsf{p} = \begin{cases} (\mathsf{G}_1\{\mathsf{G}'/\mathbf{t}\}) \upharpoonright \mathsf{p} & \text{if } \mathbf{t}' \text{ does not occur in } \mathsf{G}_1\{\mathsf{G}'/\mathbf{t}\} \\ \mu\mathbf{t}'.(\mathsf{G}_1\{\mathsf{G}'/\mathbf{t}\}) \upharpoonright \mathsf{p} & \text{if } \mathsf{p} \in \mathsf{part}(\mathsf{G}_1\{\mathsf{G}'/\mathbf{t}\}) \\ \mathsf{End} & \text{otherwise} \end{cases}$$

By induction hypothesis we have that $(\mathsf{G}_1\{\mathsf{G}'/\mathbf{t}\}) \upharpoonright \mathsf{p} = \mathsf{G}_1 \upharpoonright \mathsf{p}\{\mathsf{G}' \upharpoonright \mathsf{p}/\mathbf{t}\}$. Moreover, $\mathbf{t}'$ occurs in $\mathsf{G}_1\{\mathsf{G}'/\mathbf{t}\}$ if and only if $\mathbf{t}'$ occurs in G_1. So the result is immediate if $\mathbf{t}'$ does not occur in G_1. Otherwise by Lemma A.2 $\mathsf{part}(\mathsf{G}_1) = \mathsf{part}(\mathsf{G}_1\{\mathsf{G}'/\mathbf{t}\})$. This implies

$$(\mu\mathbf{t}'.\mathsf{G}_1\{\mathsf{G}'/\mathbf{t}\}) \upharpoonright \mathsf{p} = \mu\mathbf{t}'.(\mathsf{G}_1\{\mathsf{G}'/\mathbf{t}\}) \upharpoonright \mathsf{p} = \mu\mathbf{t}'.\mathsf{G}_1 \upharpoonright \mathsf{p}\{\mathsf{G}' \upharpoonright \mathsf{p}/\mathbf{t}\} = (\mu\mathbf{t}'.\mathsf{G}_1) \upharpoonright \mathsf{p}\{\mathsf{G}' \upharpoonright \mathsf{p}/\mathbf{t}\}$$

if $\mathsf{p} \in \mathsf{part}(\mathsf{G}_1)$ and $(\mu\mathbf{t}'.\mathsf{G}_1\{\mathsf{G}'/\mathbf{t}\}) \upharpoonright \mathsf{p} = (\mu\mathbf{t}'.\mathsf{G}_1) \upharpoonright \mathsf{p} = \mathsf{End}$ otherwise.

Lemma A.4 (Projection is preserved under folding and unfolding).

1. *If* G *is closed and* $\mathsf{p} \notin \mathsf{part}(\mathsf{G})$*, then* $\mathsf{G} \upharpoonright \mathsf{p} = \mathsf{End}$*.*
2. *If* $\mu\mathbf{t}.\mathsf{G}$ *is a closed and projectable global type, then* $\mathsf{G}\{\mu\mathbf{t}.\mathsf{G}/\mathbf{t}\}$ *is projectable too and they have the same projections modulo fold/unfold.*

Proof (1). By induction on the definition of projection.
(2). By cases on the definition of $\mu\mathbf{t}.\mathsf{G} \upharpoonright \mathsf{p}$. If $\mathbf{t}$ does not occur in G, then $\mathsf{G}\{\mu\mathbf{t}.\mathsf{G}/\mathbf{t}\} = \mathsf{G}$. If $\mathsf{p} \in \mathsf{part}(\mathsf{G})$, then $(\mu\mathbf{t}.\mathsf{G}) \upharpoonright \mathsf{p} = \mu\mathbf{t}.\mathsf{G} \upharpoonright \mathsf{p}$ and by Lemma A.3 $(\mathsf{G}\{\mu\mathbf{t}.\mathsf{G}/\mathbf{t}\}) \upharpoonright \mathsf{p} = \mathsf{G} \upharpoonright \mathsf{p}\,\{(\mu\mathbf{t}.\mathsf{G}) \upharpoonright \mathsf{p}\,/\mathbf{t}\}$. If $\mathbf{t}$ occurs in G and $\mathsf{p} \notin \mathsf{part}(\mathsf{G})$, then $\mathsf{p} \notin \mathsf{part}(\mu\mathbf{t}.\mathsf{G})$ and $\mathsf{p} \notin \mathsf{part}(\mathsf{G}\{\mu\mathbf{t}.\mathsf{G}/\mathbf{t}\})$, so by Point (1) $(\mu\mathbf{t}.\mathsf{G}) \upharpoonright \mathsf{p} = \mathsf{End}$ and $(\mathsf{G}\{\mu\mathbf{t}.\mathsf{G}/\mathbf{t}\}) \upharpoonright \mathsf{p} = \mathsf{End}$.

Theorem A.5. *Let all global types in* Θ *be closed and projectable and define*

$$\Theta \upharpoonright \mathsf{p} = \{(\mathsf{G}_1 \upharpoonright \mathsf{p}, \mathsf{G}_2 \upharpoonright \mathsf{p}) \mid (\mathsf{G}_1, \mathsf{G}_2) \in \Theta\}$$

If G *and* G' *are closed and projectable and* $\Theta \vdash^g \mathsf{G} \sim \mathsf{G}'$*, then* $\Theta \upharpoonright \mathsf{p} \vdash^s \mathsf{G} \upharpoonright \mathsf{p} \sim \mathsf{G}' \upharpoonright \mathsf{p}$*.*

Proof By induction on the derivation of $\Theta \vdash^g \mathsf{G} \sim \mathsf{G}'$. If the last applied rule is $\Theta, (\mathsf{G}, \mathsf{G}') \vdash^g \mathsf{G} \sim \mathsf{G}'$ we can derive $\Theta \upharpoonright \mathsf{p}, (\mathsf{G} \upharpoonright \mathsf{p}, \mathsf{G}' \upharpoonright \mathsf{p}) \vdash^s \mathsf{G} \upharpoonright \mathsf{p} \sim \mathsf{G}' \upharpoonright \mathsf{p}$.

Let the last applied rule be $\dfrac{\Theta \vdash^g \mathsf{G}_i \sim \mathsf{G}_i' \quad \forall i \in I}{\Theta \vdash^g \boxplus_{i\in I}\, \alpha_i;\mathsf{G}_i \sim \boxplus_{i\in I}\, \alpha_i;\mathsf{G}_i'}$ Projectability of G, G' imply projectability of $\mathsf{G}_i, \mathsf{G}_i'$ for $i \in I$. By induction hypothesis

$$\Theta \upharpoonright \mathsf{p} \vdash^s \mathsf{G}_i \upharpoonright \mathsf{p} \sim \mathsf{G}_i' \upharpoonright \mathsf{p}$$

for all $i \in I$. Let $(\boxplus_{i\in I}\, \alpha_i;\mathsf{G}_i) \upharpoonright \mathsf{p} = \bigsqcap_{i\in I}(\, \alpha_i;\mathsf{G}_i\,) \upharpoonright \mathsf{p}$, then $(\boxplus_{i\in I}\, \alpha_i;\mathsf{G}_i') \upharpoonright \mathsf{p} = \bigsqcap_{i\in I}(\, \alpha_i;\mathsf{G}_i'\,) \upharpoonright \mathsf{p}$. We define $J = \{i \in I \mid \alpha_i = \mathsf{q}\Lambda_i\mathsf{p}\}$

$$H = \{i \in I \setminus J \mid \mathsf{G}_i \upharpoonright \mathsf{p} = \mathsf{G}_i' \upharpoonright \mathsf{p} = \mathsf{End}\} \text{ and } K = I \setminus (J \cup H)$$

Notice that H includes all $\mathsf{G}_i \upharpoonright \mathsf{p}$ and $\mathsf{G}_i' \upharpoonright \mathsf{p}$ which are End, since

$$\Theta \upharpoonright \mathsf{p} \vdash^s \mathsf{G}_i \upharpoonright \mathsf{p} \sim \mathsf{G}_i' \upharpoonright \mathsf{p}$$

and $\Delta \vdash^s \mathsf{End} \sim \mathsf{T}$ implies $\mathsf{T} = \mathsf{End}$. Let $\mathsf{G}_i \upharpoonright \mathsf{p} = \bigwedge_{\ell\in L_i} \pi_\ell^{(i)};\mathsf{T}_\ell^{(i)}$ for $i \in K$, then $\Theta \upharpoonright \mathsf{p} \vdash^s \mathsf{G}_i \upharpoonright \mathsf{p} \sim \mathsf{G}_i' \upharpoonright \mathsf{p}$ and Lemma A.1(1) imply $\mathsf{G}_i' \upharpoonright \mathsf{p} = \bigwedge_{\ell\in L_i} \pi_\ell^{(i)};\mathsf{S}_\ell^{(i)}$ and $\Theta \upharpoonright \mathsf{p} \vdash^s \mathsf{T}_\ell^{(i)} \sim \mathsf{S}_\ell^{(i)}$ for $\ell \in L_i$ and $i \in K$. Then by definition of projection $\boxplus_{i\in I}\, \alpha_i;\mathsf{G}_i \upharpoonright \mathsf{p} = (\bigwedge_{i\in J} \mathsf{q}?\Lambda_i;\mathsf{G}_i \upharpoonright \mathsf{p}) \wedge (\bigwedge_{i\in K} \bigwedge_{\ell\in L_i} \pi_\ell^{(i)};\mathsf{T}_\ell^{(i)})$ and $\boxplus_{i\in I}\, \alpha_i;\mathsf{G}_i' \upharpoonright \mathsf{p} = (\bigwedge_{i\in J} \mathsf{q}?\Lambda_i;\mathsf{G}_i' \upharpoonright \mathsf{p}) \wedge (\bigwedge_{i\in K} \bigwedge_{\ell\in L_i} \pi_\ell^{(i)};\mathsf{S}_\ell^{(i)})$. We can then derive:

$$\frac{\Theta \upharpoonright \mathsf{p} \vdash^s \mathsf{G}_i \upharpoonright \mathsf{p} \sim \mathsf{G}_i' \upharpoonright \mathsf{p} \quad \forall i \in J \qquad \Theta \upharpoonright \mathsf{p} \vdash^s \mathsf{T}_\ell^{(i)} \sim \mathsf{S}_\ell^{(i)} \quad \forall \ell \in L_i\ \forall i \in K}{\Theta \upharpoonright \mathsf{p} \vdash^s (\bigwedge_{i\in J} \mathsf{q}?\Lambda_i;\mathsf{G}_i \upharpoonright \mathsf{p}) \wedge (\bigwedge_{i\in K} \bigwedge_{\ell\in L_i} \pi_\ell^{(i)};\mathsf{T}_\ell^{(i)}) \sim (\bigwedge_{i\in J} \mathsf{q}?\Lambda_i;\mathsf{G}_i' \upharpoonright \mathsf{p}) \wedge (\bigwedge_{i\in K} \bigwedge_{\ell\in L_i} \pi_\ell^{(i)};\mathsf{S}_\ell^{(i)})}$$

The proof in the case $\boxplus_{i\in I}\, \alpha_i;\mathsf{G}_i \upharpoonright \mathsf{p} = \bigvee_{i\in I}(\, \alpha_i;\mathsf{G}_i\,) \upharpoonright \mathsf{p}$ is similar and simpler.

Let the last applied rule be $\dfrac{\Theta, (\mu \mathbf{t}.\mathsf{G}, \mathsf{G}') \vdash^{g} \mathsf{G}\{\mu \mathbf{t}.\mathsf{G}/\mathbf{t}\} \sim \mathsf{G}'}{\Theta \vdash^{g} \mu \mathbf{t}.\mathsf{G} \sim \mathsf{G}'}$ Projectability of $\mu \mathbf{t}.\mathsf{G}$ implies projectability of $\mathsf{G}\{\mu \mathbf{t}.\mathsf{G}/\mathbf{t}\}$ by Lemma A.4(2). By induction hypothesis $\Theta \upharpoonright \mathsf{p}, ((\mu \mathbf{t}.\mathsf{G}) \upharpoonright \mathsf{p}, \mathsf{G}' \upharpoonright \mathsf{p}) \vdash^{s} (\mathsf{G}\{\mu \mathbf{t}.\mathsf{G}/\mathbf{t}\}) \upharpoonright \mathsf{p} \sim \mathsf{G}' \upharpoonright \mathsf{p}$. If $\mathsf{p} \notin \mathsf{part}(\mu \mathbf{t}.\mathsf{G}) = \mathsf{part}(\mathsf{G}\{\mu \mathbf{t}.\mathsf{G}/\mathbf{t}\})$, then by Lemma A.4(1) $(\mu \mathbf{t}.\mathsf{G}) \upharpoonright \mathsf{p} = \mathsf{End}$ and $(\mathsf{G}\{\mu \mathbf{t}.\mathsf{G}/\mathbf{t}\}) \upharpoonright \mathsf{p} = \mathsf{End}$. Then $\mathsf{G}' \upharpoonright \mathsf{p} = \mathsf{End}$ and we are done. Otherwise it is easy to check that $(\mu \mathbf{t}.\mathsf{G}) \upharpoonright \mathsf{p} = \mu \mathbf{t}.\mathsf{G} \upharpoonright \mathsf{p}$. By Lemma A.3

$$(\mathsf{G}\{\mu \mathbf{t}.\mathsf{G}/\mathbf{t}\}) \upharpoonright \mathsf{p} = \mathsf{G} \upharpoonright \mathsf{p}\{(\mu \mathbf{t}.\mathsf{G}) \upharpoonright \mathsf{p}/\mathbf{t}\}$$

We can then derive:

$$\frac{\Theta \upharpoonright \mathsf{p}, (\mu \mathbf{t}.\mathsf{G} \upharpoonright \mathsf{p}, \mathsf{G}' \upharpoonright \mathsf{p}) \vdash^{s} \mathsf{G} \upharpoonright \mathsf{p}\{\mu \mathbf{t}.\mathsf{G} \upharpoonright \mathsf{p}/\mathbf{t}\} \sim \mathsf{G}' \upharpoonright \mathsf{p}}{\Theta \upharpoonright \mathsf{p} \vdash^{s} \mu \mathbf{t}.\mathsf{G} \upharpoonright \mathsf{p} \sim \mathsf{G}' \upharpoonright \mathsf{p}}$$

B. Proof of Theorem 6.6

Proof Structurally equivalent networks can be typed by global types generating the same tree, since the projection of global types with the same tree yields session types with the same tree (as shown in Appendix A).

The rest of the proof is done by case analysis on the reduction rules for networks. There are three rules to consider, Rule [Com], Rule [BDel] and Rule [EDel]. Notice each of these rules are applied after zero or more applications of the Rule [Ct].

First we complete the proof for the rule [Com]. As in the proof given after the statement of the theorem $\mathbb{N} = \mathsf{p}[\![P]\!] \parallel \mathsf{q}[\![Q]\!] \parallel \mathbb{N}_0$ and

$$\frac{P \xrightarrow{\mathsf{q}!\Lambda} P' \qquad Q \xrightarrow{\mathsf{p}?\Lambda} Q'}{\mathsf{p}[\![P]\!] \parallel \mathsf{q}[\![Q]\!] \xrightarrow{\mathsf{p}\Lambda\mathsf{q}} \mathsf{p}[\![P']\!] \parallel \mathsf{q}[\![Q']\!]}$$

and $\mathbb{N}' = \mathsf{p}[\![P']\!] \parallel \mathsf{q}[\![Q']\!] \parallel \mathbb{N}_0$. Moreover:
- $P = \oplus_{i \in I} \pi_i; P_i$, where $\pi_j = \mathsf{q}!\Lambda$ and $P' = P_j$ for some $j \in I$;
- $Q = \Sigma_{h \in H} \pi'_h; Q_h$, where $\pi'_k = \mathsf{p}?\Lambda$ and $Q' = Q_k$ for some $k \in H$;
- $\vdash P : \mathsf{T}$ and $\vdash Q : \mathsf{S}$ for some T and S;
- $\mathsf{T} = \bigvee_{i \in I} \pi_i; \mathsf{T}_i$ and $\vdash P_i : \mathsf{T}_i$ for $i \in I$ and $\mathsf{S} = \bigwedge_{h \in H} \pi'_h; \mathsf{S}_h$ and $\vdash Q_h : \mathsf{S}_h$ for $h \in H$.

Neither p nor q may be the deputy of some open delegation, since, by Rule [BDel] of Figure 3, a deputy takes the name of the principal associated to its process during delegation (and no process is associated to its name). By Lemma 6.1(7) and the definition of projection there are four cases:

1. $\mathsf{T} \leq \mathsf{G} \upharpoonright \mathsf{p}$ and $\mathsf{S} \leq \mathsf{G} \upharpoonright \mathsf{q}$, i.e. both p and q are not principals;
2. $\mathsf{T} \leq \mathsf{G} \upharpoonright \mathsf{p}$ and $\mathsf{S} \leq \mathsf{G} \upharpoonright_2 (\mathsf{q}, \mathsf{r})$, i.e. p is not a principal but q is the principal of r;
3. $\mathsf{T} \leq \mathsf{G} \upharpoonright_2 (\mathsf{p}, \mathsf{s})$ and $\mathsf{S} \leq \mathsf{G} \upharpoonright \mathsf{q}$, i.e. p is the principal of s and q is not a principal;

4. $\mathsf{T} \leq \mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{s})$ and $\mathsf{S} \leq \mathsf{G}\upharpoonright_2(\mathsf{q},\mathsf{r})$, i.e. both p and q are principals, of s and r, respectively.

In the first case, by definition of $\leq$, $\mathsf{G}\upharpoonright\mathsf{p} = \bigvee_{i\in I'} \pi_i;\mathsf{T}'_i$ with $I \subseteq I'$ and $\mathsf{T}_i \leq \mathsf{T}'_i$ for $i \in I$. By Lemma 6.4(1) $\mathsf{G} = \phi_1;\ldots;\phi_n;\boxplus_{i\in I'}\alpha_i^{\mathsf{p}};\mathsf{G}_i$, where ϕ_l for $1 \leq l \leq n$ is an atomic interaction not involving p and $(\alpha_i^{\mathsf{p}};\mathsf{G}_i)\upharpoonright\mathsf{p} = \pi_i;\mathsf{T}'_i$ for $i \in I'$. From $\pi_j = \mathsf{q}!\Lambda$ we get $\alpha_j^{\mathsf{p}} = \mathsf{p}\Lambda\mathsf{q}$. By definition of $\leq$, $\mathsf{G}\upharpoonright\mathsf{q} = \bigwedge_{h\in H'}\pi'_h;\mathsf{S}'_h$ with $H' \subseteq H$ and $\mathsf{S}_h \leq \mathsf{S}'_h$ for $h \in H'$. From $\pi'_k = \mathsf{p}?\Lambda$ and $\alpha_j^{\mathsf{p}} = \mathsf{p}\Lambda\mathsf{q}$ it follows that $(\alpha_j^{\mathsf{p}};\mathsf{G}_j)\upharpoonright\mathsf{q} = \pi'_k;\mathsf{S}'_k$ and that ϕ_l for $1 \leq l \leq n$ is an atomic interaction not involving q. We can then choose $\mathsf{G}' = \phi_1;\ldots;\phi_n;\mathsf{G}_j$.
We prove that $\vdash \mathsf{p}[\![\, P \,]\!] \parallel \mathsf{q}[\![\, Q \,]\!] \parallel \mathbb{N}_0 : \mathsf{G}$ implies $\vdash \mathsf{p}[\![\, P' \,]\!] \parallel \mathsf{q}[\![\, Q' \,]\!] \parallel \mathbb{N}_0 : \mathsf{G}'$.
By definition of projection $\mathsf{G}'\upharpoonright\mathsf{p} = \mathsf{T}'_j$ and $\mathsf{G}'\upharpoonright\mathsf{q} = \mathsf{S}'_k$. So $\vdash P' : \mathsf{T}_j$ and $\vdash Q' : \mathsf{S}_k$, where $\mathsf{T}_j \leq \mathsf{G}'\upharpoonright\mathsf{p}$ and $\mathsf{S}_k \leq \mathsf{G}'\upharpoonright\mathsf{q}$. From Lemma 6.5(2) we get that for all r,s such that $\{\mathsf{r},\mathsf{s}\}\cap\{\mathsf{p},\mathsf{q}\} = \emptyset$, if $\mathsf{G}\upharpoonright_1(\mathsf{r},\mathsf{s})$ is defined then $\mathsf{G}\upharpoonright_1(\mathsf{r},\mathsf{s}) \leq \mathsf{G}'\upharpoonright_1(\mathsf{r},\mathsf{s})$. So, letting R be the process associated with participant r, if $\vdash R : \mathsf{T}_\mathsf{r}$ and $\mathsf{T}_\mathsf{r} \leq \mathsf{G}\upharpoonright_1(\mathsf{r},\mathsf{s})$, then $\mathsf{T}_\mathsf{r} \leq \mathsf{G}'\upharpoonright_1(\mathsf{r},\mathsf{s})$. A similar argument may be used for $\mathsf{G}\upharpoonright_2(\mathsf{r},\mathsf{s})$ and $\mathsf{G}\upharpoonright\mathsf{r}$. Therefore applying rule [T-NET] we can derive $\vdash \mathsf{p}[\![\, P' \,]\!] \parallel \mathsf{q}[\![\, Q' \,]\!] \parallel \mathbb{N}_0 : \mathsf{G}'$.

The proof of the second case is given after the statement of the theorem. The proofs of remaining two cases are similar to that of the second case.

Consider now rule [BDEL]. Then $\mathbb{N} = \mathsf{p}[\![\, \circ\langle\!\langle\bullet\mathsf{q};P \,]\!] \parallel \mathsf{q}[\![\, \mathsf{p}\circ\langle\!\langle\bullet;Q \,]\!] \parallel \mathbb{N}_0$ and

$$\mathsf{p}[\![\, \circ\langle\!\langle\bullet\mathsf{q};P \,]\!] \parallel \mathsf{q}[\![\, \mathsf{p}\circ\langle\!\langle\bullet;Q \,]\!] \xrightarrow{\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q}} \overset{*}{\mathsf{p}}[\![\, P \,]\!] \parallel \mathsf{p}[\![\, Q \,]\!]$$

and $\mathbb{N}' = \overset{*}{\mathsf{p}}[\![\, P \,]\!] \parallel \mathsf{p}[\![\, Q \,]\!] \parallel \mathbb{N}_0$.
By Lemma 6.1(7c) we have $\vdash \circ\langle\!\langle\bullet\mathsf{q};P : \mathsf{T}$ and $\vdash \mathsf{p}\circ\langle\!\langle\bullet;Q : \mathsf{S}$ for some T and S such that $\mathsf{T} \leq \mathsf{G}\upharpoonright\mathsf{p}$ and $\mathsf{S} \leq \mathsf{G}\upharpoonright\mathsf{q}$.
Then by Lemma 6.1(3) we get $\mathsf{T} = \circ\langle\!\langle\bullet\mathsf{q};\mathsf{T}'$ where $\vdash P : \mathsf{T}'$ and $\mathsf{S} = \mathsf{p}\circ\langle\!\langle\bullet;\mathsf{S}'$ where $\vdash Q : \mathsf{S}'$. By definition of $\leq$ it must be $\mathsf{G}\upharpoonright\mathsf{p} = \circ\langle\!\langle\bullet\mathsf{q};\mathsf{T}''$ and $\mathsf{G}\upharpoonright\mathsf{q} = \mathsf{p}\circ\langle\!\langle\bullet;\mathsf{S}''$ where $\mathsf{T}' \leq \mathsf{T}''$ and $\mathsf{S}' \leq \mathsf{S}''$. By Lemma 6.4(2) $\mathsf{G} = \phi_1;\ldots;\phi_n;\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q};\mathsf{G}_0$, where ϕ_l for $1 \leq l \leq n$ is an atomic interaction not involving p and q. Note that $\mathsf{T}'' = \mathsf{G}_0\upharpoonright_1(\mathsf{p},\mathsf{q})$ and $\mathsf{S}'' = \mathsf{G}_0\upharpoonright_2(\mathsf{p},\mathsf{q})$. We can then choose $\mathsf{G}' = \phi_1;\ldots;\phi_n;\mathsf{G}_0$. In fact we get by definition of projection $\mathsf{G}'\upharpoonright_1(\mathsf{p},\mathsf{q}) = \mathsf{T}''$ and $\mathsf{G}'\upharpoonright_2(\mathsf{p},\mathsf{q}) = \mathsf{S}''$. Moreover, by Lemma 6.5(1) we have $\mathsf{G}\upharpoonright\mathsf{r} = \mathsf{G}'\upharpoonright\mathsf{r}$, $\mathsf{G}\upharpoonright_1(\mathsf{r},\mathsf{s}) = \mathsf{G}'\upharpoonright_1(\mathsf{r},\mathsf{s})$, $\mathsf{G}\upharpoonright_2(\mathsf{r},\mathsf{s}) = \mathsf{G}'\upharpoonright_2(\mathsf{r},\mathsf{s})$ for all r,s such that $\{\mathsf{p},\mathsf{q}\}\cap\{\mathsf{r},\mathsf{s}\} = \emptyset$. Therefore we may apply rule [T-NET] to derive $\vdash \overset{*}{\mathsf{p}}[\![\, P \,]\!] \parallel \mathsf{p}[\![\, Q \,]\!] \parallel \mathbb{N}_0 : \mathsf{G}'$.

Let us consider rule [EDEL], then $\mathbb{N} = \overset{*}{\mathsf{p}}[\![\, \mathsf{q}\bullet\rangle\!\rangle\circ;P \,]\!] \parallel \mathsf{p}[\![\, \bullet\rangle\!\rangle\circ\mathsf{p};Q \,]\!] \parallel \mathbb{N}_0$ and

$$\overset{*}{\mathsf{p}}[\![\, \mathsf{q}\bullet\rangle\!\rangle\circ;P \,]\!] \parallel \mathsf{p}[\![\, \bullet\rangle\!\rangle\circ\mathsf{p};Q \,]\!] \xrightarrow{\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p}} \mathsf{p}[\![\, P \,]\!] \parallel \mathsf{q}[\![\, Q \,]\!]$$

and $\mathbb{N}' = \mathsf{p}[\![\, P \,]\!] \parallel \mathsf{q}[\![\, Q \,]\!] \parallel \mathbb{N}_0$.
By Lemma 6.1(7) $\vdash \mathsf{q}\bullet\rangle\!\rangle\circ;P : \mathsf{T}$ and $\vdash \bullet\rangle\!\rangle\circ\mathsf{p};Q : \mathsf{S}$ for some T and S.
By Lemma 6.1(3) $\mathsf{T} = \mathsf{q}\bullet\rangle\!\rangle\circ;\mathsf{T}'$ and $\vdash P : \mathsf{T}'$ and $\mathsf{S} = \bullet\rangle\!\rangle\circ\mathsf{p};\mathsf{S}'$ and $\vdash Q : \mathsf{S}'$.

By Lemma 6.1(7b) and the fact that only delegation projection $\mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q})$ and $\mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{q})$ generate types starting with $\mathsf{q}\bullet\rangle\!\rangle\circ$ and $\bullet\rangle\!\rangle\circ\mathsf{p}$, respectively, we derive that $\mathsf{T} \leq \mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q})$ and $\mathsf{S} \leq \mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{q})$. This implies $\mathsf{G}\upharpoonright_1(\mathsf{p},\mathsf{q}) = \mathsf{q}\bullet\rangle\!\rangle\circ;\mathsf{T}''$ and $\mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{q}) = \bullet\rangle\!\rangle\circ\mathsf{p};\mathsf{S}''$, where $\mathsf{T}' \leq \mathsf{T}''$ and $\mathsf{S}' \leq \mathsf{S}''$, by definition of $\leq$. By Lemma 6.4(3) $\mathsf{G} = \phi_1;\ldots;\phi_n;\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p};\mathsf{G}_0$, where ϕ_l for $1 \leq l \leq n$ is an atomic interaction not involving p and q. We can then choose $\mathsf{G}' = \phi_1;\ldots;\phi_n;\mathsf{G}_0$. In fact we get by definition of projection $\mathsf{G}'\upharpoonright\mathsf{p} = \mathsf{T}''$ and $\mathsf{G}'\upharpoonright\mathsf{q} = \mathsf{S}''$. Moreover, by Lemma 6.5(1) $\mathsf{G}\upharpoonright\mathsf{r} = \mathsf{G}'\upharpoonright\mathsf{r}$, $\mathsf{G}\upharpoonright_1(\mathsf{r},\mathsf{s}) = \mathsf{G}'\upharpoonright_1(\mathsf{r},\mathsf{s})$, $\mathsf{G}\upharpoonright_2(\mathsf{r},\mathsf{s}) = \mathsf{G}'\upharpoonright_2(\mathsf{r},\mathsf{s})$ for all r,s such that $\{\mathsf{p},\mathsf{q}\}\cap\{\mathsf{r},\mathsf{s}\} = \emptyset$. Therefore applying rule [T-Net] we can derive $\vdash \mathsf{p}[\![\, P \,]\!] \parallel \mathsf{q}[\![\, Q \,]\!] \parallel \mathbb{N}_0 : \mathsf{G}'$.

C. Proof of Theorem 6.7

Proof (1). By Lemma 6.3(3) $\mathbb{N} \xrightarrow{\mathsf{p}\Lambda\mathsf{q}} \mathbb{N}'$ implies $\mathbb{N} = \mathsf{p}[\![\, P \,]\!] \parallel \mathsf{q}[\![\, Q \,]\!] \parallel \mathbb{N}_0$ and $P \xrightarrow{\mathsf{q}!\Lambda} P'$ and $Q \xrightarrow{\mathsf{p}?\Lambda} Q'$ and $\mathbb{N}' = \mathsf{p}[\![\, P' \,]\!] \parallel \mathsf{q}[\![\, Q' \,]\!] \parallel \mathbb{N}_0$. As in the proof of Theorem 6.6 we obtain $\mathsf{G} = \phi_1;\ldots;\phi_n;\boxplus_{i\in I}\alpha_i^{\mathsf{p}};\mathsf{G}_i$, where $\alpha_j^{\mathsf{p}} = \mathsf{p}\Lambda\mathsf{q}$ for some $j \in I$ and $\vdash \mathbb{N}' : \phi_1;\ldots;\phi_n;\mathsf{G}_j$ and ϕ_h for $1 \leq h \leq n$ is an atomic interaction involving neither p nor q.

(2). By Lemma 6.3(4) $\mathbb{N} \xrightarrow{\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q}} \mathbb{N}'$ implies $\mathbb{N} = \mathsf{p}[\![\, \circ\langle\!\langle\bullet\mathsf{q};P \,]\!] \parallel \mathsf{q}[\![\, \mathsf{p}\circ\langle\!\langle\bullet;Q \,]\!] \parallel \mathbb{N}_0$ and $\mathbb{N}' \overset{\ast}{=} \mathsf{p}[\![\, P \,]\!] \parallel \mathsf{p}[\![\, Q \,]\!] \parallel \mathbb{N}_0$. As in the proof of Theorem 6.6 we obtain $\mathsf{G} = \phi_1;\ldots;\phi_n;\mathsf{p}\circ\langle\!\langle\bullet\mathsf{q};\mathsf{G}'$and $\vdash \mathbb{N}' : \phi_1;\ldots;\phi_n;\mathsf{G}'$, where ϕ_i for $1 \leq i \leq n$ is an atomic interaction involving neither p nor q.

(3). By Lemma 6.3(5) $\mathbb{N} \xrightarrow{\mathsf{q}\bullet\rangle\circ\mathsf{p}} \mathbb{N}'$ implies $\mathbb{N} = \mathsf{p}[\![\, \mathsf{q}\bullet\rangle\!\rangle\circ;P \,]\!] \parallel \mathsf{p}[\![\, \bullet\rangle\!\rangle\circ\mathsf{p};Q \,]\!] \parallel \mathbb{N}_0$ and $\mathbb{N}' = \mathsf{p}[\![\, P \,]\!] \parallel \mathsf{q}[\![\, Q \,]\!] \parallel \mathbb{N}_0$. As in the proof of Theorem 6.6 we obtain $\mathsf{G} = \phi_1;\ldots;\phi_n;\mathsf{q}\bullet\rangle\!\rangle\circ\mathsf{p};\mathsf{G}'$ and $\vdash \mathbb{N}' : \phi_1;\ldots;\phi_n;\mathsf{G}'$, where ϕ_i for $1 \leq i \leq n$ is an atomic interaction involving neither p nor q.

(4). Let $\mathsf{G} = \boxplus_{i\in I}\mathsf{p}\Lambda_i\mathsf{q}_i;\mathsf{G}_i$. By Lemma 6.2(7b) or (7c)

$$\mathbb{N} = \mathsf{p}[\![\, P \,]\!] \parallel \Pi_{i\in I}\mathsf{q}_i[\![\, Q_i \,]\!] \parallel \mathbb{N}_0$$

and $\vdash P : \mathsf{T}$ with either $\mathsf{T} \leq \mathsf{G}\upharpoonright\mathsf{p}$ or $\mathsf{T} \leq \mathsf{G}\upharpoonright_2(\mathsf{p},\mathsf{r})$ and $\vdash Q_i : \mathsf{S}_i$ with either $\mathsf{S}_i \leq \mathsf{G}\upharpoonright\mathsf{q}_i$ or $\mathsf{S}_i \leq \mathsf{G}\upharpoonright_2(\mathsf{q}_i,\mathsf{s}_i)$ for $i \in I$. In both cases by definition of $\leq$ and projection $\mathsf{T} = \bigvee_{i\in I'}\mathsf{q}_i!\Lambda_i;\mathsf{T}_i$ with $I' \subseteq I$ and $\mathsf{S}_i = \mathsf{p}?\Lambda_i;\mathsf{S}_i' \wedge \mathsf{S}_i''$ for $i \in I$. Lemma 6.2(2) gives $P = \oplus_{i\in I'}\mathsf{q}_i!\Lambda_i;P_i$. Lemma 6.2(1) gives $Q_i = \mathsf{p}?\Lambda_i;Q_i' + Q_i''$ for $i \in I$. By the reduction rules [Com] and [Ct] $\mathbb{N} \xrightarrow{\mathsf{p}\Lambda_i\mathsf{q}_i} \mathsf{p}[\![\, P_i \,]\!] \parallel \mathsf{q}_i[\![\, Q_i' \,]\!] \parallel \Pi_{j\in I\setminus\{i\}}\mathsf{q}_j[\![\, Q_j \,]\!] \parallel \mathbb{N}_0$ for all $i \in I'$. As in the proof of Subject Reduction we can show that

$$\vdash \mathsf{p}[\![\, P_i \,]\!] \parallel \mathsf{q}_i[\![\, Q_i' \,]\!] \parallel \Pi_{j\in I\setminus\{i\}}\mathsf{q}_j[\![\, Q_j \,]\!] \parallel \mathbb{N}_0 : \mathsf{G}_i$$

for all $i \in I'$.

(5). If ϕ is a communication the statement is a particular case of the previous point.
Let $\phi = \mathsf{p}\circ\langle\!\langle\bullet\mathsf{q}$. By Lemma 6.2(7c) and the definition of projection

$$\mathbb{N} = \mathsf{p}[\![\, P \,]\!] \parallel \mathsf{q}[\![\, Q \,]\!] \parallel \mathbb{N}_0$$

and $\vdash P : \mathsf{T}$ with $\mathsf{T} \leq \mathsf{G} \upharpoonright \mathsf{p}$ and $\vdash Q : \mathsf{S}$ with $\mathsf{S} \leq \mathsf{G} \upharpoonright \mathsf{q}$ by definition of projection. By definition of $\leq$ we get $\mathsf{T} = \circ\langle\!\langle \bullet \mathsf{q}; \mathsf{T}'$ and $\mathsf{S} = \mathsf{p} \circ \langle\!\langle \bullet; \mathsf{S}'$. Lemma 6.2(3) implies $P = \circ\langle\!\langle \bullet \mathsf{q}; P'$ and $Q = \mathsf{p} \circ \langle\!\langle \bullet; Q'$. By applying the reduction rules [BDEL] and [CT] $\mathbb{N} \xrightarrow{\mathsf{p} \circ \langle \bullet \mathsf{q}} \overset{*}{\mathsf{p}}[\![P']\!] \parallel \mathsf{p}[\![Q']\!] \parallel \mathbb{N}_0$. As in the proof of Subject Reduction we can show that $\vdash \overset{*}{\mathsf{p}}[\![P']\!] \parallel \mathsf{p}[\![Q']\!] \parallel \mathbb{N}_0 : \mathsf{G}$.
If $\phi = \mathsf{p} \bullet \rangle\!\rangle \circ \mathsf{q}$ the proof is similar.