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Bisimulations and Abstraction Homomorphisms

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We show that the notion of bisimulation equivalence for a class of labelled transition systems (the class of nondeterministic processes) may be restated as one of "reducibility to a same system" via a simple reduction relation. This relation is proved to enjoy some desirable properties, notably the Church–Rosser property. We also show that, when restricted to finite nondeterministic processes, the relation yields unique minimal forms for processes and can be characterised algebraically by a set of reduction rules. © 1987 Academic Press, Inc.

1. INTRODUCTION

Labelled transition systems [K, P] are generally recognised as an appropriate model for nondeterministic computations. The motivation for studying such computations stems from the increasing interest in concurrent programming.

When modelling communication between concurrent programs, some basic difficulties have to be faced. A concurrent program is inherently part of a larger environment, with which it interacts *in the course of* its computation. Therefore a simple input–output function is not an adequate model. The model should retain some information about the internal states of a program, so as to be able to express the program's behaviour in any interacting environment. Also, *nondeterminacy* arises when abstracting from such parameters as the relative speeds of concurrent programs: as a consequence, we need to regard any concurrent program as being nondeterministic.

The question is then to find a model for nondeterministic programs that somehow accounts for intermediate states. On the other hand, only those intermediate states should be considered which are relevant to the "interactive" behaviour of the program. Now one can think of various criteria for selecting such significant states.

In this respect labelled transition systems provide a very flexible model: by varying the definition of the transition relation one obtains a whole range of different descriptions, going from a full account of the structure of a program to some more interesting "abstract" descriptions. However, even these abstract descriptions still need to be factored by equivalence relations (for a review see [B or DeN]).

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A natural notion of equivalence, *bisimulation equivalence*, has been recently proposed by Park [Pa] for transition systems: informally speaking, two systems are said to *bisimulate* each other if a full correspondence can be established between their sets of states in such a way that from any two corresponding states the two (sub)systems will still bisimulate each other.

In this paper we show that the notion of bisimulation equivalence for a class of labelled transition systems (the class of *nondeterministic processes*) may be restated as one of "reducibility to a same system" via a simple reduction relation. This relation is proved to enjoy some desirable properties, notably the Church–Rosser property. We also show that, when restricted to finite nondeterministic processes, the relation yields unique minimal forms for processes and can be characterised algebraically by a set of reduction rules.

The paper is organised as follows. In Section 2 we present our computational model, the class of *nondeterministic processes*. In Section 3 we argue that this basic model is not abstract enough, particularly when systems are allowed unobservable transitions as well as observable ones. We therefore introduce *abstraction homomorphisms* [CFM] as a means of simplifying the structure of a process by merging together some of its states: the result is a process with a simpler description, but "abstractly equivalent" to the original one. We can then infer a *reduction relation* between processes from the existence of abstraction homomorphisms between them. We prove some significant properties of this relation, such as substitutivity under typical operators and the Church–Rosser property. Based on the reduction relation, we define an *abstraction equivalence* relation on processes: two processes are equivalent iff they are reducible to a same process.

In Sections 4 and 5 we study the relationship between our notions of reduction and abstraction and the notion of *bisimulation* between transition systems. The criterion we use for identifying states of a process via abstraction homomorphisms is similar to the one underlying the definition of bisimulation: we show in fact that our abstraction equivalence coincides with (the *substitutive* version of) *bisimulation equivalence*, and can therefore be used as a simple alternative formulation for it.

In Section 6 we consider a small *language* for defining *finite* nondeterministic processes: essentially a subset of Milner's CCS (Calculus of Communicating Systems) [M1]. We find that our results combine neatly with some established facts about the language. On this language our equivalence is just Milner's *observational congruence*, for which a complete finite axiomatisation has been given in [HM]. So, on the one hand, we get a ready-made algebraic characterisation for abstraction equivalence; on the other hand, our characterisation proves helpful in working out a complete system of *reduction rules* for that language. We conclude by proposing a denotational *tree-model* for the language, which is isomorphic to the term-model in [HM]. The present paper is an extended version of [C], complete with proofs.

2. Nondeterministic Systems

In this section we introduce our basic computational model, the class of *nondeterministic systems*. Nondeterministic systems are essentially labelled transition systems with an initial state.

We then characterise a subclass of acyclic systems that we call *nondeterministic processes*: in the remaining sections we shall be mainly concerned with this subclass.

Let A be a set of elementary *actions* or *transitions*, containing a distinguished *unobservable transition* τ. We will use $\mu, \nu,\ldots$ to range over A, and $a, b,\ldots$ to range over $A - \{\tau\}$.

Definition 2.1. A *nondeterministic system* (NDS) over A is a triple $S = (Q \cup \{r\}, A, \rightarrow)$, where $Q \cup \{r\}$ is the set of states of S, $r \notin Q$ is the *initial state* (or root) of S, and $\rightarrow \subseteq [(Q \cup \{r\}) \times A \times (Q \cup \{r\})]$ is the *transition relation* on S.

We will use q, q' to range over $Q \cup \{r\}$, and write $q \xrightarrow{\mu} q'$ for $(q, \mu, q') \in \rightarrow$. We interpret $q \xrightarrow{\mu} q'$ as: S may evolve from state q to state q' via a transition μ.

We will make use of the transitive and reflexive closure $\rightarrow^*$ of $\rightarrow$, which we call the *derivation relation* on S. For an NDS, $S = (Q \cup \{r\}, A, \rightarrow)$, we will use Q_s, r_s, $\rightarrow_s$ instead of Q, r, $\rightarrow$, whenever an explicit reference to S is required.

According to our definition, and NDS S is a machine starting in some definite state and evolving through states by means of elementary transitions. On the other hand, each state of S may be thought of as the initial state of some NDS: We may then regard the system S as giving rise to new systems, rather than going through successive states.

In fact, the whole class $\mathscr{S}$ of NDSs may be described as a transition system (whose states are NDSs). We then say that S' is a *derivative* of S whenever $S \rightarrow^* S'$. Thus for any $S \in \mathscr{S}$, a one-to-one correspondence can be established between the states and the derivatives of S. In the following we will often use this correspondence between states and (sub)systems.

We assume the class $\mathscr{S}$ to be closed w.r.t. some simple *operators*: a nullary operator NIL, a set of unary operators μ (one for each $\mu \in A$), and a binary operator $+$. The intended meaning of these operators is the following: NIL represents *termination*, $+$ is a *free-choice* operator, and the μ's provide a simple form of sequentialisation, called *prefixing* by the action μ.

The transition relation of a compound NDS may be inferred from those of the components by means of the rules:

(i) $\mu S \xrightarrow{\mu} S$

(ii) $S \xrightarrow{\mu} S'$ implies $S + S'' \xrightarrow{\mu} S'$, $S'' + S \xrightarrow{\mu} S'$.

The operators will be given a precise definition for a subclass of $\mathscr{S}$, the class of *nondeterministic processes* that we will introduce in the next section.

2.1. *Nondeterministic Processes*

As they are, NDSs have an isomorphic representation as (rooted) *labelled directed graphs*, whose nodes and arcs represent respectively the states and the transitions of a system. On the other hand, any NDS may be unfolded into an *acyclic* graph. We shall here concentrate on a class of acyclic NDSs that we call *nondeterministic processes* (NPDs).

Basically, NDPs are NDSs whose derivation relation $\to *$ is a *partial ordering*. Each state of a process is assigned a *label*, that represents the sequence of observable actions leading from the root to that state. To make such a labelling consistent, we only allow two paths to join in the graph if they correspond to the same observable derivation sequence. The labelling is subject to the following further restriction: for any label σ, there are at most finitely many states labelled by σ. As it will be made clear subsequently, this amounts to imposing a general *image-finiteness* condition on the systems, and is a crucial hypothesis for some of our results.

In the formal definition, we will use the following notation: A^* is the set of finite sequences over A, with the usual prefix ordering, and with an empty sequence ε. The *covering relation* $-\!\!\!\subset$ associated to a partial ordering $\leqslant$ is given by: $x -\!\!\!\subset y$ iff $x < y$ and $\nexists z$ such that $x < z < y$. Also, we make the following convention: τ acts as the identity over A^* and is therefore replaced by ε when occurring in strings.

Definition 2.2. A *nondeterministic process* (NDP) over A is a triple $P = (Q \cup \{r\}, \leqslant, l)$, where

$(Q \cup \{r\}), \leqslant)$ *is a rooted poset of states*: $\forall q, r \leqslant q$

$l\colon Q \cup \{r\} \to A^*$ *is a monotonic labelling function, satisfying*:

$l(r) = \varepsilon$

$q -\!\!\!\subset q'$ implies $l(q') = l(q) \cdot \mu, \qquad \mu \in A$

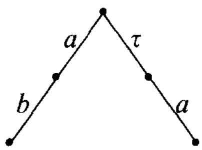
$\forall \sigma \in A^*, \qquad \{q \mid l(q) = \sigma\}$ is *finite*.

Note that an NDP is very nearly a *labelled tree*: it only differs from a labelled tree in that it might have some confluent paths. The reason we do not directly adopt labelled trees as a model is purely technical (the proof that the model is closed w.r.t. reductions would be rather tricky). However, we intend that trees are our real object of interest: in particular, our examples will always be chosen from trees.

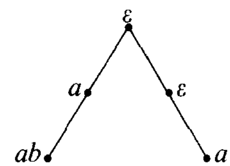
As pointed out already, we label nodes with sequences of actions, rather than labelling arcs with single actions: this minor variation w.r.t. the standard notation (see, e.g., Milner's *synchronisation trees*) will make it easier to compare different states of a process.

It is easy to see that any NDP P is also an NDS, with $\to_P$ given by $-\!\!\!\subset$. More precisely, for any $\mu \in A$, the relation $\xrightarrow{\mu}_P$ will be given by $\{(q, q') \mid q -\!\!\!\subset q'$ and $l(q') = l(q) \cdot \mu\}$.

Note that, because of our convention that $\tau = \varepsilon$, a transition τ is represented in an NDP by the repetition of the same label on the two $\xrightarrow{\tau}$ related nodes. More generally, the label of a node will now represent the sequence of *observable* actions leading to it. For example, the tree



in our notation will be



In what follows, nondeterministic processes will always be considered up to isomorphism.

DEFINITION 2.3. An *isomorphism* between two NDPs: $P_1 = (Q_1 \cup \{r_1\}, \leqslant_1, l_1)$ and $P_2 = (Q_2 \cup \{r_2\}, \leqslant_2, l_2)$ is a one-to-one correspondence: $\Phi: Q_1 \cup \{r_1\} \to Q_2 \cup \{r_2\}$ s.t.

(i) $l_2(\Phi(q)) = l_1(q)$

(ii) $\Phi(q) \leqslant_2 \Phi(q')$ iff $q \leqslant_1 q'$.

From now on we shall use: $P_1 = P_2$ to mean that P_1 is isomorphic to P_2.

We next define the operators NIL, μ, and $+$ on NDPs. Let P_i denote the NDP $(Q_1 \cup \{r_i\}, \leqslant_i, l_i)$. We have

DEFINITION 2.4. (Operators on NDPs).

$\mathrm{NIL} = (\{r_{\mathrm{NIL}}\}, \{(r_{\mathrm{NIL}}, r_{\mathrm{NIL}})\}, \{(r_{\mathrm{NIL}}, \varepsilon)\})$ is the NDP with just a root r_{NIL} and an empty set of subsequent states

μP_1 is the NDP $P = (Q \cup \{r\}, \leqslant, l)$, where r does not occur in $Q_1 \cup \{r_1\}$, and:

$$Q = Q_1 \cup \{r_1\}$$
$$\leqslant \; = \; \leqslant_1 \cup \{(r, q) | q \in Q\}$$
$$l(q) = \begin{cases} \varepsilon & \text{if } q = r, \\ \mu \cdot l_1(q) & \text{otherwise.} \end{cases}$$

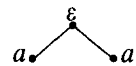
$P_1 + P_2$ is the NDP $P = (Q \cup \{r\}, \leqslant, l)$, where r does not occur in $Q_1 \cup Q_2$, and:

$$Q = Q_1 \cup Q_2 \qquad \text{(disjoint union)}$$
$$\leqslant \; = \; \leqslant_1 \upharpoonright Q_1 \cup \leqslant_2 \upharpoonright Q_2 \cup \{(r, q) | q \in Q\}$$
$$l = l_1 \upharpoonright Q_1 \cup l_2 \upharpoonright Q_2 \cup \{(r, \varepsilon)\}.$$

Let $\mathscr{P} \subseteq \mathscr{S}$ denote the class of all NDPs: in what follows, our treatment of nondeterministic systems will be confined to $\mathscr{P}$.

3. Abstraction Homomorphisms

The NDP-model, though providing a helpful conceptual simplification, does not appear abstract enough yet. It still allows, for example, for structural redundancies such as



Moreover, we want to be able, in most cases, to ignore *unobservable* transitions. Such transitions, being internal to a system, should only be detectable indirectly, on account of their capacity of affecting the *observable behaviour* of the system.

We will therefore introduce a *simplification* operation on processes, which we call *abstraction homomorphism*. Essentially an abstraction homomorphism will transform a process in a structurally simpler (but semantically equivalent) process by merging together some of its states.

The criterion for identifying states is that they be *equivalent* in some recursive sense: informally speaking, two states will be equivalent iff they have *equivalent histories* (derivation sequences) and *equivalent futures* or potentials (sets of subsequent states).

Definition 3.1. For any two NDPs $P_1 = (Q_1 \cup \{r_1\}, \leqslant_1, l_1)$, $P_2 = (Q_2 \cup \{r_2\}, \leqslant_2, l_2)$, a function

$$h\colon \begin{cases} r_1 \mapsto r_2 \\ Q_1 \to Q_2 \end{cases}$$

is an *abstraction homomorphism* (a.h.) from P_1 to P_2 iff:

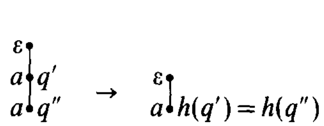
(i) $l_2(h(q)) = l_1(q)$

(ii) $\mathrm{succ}_2(h(q)) = h(\mathrm{succ}_1(q))$,

where $\mathrm{succ}(q) = \{q' \mid q \leqslant q'\}$ is the set of successors of q, *inclusive* of q.

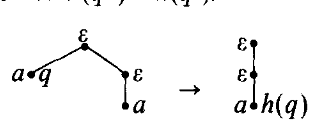
Before giving examples, we shall just remark that any a.h. is *surjective* (instance of (ii), for $q = r$) and preserves the ordering $\leqslant$ (again by (ii)).

Examples. (1)



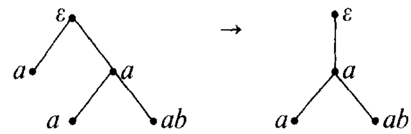
This example motivates our definition of succ (q): we want to allow q'', a *proper* successor of q', to be mapped to $h(q'') = h(q')$.

(2)

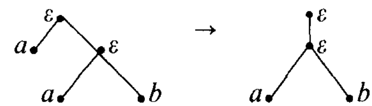


Note that the set of predecessors of q is *not* preserved by the homomorphism.

(3)



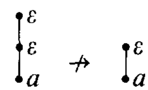
(4)



These two examples show that the relation $-\!\!\subset$ is not necessarily preserved.

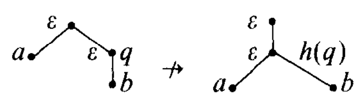
COUNTEREXAMPLES.

(5)



This is not an a.h., since by definition an a.h. never maps a proper state into a root. As a consequence, a process of the form τP can only be transformed into a process of the same form.

(6)



This is not an a.h. because it would increase the set of successors of q.

We pointed out earlier that any a.h. is surjective. In fact, it is the case that:

PROPOSITION 3.2. *Any* injective a.h. *from P_1 to P_2 is an* isomorphism *between P_1 and P_2 and vice versa.*

Proof. Let h be an injective a.h. from P_1 to P_2. To prove that h is an isomorphism between P_1 and P_2, it is sufficient to show that

$$h(q) \leqslant_2 h(q) \qquad \text{only if} \quad q \leqslant_1 q'$$

as the other properties are trivially implied by those of a.h.s.

So suppose $h(q) \leqslant_2 h(q')$, i.e., $h(q') \in \mathrm{succ}(h(q))$. Then also $h(q') \in h(\mathrm{succ}(q))$, by property (ii) of h. Therefore $\exists q'' \in \mathrm{succ}(q)$ s.t. $h(q'') = h(q')$. Since h is injective, it can only be $q'' = q'$, whence $q' \in \mathrm{succ}(q)$, i.e., $q \leqslant_1 q'$.

Conversely, let Φ be an isomorphism between P_1 and P_2. We want to show that Φ is an a.h. from P_1 to P_2 (equivalently, we could show that Φ is an a.h. from P_2 to P_1). All we have to prove is that Φ satisfies property (ii) of a.h.s, namely that $\Phi(\mathrm{succ}(q)) = \mathrm{succ}(\Phi(q))$.

We prove that $\Phi(\mathrm{succ}(q)) \supseteq \mathrm{succ}(\Phi(q))$ (the reverse inclusion is easy). Suppose $q'' \in \mathrm{succ}(\Phi(q))$. As Φ is surjective, $\exists q'$ s.t. $q'' = \Phi(q')$. So $\Phi(q) \leqslant_2 \Phi(q')$, whence $q \leqslant_1 q'$ by property (ii) of isomorphisms. Thus $q' \in \mathrm{succ}(q)$, whence $q'' = \Phi(q') \in \Phi(\mathrm{succ}(q))$. ∎

If h is an a.h. from P_1 to P_2, we write $h\colon P_1 \to P_2$. Abstraction homomorphisms induce the following *reduction relation* $\xrightarrow{\text{abs}}$ on processes:

DEFINITION 3.3. $P_1 \xrightarrow{\text{abs}} P_2$ iff $\exists$ a.h. $h\colon P_1 \to P_2$.

We next prove a few properties of the relation $\xrightarrow{\text{abs}}$.

PROPERTY 3.4. *$\xrightarrow{\text{abs}}$ is reflexive, transitive, and antisymmetric.*

Proof. Reflexivity and transitivity are easy to check. We prove here that $\xrightarrow{\text{abs}}$ is antisymmetric, namely that:

$$\text{if } h\colon P_1 \to P_2 \text{ and } h'\colon P_2 \to P_1 \text{ are a.h.'s, then } P_1 = P_2.$$

For any NDP $P = (Q \cup \{r\}, \leqslant, l)$ and for any $\sigma \in A^*$ let $Q_\sigma = \{q \mid q \in (Q \cup \{r\}), l(q) = \sigma\}$. Note that, because of our finiteness restriction on l, any such Q_σ is finite.

Now let $P_1 = (Q_1 \cup \{r_1\}, \leqslant_1, l_1)$, $P_2 = (Q_2 \cup \{r_2\}, \leqslant_2, l_2)$, and $h\colon P_1 \to P_2$, $h'\colon P_2 \to P_1$. For any $\sigma \in A^*$, define $h_\sigma = h \restriction Q_{1\sigma}$, $h'_\sigma = h' \restriction Q_{2\sigma}$. Then we have

$$h_\sigma\colon\ Q_{1\sigma} \to Q_{2\sigma} \quad \text{surjectively,} \quad \text{whence} \quad |Q_{1\sigma}| \geqslant |Q_{2\sigma}|.$$
$$h'_\sigma\colon\ Q_{2\sigma} \to Q_{1\sigma} \quad \text{surjectively,} \quad \text{whence} \quad |Q_{2\sigma}| \geqslant |Q_{1\sigma}|.$$

Summing up, we have $|Q_{1\sigma}| = |Q_{2\sigma}| < \infty$. Therefore the function h_σ is injective and thus also $h = \bigcup_{\sigma \in A^*} h_\sigma$ is injective. By Proposition 3.2 we then have that h is an isomorphism between P_1 and P_2.

PROPERTY 3.5. *$\xrightarrow{\text{abs}}$ is preserved by the operators μ and $+$.*

Proof. Let $P_1 = (Q_1 \cup \{r_1\}, \leqslant_1, l_1)$, $P_2 = (Q_2 \cup \{r_2\}, \leqslant_2, l_2)$, and $h\colon P_1 \to P_2$. We can deduce

(1) $\mu P_1 \xrightarrow{\text{abs}} \mu P_2, \forall \mu \in A$.

In fact, let $P'_1 = \mu P_1$, $P'_2 = \mu P_2$, with states $Q'_1 \cup \{r'_1\}$ and $Q'_2 \cup \{r'_2\}$, respectively. Then the function $f\colon (Q'_1 \cup \{r'_1\}) \to (Q'_2 \cup \{r'_2\})$ defined by

$$f(q) = \begin{cases} r'_2 & \text{if } q = r'_1 \\ h(q) & \text{otherwise} \end{cases}$$

is (trivially) an a.h. from P'_1 to P'_2.

(2) $P_1 + P \xrightarrow{\text{abs}} P_2 + P$, $\forall$NDP P.

In fact, let $P'_1 = P_1 + P$, $P'_2 = P_2 + P$, with states $Q'_1 \cup \{r'_1\}$ and $Q'_2 \cup \{r'_2\}$, respectively. Let $Q \cup \{r\}$ be the states of P. Then the function $f\colon (Q'_1 \cup \{r'_1\}) \to (Q'_2 \cup \{r'_2\})$ defined by

$$f(q) = \begin{cases} r'_2 & \text{if } q = r'_1, \\ h(q) & \text{if } q \in Q_1, \\ q & \text{if } q \in Q, \end{cases}$$

is (trivially) an a.h. from P'_1 to P'_2. ∎

In what follows, a relation which is preserved by our operators will often be called *substitutive*.

We turn now to what is perhaps the most interesting feature of our reduction relation, namely its *confluent* behaviour. Confluence of a.h.'s can be proved by standard algebraic techniques, once the notion of congruence associated to an a.h. is formalised.

DEFINITION 3.6. If $P = (Q \cup \{r\}, \leqslant, l)$ is an NDP, we say that an equivalence relation $\sim$ on Q is a *congruence* on P iff, whenever $q \sim q'$:

(i) $l(q) = l(q')$ (labels are preserved)

(ii) $q < p$ implies $\exists p' \sim p$ s.t. $q' \leqslant p'$ (successors are preserved)

It can be proved that any congruence $\sim$ satisfies the

convexity property: $q < p < q'$ and $q \sim q'$ implies $q \sim p \sim q'$.

(The proof is by induction on the length n of the longest chain: $q' \multimap q_1 \multimap \cdots \multimap q_n$ s.t. $l(q') = l(q_1) = \cdots = l(q_n)$. That this length is finite is ensured by our finiteness restriction on the labelling l. In fact, in absence of this restriction, the convexity property would not hold.)

We now show that, for any NDP P, there is a one-to-one correspondence between congruences and abstraction homomorphisms on P. First, some notation: If $P = (Q \cup \{r\}, \leqslant, l)$ is an NDP and h and a.h. on P, we define the equivalence $\sim_h$ on Q by

$$\sim_h = \{(q, q') \mid q, q' \in Q, h(q) = h(q')\}.$$

We can then prove the following two theorems.

THEOREM 3.7. *If P is an NDP and $\sim$ is a congruence on P, then there exists an NDP $P/\!\sim$, the* quotient of *P by $\sim$, and an a.h. $h_\sim$ from P to $P/\!\sim$ s.t. $\sim_{h_\sim} = \sim$.*

Proof. If $P=(Q\cup\{r\}, \leqslant, l)$, define $P/\!\sim\,=(Q/\!\sim\,\cup\{r'\}, \leqslant', l')$ by

$$\begin{aligned} r' &<'[q], && \forall q\in Q\\ [q]&\leqslant'[p] && \text{iff}\ \ \exists p' \text{s.t.}\ q\leqslant p'\sim p.\\ l'(r')&=\varepsilon\\ l'([q])&=l(q). \end{aligned}$$

Also, define $h_\sim: Q\cup\{r\}\to Q/\!\sim\,\cup\{r'\}$ by

$$\begin{aligned} h_\sim(r)&=r'\\ h_\sim(q)&=[q], \qquad \forall q\in Q. \end{aligned}$$

We shall prove that

(1) $P/\!\sim$ is an NDP.

(2) $h_\sim$ is an a.h. from P to $P/\!\sim$, and $\sim_{h_\sim}=\sim$.

Proof of (1). To prove that $P/\!\sim$ is an NDP: First, we check that $\leqslant'$ is a partial ordering relation. Reflexivity and transitivity follow easily from the definition. To prove antisymmetry, use the convexity property of $\sim$.

Second, we show that the labelling l' meets the requirements. The properties of monotonicity and finiteness can be easily deduced from the same properties of the labelling l. We prove here that $[q]\multimap[p]$ implies $l'([p])=l'([q])\cdot\mu$ for some $\mu\in A$. In fact, suppose $[q]\multimap[p]$: this is because $q<p'\sim p$, for some p'. That is, $\exists p_0,\dots,p_n$, $n\geqslant 1$, s.t. $q=p_0\multimap\cdots\multimap p_n=p'$. Now it can be easily shown, by induction on $n\geqslant 1$, that

$$p_0\multimap\cdots\multimap p_n \quad\text{and}\quad [p_0]\multimap[p_n] \quad\text{implies}\quad \exists p_0', p_n' \quad\text{s.t.}\quad p_0\sim p_0'\multimap p_n'\sim p_n.$$

So, from $[q]<[p']$ we deduce that $\exists q', p''$ s.t. $q\sim q'\multimap p''\sim p'$. Then

$$l'([p])=l(p'')=l(q')\cdot\mu=l'([q])\cdot\mu$$

and this ends the proof of (1).

Proof of (2). We want to show that $h_\sim$ is an a.h. from P to $P/\!\sim$, and that $\sim_{h_\sim}=\sim$. By definition, $h_\sim$ is a function s.t. $r\mapsto r'$, $Q\to Q/\!\sim$.

Now we check the properties (i) and (ii) of a.h.'s.

Property (i):

$$\begin{aligned} l'(h_\sim(r))&=l'(r')=\varepsilon=l(r),\\ l'(h_\sim(q))&=l'([q])=l(q) \qquad \text{for}\quad q\in Q. \end{aligned}$$

Property (ii):

$$\mathrm{succ}(h_\sim(r)) = \mathrm{succ}(r') = Q/\sim \cup \{r'\} = h_\sim(Q \cup \{r\}) = h_\sim(\mathrm{succ}(r))$$

$$\mathrm{succ}(h_\sim(q)) = \mathrm{succ}([q]) = \{[p] \mid [q] \leqslant' [p]\} = \{[p] \mid q \leqslant p' \sim p\}$$
$$= \{[p'] \mid q \leqslant p'\} = h_\sim(\{p' \mid q \leqslant p'\}) = h_\sim(\mathrm{succ}(q)).$$

So $h_\sim$ is indeed an a.h. from P to $P/\sim$. As for the equality $\sim_{h_\sim} = \sim$, it immediately follows from the definitions of $h_\sim$ and $\sim_h$. ∎

THEOREM 3.8. *If P, P' are NDPs and h is an a.h. from P to P', then $\sim_h$ is a congruence on P and P' is isomorphic to $P/\sim_h$.*

Proof. Again, we show the result in two steps:

(1) $\sim_h$ is a congruence on P.

(2) P' is isomorphic to $P/\sim_h$.

Proof of (1). We know that $\sim_h$ is an equivalence relation on Q. We check that it satisfies the properties (i), (ii) of congruences. Suppose $q \sim_h q'$: this is because $h(q) = h(q')$. Therefore we have

Property (i): $l(q) = l'(h(q)) = l'(h(q')) = l(q')$.

Property (ii): $q \leqslant p$ means $p \in \mathrm{succ}(q)$. Then $h(p) \in h(\mathrm{succ}(q)) = \mathrm{succ}(h(q)) = \mathrm{succ}(h(q')) = h(\mathrm{succ}(q'))$. So $\exists p' \in \mathrm{succ}(q')$ s.t. $h(p) = h(p')$. That is, $\exists p'$ s.t. $q' \leqslant p'$ and $p \sim_h p'$.

Proof of (2). If $P' = (Q' \cup \{r'\}, \leqslant', l')$ and $P/\sim_h = (Q/\sim_h \cup \{r''\}, \leqslant'', l'')$ is defined as for Theorem 3.7, let Φ: $Q/\sim_h \cup \{r''\} \to Q' \cup \{r'\}$ be the function given by

$$\Phi(r'') = r'$$
$$\Phi([q]) = h(q).$$

Then Φ is clearly well defined. We show that Φ is an injective a.h. from $P/\sim_h$ to P'. Then it will follow by Proposition 3.2 that Φ is an isomorphism between $P/\sim_h$ and P'.

It is easy to check that Φ is injective, as

$$h(q) = h(p) \qquad \text{implies} \quad [q] = [p].$$

Moreover, Φ satisfies the Properties (i) and (ii) of a.h.'s:

Property (i): $\quad l'(\Phi(r'')) = l'(r') = \varepsilon = l''(r''),$

$$l'(\Phi([q])) = l'(h(q)) = l(q) = l''([q]).$$

Property (ii): $\mathrm{succ}(\Phi([q]) = \mathrm{succ}(h(q)) = h(\mathrm{succ}(q)) = \{h(p) \mid q \leqslant p\} = \{\Phi([p]) \mid q \leqslant p\} = \{\Phi([p']) \mid q \leqslant p \sim_h p'\} = \Phi(\mathrm{succ}([q])$. ∎

To prove the confluence property of a.h.'s, we will finally make use of

LEMMA 3.9. *If* $\sim_1$, $\sim_2$ are congruences on an NDP P, then $\sim_{1,2} = [\sim_1 \cup \sim_2]^*$, the symmetric and transitive closure of the union of $\sim_1$ and $\sim_2$, is the *least congruence* $\sim$ s.t. $\sim_1 \subseteq \sim$ and $\sim_2 \subseteq \sim$.

Proof. It is a standard result that $\sim_{1,2}$ is the least equivalence on Q which includes both $\sim_1$ and $\sim_2$. Then, if $\sim_{1,2}$ is a congruence, it will also be the least congruence which includes $\sim_1$ and $\sim_2$.

We thus proceed to show that $\sim_{1,2}$ is a congruence, namely that it satisfies the required Properties (i), (ii). Note first that $q \sim_{1,2} q'$ iff $\exists n$, $\exists q_0, ..., q_n$ s.t.

$$q = q_0 \sim_{1/2} \cdots \sim_{1/2} q_n = q',$$

where $\sim_{1/2}$ means: *either* $\sim_1$ *or* $\sim_2$.

Then property (i) of congruences is easy to check. As for property (ii), suppose $q \leqslant p$ and

$$q = q_0 \sim_{1/2} \cdots \sim_{1/2} q_n = q'.$$

Since both $\sim_1$ and $\sim_2$ satisfy (ii), there exist $p_0, ..., p_n$ s.t. $q_i \leqslant p_i$ and

$$p = p_0 \sim_{1/2} \cdots \sim_{1/2} p_n.$$

Thus, if we let $p' = p_n$, we have $p \sim_{1,2} p'$ and $q' \leqslant p'$. ∎

For the coming theorems, we will need some more notation. If h, h' are two a.h.'s on the same process, we say that h is *weaker than* h' and write $h \preccurlyeq h'$, iff $\sim_h \subseteq \sim_{h'}$. The following fact is then (almost) standard.

LEMMA 3.10. (Factorisation of an abstraction homomorphism by a weaker one). *If P, P_1, P_2 are NDPs, and $h_1: P \to P_1$, $h_2: P \to P_2$, are a.h.'s s.t. $h_1 \preccurlyeq h_2$, then there exists a* unique *a.h. $h_{2/1}: P_1 \to P_2$ s.t. the following diagram commutes:*

$$\begin{tikzcd} & P \arrow[dl, "h_1"'] \arrow[dd, "h_2"] \\ P_1 \arrow[dr, dashed, "h_{2/1}"'] & \\ & P_2 \end{tikzcd}$$

Proof. Let $\sim_1$, $\sim_2$ stand for $\sim_{h_1}$, $\sim_{h_2}$. In view of Theorem 3.8, we can assume

$$P_1 = P/\sim_1, \qquad P_2 = P/\sim_2.$$

Then the *unique* mapping $h_{2/1}$ that can make the diagram commute is the one defined by

$$h_{2/1}(r_1) = r_2$$
$$h_{2/1}([q]_1) = h_2(q) = [q]_2, \qquad \forall q \in Q.$$

This mapping is a function, because

$$[q]_1 = [q']_1 \qquad \text{implies } [q]_2 = [q']_2$$

for the hypothesis that $\sim_1 \subseteq \sim_2$.

We now show that $h_{2/1}$ is an a.h. Let as usual l_i and succ_i refer to P_i. Then $h_{2/1}$ satisfies

(i) $l_1([q]_1) = l(q) = l_2([q]_2) = l_2(h_{2/1}([q]_1))$

(ii) $h_{2/1}(\text{succ}_1[q]_1)) = h_{2/1}(\text{succ}_1(h_1(q))) = h_{2/1}(h_1(\text{succ}_1(q))) = h_{2/1}(\{[q']_1 | q \leqslant q'\}) = h_2(\{q' | q \leqslant q'\}) = h_2(\text{succ}_1(q)) = \text{succ}_2(h_2(q)) = \text{succ}_2(h_{2/1}([q]_1))$. ▮

We can finally prove

THEOREM 3.11. (confluence of abstraction homomorphisms). If P, P_1, P_2 are NDPs, and $h_1: P \to P_1$, $h_2: P \to P_2$ are a.h.'s, then $\exists$ NDP P_3, $\exists$ a.h.'s, $h_{13}: P_1 \to P_3$, $h_{23}: P_2 \to P_3$ s.t. the following diagram commutes:

$$\begin{tikzcd}
& P \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\
P_1 \arrow[dr, dashed, "h_{13}"'] & & P_2 \arrow[dl, dashed, "h_{23}"] \\
& P_3 &
\end{tikzcd}$$

Proof. Let again $\sim_1$ and $\sim_2$ stand for $\sim_{h_1}$ and $\sim_{h_2}$. Define $\sim_3 = [\sim_1 \cup \sim_2]^*$. Since $\sim_3$ is a congruence (by Lemma 3.9), there exist correspondingly an NDP $P/\sim_3$ and an a.h. $h_{\sim_3}: P \to P/\sim_3$ (by Theorem 3.8).

Let P_3 be $P/\sim_3$ and $h_3 = h_{\sim_3}$. We have

$$\begin{tikzcd}
& P \arrow[dl, "h_1"'] \arrow[dr, "h_2"] \arrow[dd, "h_3"] & \\
P_1 & & P_2 \\
& P_3 &
\end{tikzcd}$$

where both the pairs (h_1, h_3) and (h_2, h_3) meet the hypothesis of Lemma 3.10, hence the result with $h_{13} = h_{3/1}$, $h_{23} = h_{3/2}$. ▮

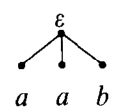
CONVENTION. In the following we will use $\xrightarrow{\text{abs}}$ instead of $\xrightarrow{\text{abs}}{}^{-1}$ whenever convenient.

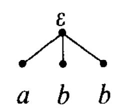
We conclude this section by stating the

COROLLARY 3.12. ($\rightarrow^{\text{abs}}$ is Church–Rosser). *If P, P_1, P_2 are NDPs s.t. $P_1 \leftarrow^{\text{abs}} P \rightarrow^{\text{abs}} P_2$, then $\exists$ NDP P_3 s.t. $P_1 \rightarrow^{\text{abs}} P_3 \leftarrow^{\text{abs}} P_2$.* ▮

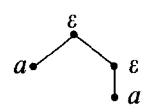
3.1. *Abstraction Equivalence*

The relation $\rightarrow^{\text{abs}}$ gives us a criterion to regard two processes as "abstractly the same." However, being essentially a simplification, $\rightarrow^{\text{abs}}$ is not symmetric and therefore does not, for example, relate the two processes





or the processes



based on $\xrightarrow{\text{abs}}$, we will then define on NDPs a more general relation $\sim_{\text{abs}}$, of *reducibility to a same* process.

DEFINITION 3.13. $\sim_{\text{abs}} =_{\text{def}} \xrightarrow{\text{abs}} \cdot \xleftarrow{\text{abs}}$.

We can immediately prove a few properties for $\sim_{\text{abs}}$.

PROPERTY 3.14. *$\sim_{\text{abs}}$ is an equivalence.*

Proof. Transitivity follows from the fact that $\rightarrow^{\text{abs}}$ is Church–Rosser, which can be restated as: $[\xrightarrow{\text{abs}} \cup \xleftarrow{\text{abs}}]^* = \xrightarrow{\text{abs}} \cdot \xleftarrow{\text{abs}}$. ▮

PROPERTY 3.15. *$\sim_{\text{abs}}$ is preserved by the operators μ and $+$.*

Proof. Direct consequence of the substitutivity of $\rightarrow^{\text{abs}}$ under μ and $+$. ▮

To sum up, we have now a *substitutive* equivalence $\sim_{\text{abs}}$ for NDPs that can be split, when required, in two reduction halves. The equivalence $\sim_{\text{abs}}$ will be called *abstraction equivalence*. In the coming section we will study how abstraction equivalence relates to bisimulation equivalence, a notion introduced by Park [Pa] for general transition systems.

4. Bisimulation Relations

A natural method for comparing different systems is to check to which extent they can *behave like* each other, according to some definition of behaviour.

Now, what is to be taken as the *behaviour* of a system need not be known a priori. One can always, in fact, having fixed a criterion for deriving subsystems, let the behaviour of a system be recursively defined in terms of the behaviours of its subsystems.

Based on such an implicit notion of behaviour, one gets an (equally implicit) notion of equivalence of behaviour, or *bisimulation equivalence*, between systems: two systems are said to be equivalent iff for any subsystem of either of the two, derived with some criterion, there exists an equivalent subsystem of the other, derived with the same criterion.

For an NDS S, the transition relation provides an obvious criterion for deriving a subsystem S': for any μ, S' is a μ-subsystem of S iff $S \xrightarrow{\mu} S'$. However, if we are to abstract from internal transitions, a weaker criterion will be needed. To this purpose the following *weak* transition relations $\overset{\mu}{\Rightarrow}$ are introduced:

$$\overset{a}{\Longrightarrow} = \xrightarrow{\tau^n} \xrightarrow{a} \xrightarrow{\tau^m} \qquad n, m \geqslant 0$$

$$\overset{\tau}{\Longrightarrow} = \xrightarrow{\tau^n} \qquad n \geqslant 0.$$

The system S' is called a *μ-derivative of* S iff $S \overset{\mu}{\Rightarrow} S'$. We then define bisimulation relations on NDSs as follows.

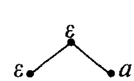
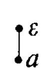
Definition 4.1. A (*weak*) *bisimulation* is a relation $R \subseteq (\mathscr{S} \times \mathscr{S})$ s.t. $R \subseteq F(R)$, where $(S_1, S_2) \in F(R)$ iff $\forall \mu \in A$:

(i) $S_1 \Rightarrow^{\mu} S_1'$ implies $\exists S_2'$ s.t. $S_2 \Rightarrow^{\mu} S_2'$, $S_1' \, R \, S_2'$

(ii) $S_2 \Rightarrow^{\mu} S_2'$ implies $\exists S_1'$ s.t. $S_1 \Rightarrow^{\mu} S_1'$, $S_1' \, R \, S_2'$.

Thus a bisimulation is exactly a post-fixed-point of the function F. As F is monotonic for relations under inclusion, it has a *largest* postfixed-point (which is also its largest fixed-point) given by $\bigcup \{R \mid R \subseteq F(R)\}$. We will denote this largest bisimulation by $\langle \approx \rangle$, and, since $\langle \approx \rangle$ turns out to be an equivalence, refer to it as the *bisimulation equivalence.*

Unfortunately, $\langle \approx \rangle$ is not preserved by all the operators. Precisely, $\langle \approx \rangle$ is not preserved by the operator $+$, as shown by the example

 $\langle \approx \rangle$ NIL, but  $\langle \not\approx \rangle$ 

On the other hand the relation $\langle \approx \rangle^+$, obtained by closing $\langle \approx \rangle$ w.r.t. the operator $+$:

$$S_1 \langle \approx \rangle^+ S_2 \qquad \text{iff} \quad \forall S\colon S + S_1 \langle \approx \rangle S + S_2$$

can be shown to be a *substitutive* equivalence, and in fact to be the largest such equivalence contained in $\langle \approx \rangle$. (For more details on $\langle \approx \rangle$ and $\langle \approx \rangle^+$ we refer to [M2, M3].)

To conclude, $\langle \approx \rangle^+$ seems a convenient restriction on $\langle \approx \rangle$ to adopt when modelling NDSs. We will see in the next section that $\langle \approx \rangle^+$ coincides, on NDPs, with our abstraction equivalence $\sim_{\text{abs}}$.

5. Relating Bisimulations to Abstraction Homomorphisms

Looking back at our relations $\rightarrow^{\text{abs}}$ and $\sim_{\text{abs}}$, we notice that they rely on a notion of *equivalence of states* which, like bisimulations, is *recursive*. Moreover, the recursion builds up on the basis of a similarity requirement (equality of *labels*) that reminds of the criterion (equality of *observable derivation sequences*) used in bisimulations to derive corresponding subsystems. All this indicates there might be a close analogy between abstraction equivalence and bisimulation equivalence.

In fact, since we know that $\sim_{\text{abs}}$ is substitutive, we shall try to relate it with the substitutive closure $\langle \approx \rangle^+$ of $\langle \approx \rangle$. To this purpose, it will be convenient to have $\langle \approx \rangle^+$ itself be defined recursively.

Note that $\langle \approx \rangle^+$ only differs from $\langle \approx \rangle$ in that it takes into account the *preemptive capacities* a system can develop when placed in a sum context. Such preemptive capacities depend on the system having some silently reachable state where, informally speaking, some of the "alternatives" offered by the sum-context are no more available. This suggests that we should adopt, when looking for a direct definition of $\langle \approx \rangle^+$, the more *restrictive* transition *relations* $\stackrel{\mu}{\Mapsto}$:

$$\stackrel{\mu}{\Longmapsto} = \xrightarrow{\tau^n}\xrightarrow{\mu}\xrightarrow{\tau^m} \qquad n, m \geqslant 0.$$

In particular, we will have $\stackrel{\tau}{\Mapsto} = \rightarrow^{\tau^n}$, $n > 0$. Note, on the other hand, that, for $a \in A$, it will be: $\stackrel{a}{\Mapsto} = \stackrel{a}{\Rightarrow}$.

However, the equivalence $\langle \approx \rangle^+$ is *restrictive* with respect to $\langle \approx \rangle$ only as for as the first $\Rightarrow^\tau$ derivation steps are concerned: at further steps $\langle \approx \rangle^+$ behaves like $\langle \approx \rangle$, as it can be seen from the example:

$$\begin{array}{ccc} \varepsilon & \langle \approx \rangle^+ & \varepsilon \\ a & \langle \approx \rangle & a \\ a & & ab \\ ab & & \end{array}$$

So, if we are to recursively define $\langle \approx \rangle^+$ in terms of the transitions $\Mapsto^\mu$, we will have to somehow counteract the strengthening effect of the $\Mapsto^\mu$ s at steps other than the first.

To this end, for any relation $R \subseteq (\mathscr{S} \times \mathscr{S})$, a relation R_a ("almost" R) is introduced: $(S_1, S_2) \in R_a$ iff $(S_1, S_2) \in R$ or $(\tau S_1, S_2) \in R$ or $(S_1, \tau S_2) \in R$.

We can then define a-bisimulation ("almost" bisimulation) relations on NDSs.

DEFINITION 5.1. A (*weak*) a-*bisimulation* is a relation $R \subseteq (\mathscr{S} \times \mathscr{S})$ such that $R \subseteq E(R_a)$, where $(S_1, S_2) \in E(R_a)$ iff $\forall \mu \in A$:

(i) $S_1 \mapsto^{\mu} S_1'$ implies $\exists S_2'$ s.t. $S_2 \mapsto^{\mu} S_2'$, $S_1' \, R_a \, S_2'$

(ii) $S_2 \mapsto^{\mu} S_2'$ implies $\exists S_1'$ s.t. $S_1 \mapsto^{\mu} S_1'$, $S_1' \, R_a \, S_2'$.

Again, the equation $R \subseteq E(R_a)$ has a *largest solution* which is an equivalence relation, and which we will denote by $\langle \approx \rangle^a$. Now the equivalence $\langle \approx \rangle^a$ has been proven to *coincide* with $\langle \approx \rangle^+$ (both the definition of $\langle \approx \rangle^a$ and the proof that $\langle \approx \rangle^a = \langle \approx \rangle^+$ are due to Hennessy).

We mention in passing that, if R is an a-bisimulation, then R_a is an ordinary bisimulation. In particular, for the maximal a-bisimulation $\langle \approx \rangle^a$, we have $\langle \approx \rangle^a_a = \langle \approx \rangle$ (this fact will be used in the proof of Theorem 5.8 below).

We proceed now to compare $\sim_{abs}$ with $\langle \approx \rangle^a$. First, μ-derivatives are redefined in terms of the new relations $\mapsto^{\mu}$: S' is a μ-*derivative of* S iff $S \mapsto^{\mu} S'$. Then we are able to prove the

THEOREM 5.2. $\rightarrow^{abs}$ *is an* a-*bisimulation relation on* NDPs.

The proof relies on the following two lemmas.

LEMMA 5.3. ($\rightarrow^{abs}$ almost preserves μ-derivatives). *If* $P_1 \rightarrow^{abs} P_2$ *and* $P_1 \mapsto^{\mu} P_1'$ *then* $\exists P_2'$ *s.t.* $P_2 \mapsto^{\mu} P_2'$ *where* either $P_1' \rightarrow^{abs} P_2'$ or $P_1' \rightarrow^{abs} \tau P_2'$.

Proof. We recall that any state of an NDP P is the initial state $r_{P'}$ of some derivative NDP P'. Note that $P \mapsto^{\mu} P'$ implies $l(r_{p'}) = \mu$ in P.

Now let $h: P_1 \rightarrow P_2$ be an a.h., and as usual let Q_i, r_i refer to P_i, $i = 1, 2$.

Suppose $P_1 \mapsto^{\mu} P_1'$. Let r_1' be the initial state of P_1' in P_1; then $r_1' \in Q_1$ (i.e., $r_1' \neq r_1$) and $l_1(r_1') = \mu$. Now let $r_2' = h(r_1')$. Then r_2' is the root of some derivative P_2' of P_2. Since h is an a.h., we have $r_2' \neq r_2$ and $l(r_2') = l(r_1') = \mu$. Therefore $P_2 \mapsto^{\mu} P_2'$.

Now let $Q_i' = \{q \mid q \in P_i, r_i' < q\}$, $i = 1, 2$. From Property (ii) of a.h.'s, we know that

$$h(Q_1' \cup r_1') = Q_2' \cup r_2'.$$

Then we are in one of two cases:

(1) $h(Q_1') = Q_2'$. In this case $h \restriction (Q_1' \cup r_1')$ is (trivially) an a.h. from P_1' to P_2'. Therefore $P_1' \rightarrow^{abs} P_2'$.

(2) $h(Q_1') = Q_2' \cup r_2'$. Here h maps some states of Q_1' to r_2'. Note that these states will be labelled by μ in P_1 and by ε in P_1'. Let Q_ε denote the set of such states

in $P'_1: Q_\varepsilon = \{q \mid q \in Q'_1, l(q) = \varepsilon\}$. Also, let $P''_2 = \tau P'_2$. Then $Q''_2 = Q'_2 \cup r'_2$ and the function: $h':(Q'_1 \cup r'_1) \to (Q''_2 \cup r''_2)$ defined by

$$h'(r'_1) = r''_2$$
$$h'(Q_\varepsilon) = r'_2$$
$$h'(q) = h(q) \qquad \text{otherwise,}$$

is (trivially) an a.h. Thus we have in this case $P'_1 \to^{\text{abs}} \tau P'_2$. ▮

LEMMA 5.4. ($\xrightarrow{\text{abs}}{}^{-1}$ almost preserves μ-derivatives). *If* $P_1 \to^{\text{abs}} P_2$ *then* $P_2 \mapsto^{\mu} P'_2$ *implies* $\exists P'_1$ *s.t.* $P_1 \mapsto^{\mu} P'_1$ *where* either $P'_1 \to^{\text{abs}} P'_2$ or $P'_1 \to^{\text{abs}} \tau P'_2$.

Proof. Suppose $P_2 \mapsto^{\mu} P'_2$. Let r'_2 be the initial state of P'_2 in P_2. Since h is surjective, $\exists\ r'_1 \neq r_1$ s.t. $h(r'_1) = r'_2$, $l(r'_1) = l(r'_2) = \mu$. Then, if P'_1 is the derivative of P_1 with root r'_1, we have $P_1 \mapsto^{\mu} P'_1$, and the rest of the proof is the same as for Lemma 5.3. ▮

COROLLARY 5.5. $\to^{\text{abs}} \subseteq \langle \approx \rangle^{\text{a}}$.

Proof. $\langle \approx \rangle^{\text{a}}$ is the largest a-bisimulation. ▮

Note that in Lemmas 5.3 and 5.4 we do not need to consider the case $\tau P'_1 \to^{\text{abs}} P'_2$. The reason this case does not arise in that a.h.'s are *single-valued* relations. In fact, our next task will be to give a characterisation of a.h.'s as relations on processes.

To this purpose, we will need some more notation. For any NDP P, let $\text{Der}(P) = \{P' \mid P \to {}^*P'\}$ and $\text{PDer}(P) = \{P' \mid P \to {}^*P',\ P \neq P'\}$. Also, we say that a bisimulation (resp. an a-bisimulation) relation R is *between* P_1 and P_2 iff $P_1\ R\ P_2$ and $R \subseteq (\text{Der}(P_1) \times \text{Der}(P_2)) \cup (\text{Der}(P_2) \times \text{Der}(P_1))$.

Let $\Rightarrow^{\sigma}$ stand for $\overset{\mu_1}{\Rightarrow} \cdots \overset{\mu_n}{\Rightarrow}$ if $\sigma = \mu_1 \cdots \mu_n$. It is easy to see that, for any two systems S_1, S_2, the following holds.

LEMMA 5.6. *If* $S_1\ R\ S_2$ *for some bisimulation* R, *then for any* $\sigma \in A^*$:

(i) $S_1 \overset{\sigma}{\Rightarrow} S'_1$ *implies* $\exists S'_2$ s.t. $S_2 \overset{\sigma}{\Rightarrow} S'_2$, $S'_1\ R\ S'_2$

(ii) $S_2 \overset{\sigma}{\Rightarrow} S'_2$ *implies* $\exists S'_1$ s.t. $S_1 \overset{\sigma}{\Rightarrow} S'_1$, $S'_1\ R\ S'_2$.

Now, if we regard a.h.'s as relations on processes, we have the following characterisation.

THEOREM 5.7. *A relation* R *on processes is an* abstraction homomorphism *from* P_1 *to* P_2 *iff* R *is a* single-valued bisimulation *between* P_1 *and* P_2 s.t. $P_2 \notin R(\text{PDer}(P_1))$.

Proof. Only if. Let R be an a.h. from P_1 to P_2. Again, we assume that any state of P_i is the root $r_{P'_i}$ of some derivative P'_i. Also, we write $P'_1\ R\ P'_2$ in place of $R(r_{P'_1}) = r_{P'_2}$ whenever we want to treat R as a relation on processes.

Now, R is by definition single-valued and s.t. $P_1\ R\ P_2$ and $R \subseteq$

$(\mathrm{Der}(P_1)) \times \mathrm{Der}(P_2))$. Also, $R(\mathrm{PDer}(P_1)) = \mathrm{PDer}(P_2)$ implies $P_2 \notin R(\mathrm{PDer}(P_1))$. What is left to show is that R is a bisimulation, namely that $R \subseteq F(R)$. Suppose $P_1' \, R \, P_2'$, i.e., $R(r_{P_1'}) = r_{P_2'}$. Then

Clause (i): If $P_1' \Rightarrow^{\mu} P_1''$, we have

$$r_{P_1''} \in \mathrm{succ}(r_{P_1'}) \qquad \text{and} \qquad l(r_{P_1''}) = l(r_{P_1'}) \cdot \mu.$$

Now, since $R(\mathrm{succ}(r_{P_1'})) = \mathrm{succ}(r_{P_2'})$ [Property (ii) of a.h.'s], there exists $r_{P_2''} \in \mathrm{succ}(r_{P_2'})$ s.t. $R(r_{P_1''}) = r_{P_2''}$, that is $P_1'' \, R \, P_2''$. Then $l(r_{P_2''}) = l(r_{P_1''}) = l(r_{P_1'}) \cdot \mu = l(r_{P_2'}) \cdot \mu$ [using Property (i) of a.h.'s]. So we have $P_2' \Rightarrow^{\mu} P_2''$, with $P_1'' \, R \, P_2''$.

Clause (ii): The proof is symmetric to that of Clause (i).

This ends the proof of the *only if* part of the theorem.

If. Now let R be a single-valued bisimulation between P_1 and P_2 s.t. $P_2 \notin R(P\,\mathrm{Der}(P_1))$. Then R can be regarded as a function $R\colon (Q_1 \cup r_1) \to (Q_2 \cup r_2)$. By hypothesis we have $R(r_1) = r_2$, $R(Q_1) = Q_2$. We check now that R satisfies Properties (i) and (ii) of a.h.'s. Suppose $R(r_{P_1'}) = r_{P_2'}$, i.e., $P_1' \, R \, P_2'$. Then

Property (i): $l(r_{P_1'}) = \sigma \in A^*$ means $P_1 \Rightarrow^{\sigma} P_1'$. Since $P_1 \, R \, P_2$, we know, by Lemma 5.3, that $\exists P_2''$ s.t. $P_2 \Rightarrow^{\sigma} P_2''$ and $P_1' \, R \, P_2''$. Since R is single-valued, it must be $P_2'' = P_2'$. Whence $l(r_{P_2'}) = \sigma$.

Property (ii): $r_{P_1''} \in \mathrm{succ}(r_{P_1'})$ means $\exists \sigma \in A^*$ s.t. $P_1' \Rightarrow^{\sigma} P_1''$. Since $P_1' \, R \, P_2'$, by Lemma 5.3 $\exists P_2''$ s.t. $P_2' \Rightarrow^{\sigma} P_2''$ and $P_1'' \, R \, P_2''$. So $r_{P_2''} = R(r_{P_1''}) \in \mathrm{succ}(r_{P_2'})$.

Thus we have shown that $R(\mathrm{succ}(r_{P_1'}) \subseteq \mathrm{succ}(r_{P_2'})$. By a symmetrical argument we can show also: $\mathrm{succ}(r_{P_2'}) \subseteq R(\mathrm{succ}(r_{P_1'}))$, and this ends the proof of the theorem. ∎

So far we have been concentrating on how bisimulations relate to the reduction relation $\to^{\mathrm{abs}}$. We now come to our main result, concerning the relationship between the abstraction equivalence $\sim_{\mathrm{abs}}$ and the substitutive bisimulation equivalence $\langle \approx \rangle^{\mathrm{a}}$. It turns out that these two equivalences coincide.

THEOREM 5.8. $\sim_{\mathrm{abs}} = \langle \approx \rangle^{\mathrm{a}}$.

Proof of $\subseteq$. From Corollary 5.5 we can infer that $\sim_{\mathrm{abs}} = [\xrightarrow{\mathrm{abs}} \cdot \xleftarrow{\mathrm{abs}}] \subseteq \langle \approx \rangle^{\mathrm{a}}$, since $\langle \approx \rangle^{\mathrm{a}}$ is symmetrically and transitively closed.

Proof of $\supseteq$. Suppose $P_1 \langle \approx \rangle^{\mathrm{a}} P_2$. We want to show that $\exists P_3$ s.t. $P_1 \xrightarrow{\mathrm{abs}} P_3 \xleftarrow{\mathrm{abs}} P_2$. For any NDP P, let $\mathrm{Der}_{\sigma}(P) = \{P' \mid P \Rightarrow^{\sigma} P'\}$ and $P\,\mathrm{Der}_{\sigma}(P) = \{P' \mid P \Rightarrow^{\sigma} P',\ P \neq P'\}$.

Note that $P_1 \langle \approx \rangle^{\mathrm{a}} P_2$ implies $P_1 \, E\, (\langle \approx \rangle^{\mathrm{a}}_{\mathrm{a}})\, P_2$, i.e., $P_1 \, E\, (\langle \approx \rangle)\, P_2$. Then it is easy to check that the relation

$$R = (P_1, P_2) \cup \langle \approx \rangle \restriction (P\,\mathrm{Der}_{\sigma}(P_1) \times P\,\mathrm{Der}_{\sigma}(P_2))$$

is a bisimulation between P_1 and P_2 s.t. $P_2 \notin R(P\,\mathrm{Der}(P_1))$.

However, R will not, in general, be single-valued. Let then $\sim$ be the equivalence induced by R on the states Q_2 of P_2,

$$r_{P_2'} \sim r_{P_2''}$$

iff $\exists P_1' \in P\,\mathrm{Der}(P_1)$ s.t. both (P_1', P_2') and $(P_1', P_2'') \in R$.

Now, it can be easily shown that $\sim$ is a congruence on P_2. Therefore, by Theorem 3.7, $\exists$NDP P_3, $\exists$ a.h. h s.t. $h: P_2 \to P_3$. So $P_2 \to^{\mathrm{abs}} P_3$.

Now, by Theorem 5.7, h can be regarded as a bisimulation between P_2 and P_3. Consider then the composition: $R' = h \circ R$. By construction R' is single-valued and s.t. $R' \subseteq (\mathrm{Der}(P_1) \times \mathrm{Der}(P_3))$. Also, R' is a bisimulation because both R and h are. Finally, since $P_2 \notin R(P\,\mathrm{Der}(P_1))$ and $P_3 \notin h(P\,\mathrm{Der}(P_2))$, we have $P_3 \notin R'(P\,\mathrm{Der}(P_1))$. Thus, by Theorem 5.7 again, R' is an a.h. from P_1 to P_3. So $P_1 \to^{\mathrm{abs}} P_3$.

Summing up, we have shown that $P_1 \to^{\mathrm{abs}} P_3 \leftarrow^{\mathrm{abs}} P_2$, that is to say, $P_1 \sim_{\mathrm{abs}} P_2$. ∎

In view of the last theorem, the equivalence $\sim_{\mathrm{abs}}$ can be regarded as an alternative definition for $\langle \approx \rangle^{\mathrm{a}} = \langle \approx \rangle^{+}$. In the next section, we will see how this new characterisation can be used to derive a set of reduction rules for $\langle \approx \rangle^{+}$ on finite processes.

6. A Language for Finite Processes

In this section, we study the subclass of *finite* NDPs, and show how it can be used to model terms of a simple language L.

The language is essentially a subset of Milner's CCS (Calculus of Communicating Systems [M1]). In [HM] a set of axioms is presented for L that exactly characterises the equivalence $\langle \approx \rangle^{\mathrm{a}}$ (and therefore $\sim_{\mathrm{abs}}$) on the corresponding transition systems. We show here that the reduction $\to^{\mathrm{abs}}$ itself can be characterised algebraically, by a set of *reduction rules*. These rules yield *normal forms* which coincide with the ones suggested in [HM].

Finally, we establish a notion of *minimality* for NDPs and use it to define a denotational model for L, a class of NDPs that we call *representation trees*. The model is shown to be isomorphic with Hennessy and Milner's term model.

We shall now introduce the language L. Following the approach of [HM], we define L as the term algebra T_Σ over the signature

$$\Sigma = A \cup \{\mathrm{NIL}, +\}.$$

If we assume the operators in Σ to denote the corresponding operators on NDPs (A will denote the set of unary operators μ), we can use *finite NDPs* to model terms in T_Σ. If t is a term of T_Σ, we will use P_t for the corresponding NDP.

We point out, however, that the denotations for terms of T_Σ in $\mathscr{P}$ will always be *trees*, that is, NDPs $P=(Q\cup\{r\}, \leqslant, l)$ obeying the further constraint:

$$\textit{confluence-freeness:} \quad \exists q'' \text{ s.t. } q\leqslant q'' \text{ and } q'\leqslant q'' \text{ implies } q\leqslant q' \text{ or } q'\leqslant q$$

Now consider the set of axioms: E_c

$$\begin{array}{ll} \text{sum-laws} & \left\{\begin{array}{ll} \text{E1.} & x+x'=x'+x \\ \text{E2.} & x+(x'+x'')=(x+x')+x'' \\ \text{E3.} & x+\text{NIL}=x \end{array}\right. \\ \tau\text{-laws} & \left\{\begin{array}{ll} \text{E4.} & \mu\tau x=\mu x \\ \text{E5.} & \tau x+x=\tau x \\ \text{E6.} & \mu(x+\tau y)+\mu y=\mu(x+\tau y) \end{array}\right. \\ \text{absorption law —} & \quad \text{E7.} \quad x+x=x. \end{array}$$

Let $=^c$ be the equality generated by E_c. It has been proved [HM] that E_c is a sound and complete axiomatisation for Milner's *observational congruence* $\approx^c$ [M1], namely that

$$t=^c t' \quad \text{iff} \quad P_t \approx^c P_{t'}.$$

The relation $\approx^c$ is defined as the closure w.r.t. the operator $+$ of the relation (Milner's *observational equivalence*):

$$\approx\, = \bigcap_n F^n(\mathscr{P}\times\mathscr{P}),$$

where $(\mathscr{P}\times\mathscr{P})$ is the universal relation on NDPs and F is the function on relations introduced in Section 4.

For *image-finite* systems, that is, systems satisfying the condition

$$\forall \text{ state } q,\ \forall\mu\text{: the set } \{q' \mid q \stackrel{\mu}{\Rightarrow} q'\} \text{ is finite}$$

the relations $\approx$ and $\approx^c$ are known to coincide with the relations $\langle\approx\rangle$ and $\langle\approx\rangle^a$ introduced in the previous sections. Now for NDPs the image-finiteness property is guaranteed by the finiteness restriction on the labelling function. So, in particular, we can assume $\approx^c$ to be defined as $\langle\approx\rangle^a$ on finite NDPs. Combining these facts together, we have that

$$t=^c t' \quad \text{iff} \quad P_t \sim_{\text{abs}} P_{t'}.$$

In other words, $=^c$ is an algebraic analogue for $\sim_{\text{abs}}$. Note on the other hand that, although each axiom of E_c could be viewed as a reduction rule (when applied from left to right), the corresponding reduction relation $\rightarrow$ would not characterise $\xrightarrow{\text{abs}}$. Consider, for example, the terms $t = a\,\text{NIL} + \tau(a\,\text{NIL} + b\,\text{NIL})$, $t' = \tau(a\,\text{NIL} + b\,\text{NIL})$. We have $P_t \xrightarrow{\text{abs}} P_{t'}$ but we cannot infer $t \rightarrow t'$.

However, using the axiomatisation E_c as a reference, we are able to derive a new system of reduction rules, which characterises $\xrightarrow{\text{abs}}$. To this end, let us define the relations $\xrightarrow{\mu}$ on the terms of T_Σ: $\forall \mu \in A^*$, let $\xrightarrow{\mu}$ be the *least* relation on terms that satisfies

(i) $\mu t \xrightarrow{\mu} t$

(ii) $t \xrightarrow{\mu} t'$ implies $t + t'' \xrightarrow{\mu} t'$, $t'' + t \xrightarrow{\mu} t'$.

The weak relations $\Rightarrow^\mu$ and $\Mapsto^\mu$ are defined in terms of the $\rightarrow^\mu$s just as in Section 4.

Now let $\rightarrow^c$ be the reduction relation generated by the following set of reduction rules R_c (where $\leftrightarrow$ stands for $(\rightarrow \cap \rightarrow^{-1})$): R_c

$$
\begin{array}{lll}
\text{sumlaws} & \left\{\begin{array}{l} \text{R1.} \quad x + x' \leftrightarrow x' + x \\ \text{R2.} \quad (x + x') + x'' \leftrightarrow x + (x' + x'') \\ \text{R3.} \quad x + \text{NIL} \rightarrow x \end{array}\right. & \\
\text{1st } \tau\text{-law —} & \text{R4.} \quad \mu\tau x \rightarrow \mu x & \\
\text{generalised absorption law —} & \text{R5.} \quad x + \mu x' \rightarrow x, & \text{whenever} \quad x \Mapsto^\mu x'.
\end{array}
$$

In what follows, we will often consider terms modulo the congruence induced by R1–R3. When taken modulo R1–R3, a term t can be expressed in the form $\sum_i \mu_i t_i$, where $i \in I$ for some finite set of indices I. By convention $\sum_i \mu_i t_i = \text{NIL}$ if $I = \varnothing$.

It is easy to check that the rules R_c are *sound* for NDPs under $\rightarrow^{\text{abs}}$, in the sense that $t \rightarrow^c t'$ implies $P_t \rightarrow^{\text{abs}} P_{t'}$. We proceed now to show that the rules R_c are also *complete* for $\rightarrow^{\text{abs}}$, namely that if $P_t \rightarrow^{\text{abs}} P_{t'}$ then we can infer $t \rightarrow^c t'$. To this purpose, we first prove a stronger version of Lemma 5.4.

LEMMA 6.1. ($\xrightarrow{\text{abs}}{}^{-1}$ almost preserves μ-summands). *If* $P_1 \rightarrow^{\text{abs}} P_2$ *and*

$P_2 \rightarrow^\mu P_2'$ *then* $\exists P_1'$ s.t. $P_1 \rightarrow^\mu P_1'$, *where* either $P_1' \rightarrow^{\text{abs}} P_2'$ or $P_1' \rightarrow^{\text{abs}} \tau P_2'$.

Proof. Let $h: P_1 \rightarrow P_2$ be an a.h. and suppose $P_2 \rightarrow^\mu P_2'$. Then, if r_2' is the root of P_2' in P_2, we have $l(r_2') = \mu$. Also, since h is surjective, $r_2' = h(r_1')$ for some state $r_1' \in Q_1$. Now in general it will be

$$r_1 \multimap q_1 \multimap \cdots \multimap q_n = r_1'.$$

Since h is order-preserving, this implies

$$r_2 = h(r_1) \leqslant h(q_1) \leqslant \cdots \leqslant h(q_n) = h(r_1') = r_2'.$$

Now, we know that $r_2 \multimap r_2'$ and $r_2 \notin h(Q_1)$. Therefore it must be

$$r_2 = h(r_1) \multimap h(q_1) = \cdots = h(q_n) = h(r_1') = r_2'.$$

This implies

$$l(q_1) = \cdots = l(q_n) = l(r_2') = \mu.$$

Let P_1' be the derivative of P_1 whose root is q_1'. Then we have $P_1 \to^{\mu} P_1'$, and the rest of the proof is the same as for Lemma 5.4. ∎

We now have the following (soundness and) completeness result.

THEOREM 6.2. *$t \to^{c} t'$ iff $P_t \to^{\text{abs}} P_{t'}$.*

Proof of Completeness. We show, by *induction* on the sum of the sizes of P_t and $P_{t'}$, that $P_t \to^{\text{abs}} P_{t'}$ implies $t \to^{c} t'$.

Assume $t = \sum_i \mu_i t_i$, $t' = \sum_j \nu_j t_j'$, where $i \in I$, $j \in J$. In the rest of the proof, we shall use P, P' for P_t, $P_{t'}$ and P_i, P_j' for P_{t_i}, $P_{t_j'}$. Let $\nu_j t_j'$ be a summand of t'. By Lemma 6.1, $\exists i \in I$ s.t. $\mu_i = \nu_j$, and *either* $P_i \to^{\text{abs}} P_j'$ *or* $P_i \to^{\text{abs}} \tau P_j'$. By induction we have correspondingly *either* $t_i \to^{c} t_j'$ *or* $t_i \to^{c} \tau t_j'$. In both cases we can deduce: $\mu_i t_i \to^{c} \nu_j t_j'$ (using R4 for the latter case).

So, corresponding to any $j \in J$, we can find $i \in I$ s.t. $\mu_i t_i \to^{c} \nu_j t_j'$. Let $I_J \subseteq I$ be the set of all indices i thus chosen to match some $j \in J$. Then

$$\sum_{i \in I_j} \mu_i t_i \to^{c} \sum_j \nu_j t_j'.$$

Hence, substituting in t,

$$t = \left(\sum_{i \in I_j} \mu_i t_i + \sum_{k \in I - I_j} \mu_k t_k \right) \to^{c} \left(\sum_j \nu_j t_j' + \sum_{k \in I - I_J} \mu_k t_k \right) = t' + \sum_{k \in I - I_J} \mu_k t_k.$$

We show now that $(t' + \sum_{k \in I - I_J} \mu_k t_k) \to^{c} t'$, and this will end the proof of the theorem. Each $\mu_k t_k$ is a summand of t. Thus for $P = P_t$, $P_k = P_{t_k}$, we have $P \mapsto^{\mu_k} P_k$. Since $P \to^{\text{abs}} P'$, we can deduce (by Lemma 5.3) that $\exists P''$ s.t. $P' \mapsto^{\mu_k} P''$, where *either* $P_k \to^{\text{abs}} P''$ *or* $P_k \to^{\text{abs}} \tau P''$.

Let now t'' be s.t. $P'' = P_{t''}$. Note that $t' \mapsto^{\mu_k} t''$. By induction we have *either* $t_k \to^{c} t''$ or $t_k \to^{c} \tau t''$. In any case we can deduce $\mu_k t_k \to^{c} \mu_k t''$. Thus we have

$$(t + \mu_k t_k) \to^{c} (t' + \mu_k t'').$$

Since $t' \mapsto^{\mu_k} t''$, we can now use R5 to get

$$(t' + \mu_k t'') \to^{c} t'.$$

As this can be repeated for all $k \in (I - I_J)$, we can conclude that

$$\left(t' + \sum_{k \in I - I_J} \mu_k t_k\right) \to^{c} t'.$$

To sum up, we have shown that

$$\begin{aligned} t &= \left(\sum_{i \in I_J} \mu_i t_i + \sum_{k \in I - I_J} \mu_k t_k\right) \to^{c} \left(\sum_j \nu_j t'_j + \sum_{k \in I - I_J} \mu_k t_k\right) \\ &= \left(t' + \sum_{k \in I - I_J} \mu_k t_k\right) \to^{c} t'. \quad \blacksquare \end{aligned}$$

COROLLARY 6.3. *R_c is a rewriting system for the equational theory E_c.*

We can make use of our new axiomatisation for $=^{c}$ to characterise normal forms for terms in T_Σ. We say that a term is in *normal form* if no proper reduction (R3, R4, or R5) can be applied to it. It can be shown that

THEOREM 6.4. *A term $t = \sum_i \mu_i t_i$ is a* normal form *iff (Hennessy–Milner characterisation):*

(i) *no t_i is of the form $\tau t'$*

(ii) *each t_i is a normal form*

(iii) *for $i \neq j$, $t_i \not\leftrightarrow t'_j$ $\forall t'_j$* s.t. $\mu_j t_j \Rightarrow^{\mu_i} t'_j$.

Corresponding to normal forms, we have a notion of *minimality* for processes. We say that a process P is *irreducible* or *minimal* iff $\forall P'$: $P \to^{abs} P'$ implies $P = P'$. Then it is easy to see that

THEOREM 6.5. *For any finite NDP P, $\exists!$* minimal *NDP P'* s.t. $P \sim_{abs} P'$.

Proof. For uniqueness, use $\to^{abs}$'s Church–Rosser property. $\blacksquare$

We shall denote by $\hat{P}$ the unique minimal process corresponding to the *NDP P*.

COROLLARY 6.5. *$P \sim_{abs} P'$ iff $\hat{P} = \hat{P}'$.*

As we mentioned earlier, the denotation P_t of a term t is always a *tree*. However, its "abstract" denotation $\hat{P}_t$ might not be a tree. We shall now propose a *tree model* for terms of T_Σ, which is isomorphic to the term model $T_\Sigma / =^{c}$.

Note first that any NDP which is not a tree has a *unique unwinding* into a tree. The tree-unwinding of an NDP P (which is not defined formally here) will be denoted by $U(P)$.

Now let $\mathscr{RT}$ (*representation trees*) be the class: $\mathscr{RT} = \{U(P) | P$ is a minimal NDP$\}$. The *denotation* T_t of a term $t \in T_\Sigma$ in $\mathscr{RT}$ is defined by: $T_t = U(\hat{P}_t)$.

It can be easily shown that:

THEOREM 6.6. $t =^{c} t'$ *iff* $T_t = T_{t'}$.

We shall finally argue that our model $\mathscr{RT}$ is *isomorphic* to the term model $T_\Sigma/=^{c}$. $\mathscr{RT}$ is a Σ-algebra satisfying the axioms E_c (by Theorem 6.6), with the operators defined by

$$\mu U(P) = U(\widehat{\mu P})$$

$$U(P_1) + U(P_2) = U(\widehat{P_1 + P_2}).$$

Therefore, since $T_\Sigma/=^{c}$ is the initial Σ-algebra satisfying the axioms E_c, we know that

$$\exists!\ \Sigma\text{-homomorphism } \Psi\colon T_\Sigma/=^{c} \to \mathscr{RT}.$$

It is easily seen that Ψ is given by: $\Psi([t]) = U(\hat{P}_t) = T_t$. Also, by Theorem 6.6 again, Ψ is a *bijection* between T_Σ and $\mathscr{RT}$.

7. CONCLUSION

We have proposed an alternative definition for the (substitutive) bisimulation equivalence $\langle \approx \rangle^{+}$ for a class of transition systems. Note that the ordinary bisimulation equivalence could be characterised just as easily, by slightly changing the definition of homomorphism; in fact it would be enough to drop the requirement that *proper* states should be preserved. Also, using our definition, we have been able to derive a denotational model for the language L, which is isomorphic to Hennessy and Milner's term model for the same language.

Our approach is intended to extend to richer languages, for programs which are *both* nondeterministic *and* concurrent (meaning that the actual concurrency is not interpreted nondeterministically). Some simple results have already been reached in that direction.

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