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Intersection modeling using a convergent scheme based on Hamilton-Jacobi equation

G. Costeseque[a,b,*], J.P. Lebacque[b]

[a]*Université ParisEst, CERMICS, 6 & 8 avenue Blaise Pascal, Cité Descartes, 77455 Marne la Vallée Cedex 2, France*
[b]*Université ParisEst, GRETTIA, 2 rue de la Butte Verte, Bâtiment Le Descartes II, 93166 Noisy-le-Grand Cedex, France*

Abstract

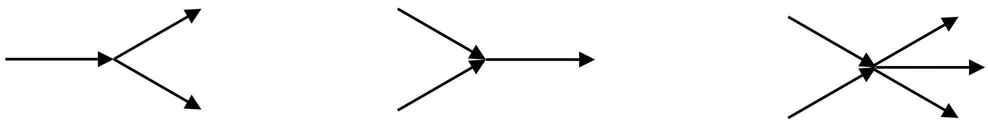
This paper presents a convergent scheme for Hamilton-Jacobi (HJ) equations posed on a junction. The general aim of the approach is to develop a framework using similar tools to the variational principle in traffic theory to model intersections taking in account many incoming and outgoing roads. Then a time-explicit numerical scheme is proposed. It is based on the very classical Godunov scheme. The proposed model could be characterized as a pointwise model of intersection without any internal state. Moreover, our model respects the invariance principle. This scheme is applied to the cases of diverge junctions. The goal is to know how to manage the fluxes in order to maximize the flow through the junction.

Keywords: Hamilton-Jacobi equation, discontinuous Hamiltonians, numerical scheme, traffic modeling, intersection, junction.

1. Introduction

This paper focus on a convergent scheme for Hamilton-Jacobi (HJ for short in the remainder) equations posed on a junction. The motivation comes from traffic modeling and more precisely from modeling of network intersections. The distinction is made between the words *junction* and *intersection*. A *junction* classically defines either merge or diverge while *intersection* is a more general configuration with many incoming and outgoing roads. As illustrated below, diverge models deal with a branch separating between at least two others branches. Merge could be seen as the inverse that is at least two roads converging to another road. The general goal of the approach is to develop a general framework to model complex intersections.

[*] Corresponding author. Tel.: +3-316-415-3517; fax: +3-316-415-3586.
E-mail address: guillaume.costeseque@cermics.enpc.fr.



a. Diverge junction b. Merge junction c. “Complex” intersection

Figure 1. Illustrations of junctions and intersection

The aim of the paper is to study how incoming fluxes behave on the network at the junction point in order to maximize the flow through the junction. In most of cases the particles flow at such singular points stay uneasy to understand. Any type of vehicles is considered and it is just assumed that vehicles are almost homogeneous. To manage the problem, we carry out a macroscopic point of view i.e. particles density on the branches allowing to do not care of the microscopic one i.e. the individual dynamics.

The paper first deals with a quick overview of the classical formulation of traffic models. Then the variational formulation in traffic theory is presented. Indeed traffic behavior is now well-known on highways i.e. on long edges without spatial discontinuities. The classical way to deal with traffic patterns at junctions is based on a hyperbolic approach of conservation laws [GP06]. Here a new point of view using a Hamilton-Jacobi approach is proposed. The HJ equation could be seen as an integrated equation from the hyperbolic system. The purpose is to show that Hamilton-Jacobi equations provide a great frame to get rigorous results in the case of road junctions.

Then, the numerical scheme adapted to junction problems is fully described and the results of convergence are recalled. We propose a time-explicit numerical scheme which is based on the very classical Godunov scheme [GO59]. In the remainder, the model could be characterized as a pointwise model of intersection without any internal state. Our model respects the invariance principle [LK05]. As it seems natural to impose different dynamics on each branch of the junction, the resulting Hamiltonian is by nature discontinuous at the junction point. The representation of the discontinuous Hamiltonians deals with the notion of local traffic demand and supply functions [LE93], [LE96]. Our main variable could be assimilated to the index of vehicles which cross the junction. The main results we already have in [CML12] are first space and time gradients estimates and then the convergence of the numerical scheme to the continue solution of the HJ equation (see equation (11)). Note that existence and uniqueness results for HJ equations at a junction point have been already proved in such a case by applying control framework [IMZ11]. It is pointed out that the numerical scheme is adapted to solve such a problem. Finally, it is applied to a diverge junction.

2. Background

The purpose in this paper is to present a numerical scheme which goal is to model the behavior of road traffic flow at a junction point. The approach is based on the resolution of Hamilton-Jacobi (HJ) equation while in traffic theory it is more current to deal with hyperbolic conservation laws. Indeed historically one of the most important traffic flow model was proposed by Lighthill and Whitham [LW55] and independently by Richards [RI56]. This model so-called LWR model was built on the analogy between road traffic flow and hydrodynamics.

2.1. Classical hydrodynamic formulation

Based on B. Greenshields works [GR35] introducing a phenomenological law describing the relation between the density denoted by $\rho(x,t)$ and the flow $Q(x,t)$ for road traffic, Lighthill, Whitham [LW55] and independently Richards [RI56] proposed a hyperbolic system of conservation law in the middle of 1950’s. This model became the seminal model for traffic flow theory. The model is given by the following system:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial Q}{\partial x} = 0 \\ Q = \rho V \\ Q = Q_e(\rho, x) \end{cases} \tag{1}$$

The first equation traduces the conservation of vehicles for a given section. This equation is valid for continuum description of any conserved quantity. The second one gives the definition of the flux speed V. The last relationship i.e. $Q = Q_e(\rho, x)$ is now classically called the Fundamental Diagram (FD for short) and it states that the traffic is always on an equilibrium situation. This assumption could be dropped out, giving birth to higher order models that all could be represented by the Generic Second Order Models [LMH07]. However the paper only focuses on first order models.

The velocity function assumes some reasonable properties: first when the density is small enough the velocity is maximal and when the density increases up to some maximum capacity ρ_{max} the velocity decreases. The flow function $Q_e(\rho)$ is thus concave with a unique maximum and $Q_e(0) = Q_e(\rho_{max}) = 0$. Associated to the equilibrium flow-density relationship, there are two more equilibrium relationships i.e. the local equilibrium supply and demand functions (resp. $\Sigma(\rho, x)$ and $\Delta(\rho, x)$) which were described by [DA95], [LE93] and [LE96]. Both functions are defined above:

$$\Sigma(\rho, x) \overset{\text{def}}{=} \begin{cases} Q_{max} & \text{if } \rho \leq \rho_{crit} \\ Q_e(\rho, x) & \text{if } \rho \geq \rho_{crit} \end{cases} \tag{2}$$

And

$$\Delta(\rho, x) \overset{\text{def}}{=} \begin{cases} Q_e(\rho, x) & \text{if } \rho \leq \rho_{crit} \\ Q_{max} & \text{if } \rho \geq \rho_{crit} \end{cases} \tag{3}$$

These functions express the greatest possible inflow and outflow at any point $x \geq 0$, and result from the fundamental diagram as illustrated on figure 2. At any point, the flow is equal to the minimum between upstream demand and downstream supply:

$$Q(x,t) = \min\big(\Sigma(\rho(x^+, t), x), \Delta(\rho(x^-, t), x)\big) \tag{4}$$

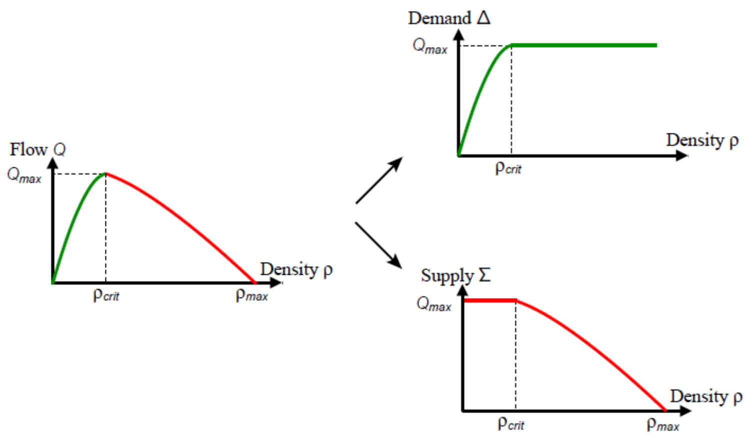


Figure 2. Illustration of the supply and demand functions

2.2. Hamilton-Jacobi formulation

The concept of vehicle number function $N(x,t)$ standing for the cumulative number of vehicles or cumulated vehicles curves (CVC) passing at a given position after a given time on a one-dimensional road was originally proposed by K. Moskowitz, an engineer who first used this concept to investigate properties of traffic. This function was introduced to kinematic wave theory in the early nineties by G. Newell's trilogy [NE93]. The main proofs for selecting the unique and correct value at every point in space-time were proposed later by Daganzo ([DA05a], [DA05b] , [DM05] and [DA06]). These papers have shown that the function $N(x,t)$ is solution of the following Hamilton–Jacobi equation:

$$\frac{\partial N}{\partial t} - Q\left(-\frac{\partial N}{\partial x}\right) = 0 \tag{5}$$

This HJ equation comes from a variational formulation of kinematic waves used in the classical resolution of the LWR model. It is commonly known that dealing with the CVC are more interesting in the traffic engineering framework than dealing with the common-used density and flow notions. Indeed Moskowitz function is most easily described in terms of integer labels that are carried by vehicles. Each vehicle carries a unique label for all time and that the labels have been numbered consecutively at their enter time in the section. Thus, the number on each label indicates at all times the position of its current vehicle in the traffic stream. If traffic flows in the direction of increasing x axis assumed in this paper, then the Moskowitz function describes a geometric surface in (t,x,n) space.

The intersection of this surface and the plane for a given x-coordinate gives a (t,n) curve. A collection of N-curves at different abscissas along a one-dimensional road is useful for traffic engineering because the horizontal separation between two N-curves for a given ordinate n represents the trip time of the n-th labeled vehicle. In the same way, the vertical separation between both N-curves for a given abscissa t represents the vehicles accumulation at time t. The study of the evolution of the surface of Moskowitz can be based on travel in each of planes (t,x) or (t,n). The variational principle allows developing simple solution techniques based on finding shortest paths in both planes. It states that the traffic function is the value function of an optimal control problem. For example, the variation of N from the point A to the point B is given by the following equation:

$$\Delta N_{AB} = N(x_B,t_B) - N(x_A,t_A) = \int_{t_A}^{t_B} (Q(t) - \rho(t)u_{AB})dt \tag{6}$$

In the equation (6) the quantity u_{AB} represents the slope of the line between both points in the (t,x) plane. According to Daganzo works, it has been shown that the position of the surface can be calculated at any point in space from a simple translation of the boundary conditions. The actual surface is then the lower envelope of these translations, given by the set of Fundamental Diagram.

Notice that some articles like [DA06] and [LLC07] have deeply investigated the links between variational theory and Lagrangian coordinates. It is another way of expressing the LWR model. However in the remainder the variables stay expressed from an Eulerian point of view.

2.3. Overview of junctions modeling

So far, traffic on networks has mostly been modeled with conservation laws. Flow propagation on links has been extensively studied and various adequate link models exist. For example the modeling of traffic flow by conservation laws was studied by the LWR model described previously in subsection 2.1. Intersection models for macroscopic simulation have attracted relatively little attention in traffic theory literature. Many macroscopic intersection models are generalizations of simple merge and diverge models. These intersection models are typically pointwise i.e. the intersection is reduced to a point without any physical dimensions. However notice that even if for pointwise model a complex intersection model could not be a simple superposition of merge and diverge models because of the management of conflicts at the node.

It is also classical to distinguish first order link models which consider only one independent variable (often the flow) and higher models which transfer not only flow but also speeds or momentum. First order models may not allow relevantly representing traffic flow phenomena like instabilities, capacity drop or stop-and-go waves. In first order macroscopic models, the link model provides the demand of incoming links and the supply of outgoing links as constraints to the intersection model. While it is possible to quote the intersection models of Haut, Bastin [HB07] or Herty and Klar [HK03] which are based on second order models, the remainder just focuses on first order models. The following overview is straightforward adapted from Tampère et al. [TCCI11].

Merge models have been described by e.g. Daganzo [DA95], Lebacque [LE96] and Jin and Zhang [JZ03]. It is also possible to quote the merge distribution schemes of Ni and Leonard [NL05]. Daganzo proposed to maximize the total flow i.e. $\max q = \sum_i q_i$ for incoming branches $i \in \mathbb{N}$. Analogously than in (2)-(3), let us define demand constraints $\varDelta_i$ formally as the maximum flow that incoming link (i) could possibly send if the node and outgoing link would impose no constraints on the outflow of link (i). Respectively the supply constraint constraints Σ_j is the maximum inflow that outgoing link (j) could receive. While Daganzo introduced fixed distribution fractions d_i to reflect priorities, Jin and Zhang considered a demand proportional distribution incorporated $d_i = \varDelta_i / \sum_k \varDelta_k$ and Ni and Leonard proposed a capacity proportional distribution $d_i = Q_i^{max} / \sum_k Q_k^{max}$ for the incoming links.

For diverge models, it is classically assumed that the total flow q is maximized. This total flux through the junction point corresponds to the outflow of the incoming link which divides over the outgoing links according to some turning fractions which could be fixed or variable. The traffic flow is classically assumed to be FIFO (First-In-First-Out) i.e. the vehicles keep ordered and exit the junction point in a FIFO sequence. Notice that the FIFO assumption neglects the separation of traffic in different turning lanes. Congestion on one lane will immediately affects all the lane of the road.

Some models propose an overview of macroscopic intersection models. For models like presented in [HR95] or [CGP05], one needs to introduce partial demands i.e. proportion derived from the total demand of an incoming link (i) whishing to outflow on the outgoing link (j).

According to [TCCI11], a macroscopic intersection model should achieve general consistency with the fundamental modeling principles of first order traffic flow theory if it complies what follows:

- General applicability of the model to any number of incoming links and outgoing links;
- Non-negativity of flows;
- Conservation of vehicles;
- Compliance with demand and supply constraints;
- Flow maximization from the users' perspective. Indeed notice that global flow maximization would imply a behaviorally unrealistic cooperation of drivers;
- Conservation of turning fractions coming from the FIFO rule at the intersection level;
- Compliance with the invariance principle of Lebacque and Khoshyaran [LK05]. The invariance principle sets that the intersection model should be invariant by replacing Δ_i by the capacity of link (i) Q_i^{max} if q_i is supply constrained i.e. $q_i < \Delta_i$ or analogously invariant to an increase of Σ_j up to Q_i^{max} if $q_j < \Sigma_j$.

The intersection model proposed in the next section should satisfy each condition of this list.

3. Intersection model

3.1. Getting of the equations

Before all, let us recall that the presented model is pointwise, without internal dynamics. The axis are considerate for positive abscissas i.e. that there is no specific consideration for incoming or outgoing roads. For analogy, it is just like if the space has been divided into two half-planes and the negative one has been cut and returned on the positive side. Each branch α is denoted by branch J_α. Notice that J_α^* stands for the half-line minus the point of abscissa $x = 0$. It is illustrated on the figure 3 below.

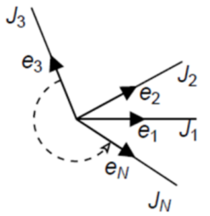


Figure 3. Illustration of the intersection model

It is classical to consider that the well-known LWR equation is satisfied on each branch α of the model i.e. that the densities are conserved on each branch α according to the following equation:

$$\rho_t^\alpha + \left(Q^\alpha(\rho^\alpha)\right)_x = 0 \tag{7}$$

Let us consider the new variable denoted by $U^\alpha(x,t) = \frac{1}{\gamma^\alpha}\int_0^x \rho^\alpha(y,t)dy + g^\alpha(t)$ where γ^α describes the mix fraction on each incoming road or the turning fraction on each outgoing road of the total flow through the junction. These coefficients are such that:

$$0 \le \gamma^\alpha \le 1 \quad \text{and} \quad \begin{cases} \sum_{\alpha=1}^{n} \gamma^\alpha = 1 \\ \sum_{\alpha=n+1}^{n+m} \gamma^\alpha = 1 \end{cases} \tag{8}$$

For each branch, the function $g^\alpha(t) = U^\alpha(0,t)$ represents the vehicle index at the junction point and the function is taken such that $g^{\alpha\prime}(t) + \frac{1}{\gamma^\alpha} Q^\alpha\left(\rho^\alpha(x,0)\right) = 0$. Thus the variable $U^\alpha(x,t)$ which is very similar to the cumulated number of vehicles on the branch α verifies the following Hamilton-Jacobi equation:

$$U_t^\alpha + \frac{1}{\gamma^\alpha} Q^\alpha(\gamma^\alpha U_x^\alpha) = 0 \tag{9}$$

For taking in account the assumption on positively oriented axis it is necessary to little modify the variable such that:

$$u^\alpha(x,t) = \begin{cases} -U^\alpha(-x,t) & \text{for } x > 0, \text{ for } \alpha \in \{\text{incoming roads}\} \\ -U^\alpha(x,t) & \text{for } x > 0, \text{ for } \alpha \in \{\text{outgoing roads}\} \end{cases} \tag{10}$$

The reader can notice that the vehicles are numbered from a mathematical point of view and not from a traffic eye. Indeed while in traffic engineering it is more accurate to index the vehicles according to their passage time at a given position, in mathematics it is simpler to consider vehicles positively numbered according to the positive direction of the spatial axis. In practice, both indexes just differ from a negative sign.

The variable u^α which is now available for incoming and outgoing roads satisfies the following HJ equation:

$$\begin{cases} u_t^\alpha + H_\alpha(u_x^\alpha) = 0 & \text{on } (0,T) \times J_\alpha^* \text{ for } \alpha = 1, \dots, N \\ u_t^\alpha + \max_\alpha H_\alpha^-(u_x^\alpha) = 0 & \text{on } (0,T) \times \{0\} \end{cases} \tag{11}$$

Submitted to the initial condition

$$u^\alpha(x,0) = u_0^\alpha(x) \quad \text{for all } \alpha = 1, \dots, N \tag{12}$$

The convex Hamiltonians H_α are defined more or less as the opposite of the Fundamental Diagram i.e. they are given by what follows for all α:

$$H_\alpha(p) \stackrel{\text{def}}{=} \begin{cases} -\dfrac{1}{\gamma^\alpha} Q^\alpha(\gamma^\alpha p), & \text{for } \alpha \in \{\text{incoming roads}\} \\ -\dfrac{1}{\gamma^\alpha} Q^\alpha(-\gamma^\alpha p), & \text{for } \alpha \in \{\text{outgoing roads}\} \end{cases}, \quad \forall p \tag{13}$$

In the previous HJ equation (11) the function H_α^- represents the non-increasing part of the Hamiltonian. The non-decreasing part is respectively denoted by H_α^+. In this way it is possible to identify the function H_α^- (resp. H_α^+) to the opposite of the Demand function Δ^α (resp. the Supply function Σ^α) modulo a coefficient equal to the inverse of the turning fraction γ^α. This is not trivial because the construction of the numerical scheme on the Supply and Demand functions allows satisfying the invariance principle of [LK05].

3.2. Presentation of the scheme

Let us consider a discretization of the time and space axis and let us set Δx and Δt the space and time steps with $\Delta x, \Delta t > 0$. In the remainder let us consider $U_i^{\alpha,n}$ the discrete value of the function u^α at the grid point $x = i\,\Delta x$ and $t = n\,\Delta t$ (with $i, n \in \mathbb{N}$) on the considered branch α, such that:

$$U_i^{\alpha,n} \stackrel{\text{def}}{=} u^\alpha(i\,\Delta x, n\,\Delta t) \tag{14}$$

We now recall the HJ equation verified by our variable $u^\alpha(x,t)$ which was given previously by (11) with the initial condition (12). Let us defined the discrete gradient forward (respectively backward) as follows:

$$p_{i,+}^{\alpha,n} \stackrel{\text{def}}{=} \frac{U_{i+1}^{\alpha,n} - U_i^{\alpha,n}}{\Delta x}, \quad n \geq 0, \quad i \geq 0 \tag{15}$$

$$\left(\text{and respectively} \quad p_{i,-}^{\alpha,n} \stackrel{\text{def}}{=} \frac{U_i^{\alpha,n} - U_{i-1}^{\alpha,n}}{\Delta x}, \quad n \geq 0, \quad i \geq 1\right) \tag{16}$$

We also defined the discrete time derivative such that:

$$W_i^{\alpha,n} \stackrel{\text{def}}{=} \frac{U_i^{\alpha,n+1} - U_i^{\alpha,n}}{\Delta t}, \quad n \geq 0, \quad i \geq 0 \tag{17}$$

Then the numerical scheme is given by the following equation:

$$\begin{cases} W_i^{\alpha,n} = \min\left(-H_\alpha^-\left(p_{i,+}^{\alpha,n}\right), -H_\alpha^+\left(p_{i,-}^{\alpha,n}\right)\right) & \text{for } i \geq 1 \\ \begin{cases} U_0^{\alpha,n} \stackrel{\text{def}}{=} U_0^n \\ W_0^{\alpha,n} = \min\limits_{\alpha}\left(-H_\alpha^-\left(p_{0,+}^{\alpha,n}\right)\right) \end{cases} & \text{for } i = 0 \end{cases} \tag{18}$$

With the initial condition

$$U_i^{\alpha,0} \stackrel{\text{def}}{=} u_0^\alpha(i\,\Delta x) \tag{19}$$

Our numerical scheme is adapted for HJ equations posed on junction. This scheme is equivalent to the very classical Godunov scheme which is already used in the framework of traffic schemes (see e.g. the Daganzo cell transmission model [DA95a], [DA95b] or Lebacque models [LE96]). Our model takes a general point of view assuming that the drivers try to maximize the flow locally at each intersection.

4. Simulations for a divergent

4.1. Settings

The modeling of road diverge junction has no longer interested the traffic scientific community. The main publications available on junction modeling are mainly oriented for merge models. Indeed the main process at the junction point is more complex for a merge model for which the flows from each incoming road have to mix to go through the node. There is no such a complex behavioral law for diverge since the flow which can pass through the junction may divide into each outgoing road without generating any conflict between each other.

In traffic theory the simplest case of a diverge junction corresponds to the case when a road splits into two. The cars move on the single road in the direction of the junction at which they can either go left or right. In their recent work [IMZ11], Imbert, Monneau and Zidani have shown that for convex Hamiltonians, among the admissible solutions (i.e. which are compatible with the conservation of cars) there is a unique optimal solution which allows the maximum flow of cars at the junction. Indeed this solution is the value function of an optimal control problem associated to Lagrangians which are discontinuous at the junction point. The junction condition is similar to the Lebacque condition of Demand and Supply usually used in traffic. This condition is also similar to the one used for the Riemann solver at the junction in [CGP05] and has deep relations with the Bardos-Le Roux-Nédélec boundary condition for conservation laws [BLN79].

In the following numerical experiments, the boundary conditions have to be properly defined. On the upstream boundary of the link, the flow entering the section $QI(t)$ is taken as the minimum between the supply of the first cell $\Sigma_{a^+}(t)$ and the demand upstream $\Delta_u(t)$ which is defined equal to the initial chosen flow on the branch. On downstream boundaries of the links, the exiting flow $QO(t)$ is equal to the minimum between the demand of the last cell $\Delta_{b^-}(t)$ and the downstream supply $\Sigma_d(t)$ which is defined equal to the supply of the last cell. The supply outside could be supposed to be infinite but in this case the demand is always satisfied and a fictitious kinematic wave could be generated.

Figure 4. Illustration of the boundary conditions

$$\begin{cases} QI(t) \stackrel{\text{def}}{=} Min\left(\Delta_u(t), \Sigma_{a^+}(t)\right) \\ QO(t) \stackrel{\text{def}}{=} Min\left(\Delta_{b^-}(t), \Sigma_d(t)\right) \end{cases} \tag{20}$$

4.2. Riemann problems

In traffic modeling generalized Riemann problems are often considered because they reproduce the main cases occurring on real situations. A generalized Riemann problem is problem for which we have to find the solution $\rho(x,t)$, for all $x,t \geq 0$ of the LWR system (1) with the following initial conditions:

$$\rho(x,0) = \begin{cases} \rho_l & if \quad x < 0 \\ \rho_r & if \quad x > 0 \end{cases} \tag{21}$$

And Q_e given by

$$Q_e(\rho,x) = \begin{cases} Q_e(\rho,l) & if \quad x < 0 \\ Q_e(\rho,r) & if \quad x > 0 \end{cases} \tag{22}$$

Let us set for sake of clarity what follows

$$\begin{cases} Q_l \stackrel{\text{def}}{=} Q_e(\rho_l, l) \\ Q_r \stackrel{\text{def}}{=} Q_e(\rho_r, r) \end{cases} \tag{23}$$

Let us deal with inhomogeneous Riemann problems. Here the flow functions up and downstream are not equal. The discontinuities of the flow function depending on the spatial variable may represent physically meaningful cases such as:

- A change of maximal speed on a road by example passing from an urban road to a ring road. Let us also assume that one driver is on an inside lane of a roundabout and that he has the option to either exit at the next road on the right or to continue to turn by taking the left lane to exit later. It is assumed that the features on the roundabout inside lane are such that the speed is fairly low (less than or equal to 50 km/h) while on the outgoing branch, the speed limit may be higher.
- A change of the number of lanes on a road with equal upstream and downstream speed properties. There are two possible illustrations with either a decrease occurred because of a fixed or mobile bottleneck or a capacity increase e.g. by passing from two to three lanes;
- Both changes (maximal speed and number of lanes). Let us consider the exit road of a highway. While the main section can have a fairly large number of lanes (e.g. higher than two lanes) and high speed (90, 110 or 130 km/h) the exit road has a single lane and a lower maximal speed (typically 50, 70 or 90 km/h).

In these simulation experiments, a wide range of situation for which the speed and / or capacity can vary was considered including the most interesting cases like (i) junction, (ii) sections with a different number of lanes, (iii) the presence of an incident resulting in a restriction of local and temporary capacity or speed. According to [LE96] the values of the flow at the singularity point that result for the solution of the generalized Riemann problem are given in the following tables:

$\Sigma(\rho_r, r)$ / $\Delta(\rho_l, l)$	$Q_{max}(r)$	$Q_e(\rho_r, r)$
$Q_e(\rho_l, l)$	Q_l	$Min(Q_r, Q_l)$
$Q_{max}(l)$	$Q_{max}(l)$	$Min\big(Q_r, Q_{max}(l)\big)$

Table 1. Flow resulting for the generalized Riemann problem with $Q_{max}(l) \geq Q_{max}(r)$

$\Delta(\rho_l, l)$ \ $\Sigma(\rho_r, r)$	$Q_{max}(r)$	$Q_e(\rho_r, r)$
$Q_e(\rho_l, l)$	$Min(Q_r, Q_{max}(l))$	$Min(Q_r, Q_l)$
$Q_{max}(l)$	$Q_{max}(r)$	Q_r

Table 2. Flow resulting for the generalized Riemann problem with $Q_{max}(l) \leq Q_{max}(r)$

Let us then focus on the so-called homogeneous Riemann problem: in this case the fundamental diagram on upstream links is identical to the one on downstream so the maximum speed, the critical concentration and the maximum concentration are the same upstream and downstream. Hence we consider in this particular case that the flow function is the same on incoming roads and outgoing roads. It is the simplest case because there is no diffraction-like phenomenon at the junction point. This also implies that the number of lanes per incoming roads is equal to the number of lanes per outgoing ones. One may wonder if it is really appropriate to consider such a homogeneous Riemann problem in traffic case. At our own knowledge it could be mainly found for urban intersections with the example of a fork. According to Lebacque [LE96] seven cases could be distinguished for homogeneous Riemann problems:

- Cases of ρ_l and ρ_r under-critical: first subcase with $\rho_l \leq \rho_r$; second subcase with $\rho_l \geq \rho_r$;
- Cases of ρ_l and ρ_r overcritical: first subcase with $\rho_l \leq \rho_r$; second subcase $\rho_l \geq \rho_r$;
- Case of ρ_l overcritical and ρ_r under-critical;
- Case of ρ_l undercritical and ρ_r overcritical: first subcase $Q_l \leq Q_r$; second subcase $Q_l \geq Q_r$.

In all of these cases and whatever the initial conditions and the turning coefficients, the numerical scheme allows to reproduce queues and kinematic waves through the junction point. For each case, the scheme fits with the analytical solution to the Riemann problem. In main cases, there is apparition of a kinematic wave which speed is given by the classical Rankine-Hugoniot jump condition i.e.

$$c = \frac{[Q]}{[\rho]} = \frac{Q_r - Q_l}{\rho_r - \rho_l} \tag{24}$$

According to the sign of the wave celerity c, either there is a shock wave which travels through the junction point and which affect the flow upstream leading to the apparition of a queue or there is apparition of a kinematic wave (which could be a rarefaction fan in certain cases) which travels from the junction point on the downstream branches.

The numerical scheme seems to be able to reproduce the features of the analytical solutions for the Riemann problems. However no estimation error between numerical and analytical solutions has been obtained until now.

One of the main shortcomings of the model seems to be the non-specialization of lanes. Indeed there is no particular consideration of a multi-lanes branch. In practice if an outgoing road is congested and if the proportion of entering vehicles occupy a single lane on the incoming branch, the model would consider that any incoming branch is congested and that the flow through the junction is null. Nevertheless it is possible to consider that the junction actually starts at the real flow separation point which may be upstream of the physical node.

5. Conclusion

To gather microscopic and macroscopic patterns for intersection modeling, a numerical scheme built on Hamilton-Jacobi traffic equation is proposed. This numerical scheme was shown to be able to reproduce delays and queue for a diverge model. The procedure is straightforward to implement and has a low computational cost.

However some simplifying assumptions were made for e.g. not considering the separation of turning movements into lanes on the upstream branches.

The main wished improvements are (i) to extend the results of [IMZ11] to more general and realistic junction conditions including suboptimal traffic flows like for instance signalized intersections or queues of cars (ii) to explore the link with conservation laws by showing that the derivative of the HJ solution on a network is a solution of the conservation law with a scheme for which the convergence of the numerical solution is well-known and (iii) to stress on a micro-macro approach for the divergent junction modeling by recovering that the natural junction condition could be obtained by considering the macroscopic limit of the discrete dynamics model of individual cars. Then convergent junctions could be considered as well as priority laws for recovering a reasonable junction condition and uniqueness of the corresponding solution for Hamilton-Jacobi equation.

Further research should be conducted in order to extend the model to merges. In the particular case of merge model, a phenomenological priority law should be introduced to determine how the fluxes behave to mix them at the junction point. This law may come from a passage from the microscopic scale to the macroscopic one. Indeed it seems interesting to define a macroscopic behavior law from the way that vehicles interact at the microscopic level.

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References

[BLN79] C. Bardos, A.Y. Le Roux and J.C. Nédélec, First order quasilinear equations with boundary conditions, Comm. Partial Differ. Equations, 4 (1979), pp. 1017- 1034.
[CGP05] G. M. Coclite, M. Garavello, and B. Piccoli, Traffic Flow on a Road Network, SIAM J. Math. Anal. 36, (2005), pp. 1862-1886.
[CML12] G. Costeseque, R. Monneau and J.P. Lebacque, A convergent scheme for Hamilton-Jacobi equation on a junction: application to traffic, work in progress.
[DA95] C. F. Daganzo, The cell transmission model, Part II: Network traffic. Transportation Research, Vol. 29B, No. 2, (1995), pp. 79-93.
[DA05a] C.F. Daganzo, A variational formulation of kinematic waves: Basic theory and complex boundary conditions, Transportation Research 39B (2005) pp. 187–196.
[DA05b] C.F. Daganzo, A variational formulation of kinematic waves: Solution method, Transportation Research 39B (2005) pp. 934–950.
[DA06] C.F. Daganzo, On the variational theory of traffic flow: well-posedness, duality and aplications, Networks and Heterogeneous Media, AIMS, vol.1, n°4, (2006), pp. 601-619.
[DM05] C.F. Daganzo and M. Menendez, A variational formulation of kinematic waves: Bottleneck properties and examples, In H.S. Mahmassani, editor, *Proceedings of the 16th International Symposium on the Transportation and Traffic Theory*, College Park, Maryland, USA, Elsevier, Oxford, (2005), pp. 345-364.
[GP06] M. Garavello and B. Piccoli, Traffic flows on networks, vol. 1 of AIMS Series on Applied Mathematics, *American Institute of Mathematical Sciences* (AIMS), Springfield, MO, (2006).
[GO59] S. Godunov, A Difference Scheme for Numerical Solution of Discontinuous Solution of Hydrodynamic Equations, Math. Sbornik, 47, (1959), pp. 271-306.
[GR35] B.D. Greenshields, A study of traffic capacity, Highway Research Board Proceedings, Vol. 14. Washington: Transportation Research Board, (1935), p. 448-477.
[HB07] B. Haut and G. Bastin, A second order model of road junctions in fluid models of traffic networks, Networks and Heterogeneous Media, Vol. 2(2), (2007), pp. 227 – 253.
[HK03] M. Herty and A. Klar, Modelling, Simulation and Optimization of Traffic Flow Networks - SIAM J. Sci. Comp., Vol. 25(3), (2003), pp. 1066-1087.
[HR95] H. Holden and N. H. Risebro, A Mathematical Model of Traffic Flow on a Network of Unidirectional Roads, SIAM J. Math. Anal., 26 (1995), pp. 999-1017.
[IMZ11] C. Imbert, R. Monneau and H. Zidani, A Hamilton-Jacobi approach to junction problems and application to traffic flows, *accepted*

to ESAIM-COCV, (2011), 38p.

[JZ03] W.-L. Jin and H. M. Zhang, On the distribution schemes for computing fluxes through a highway merge, Transportation Research Part B, 37(6), (2003), pp. 521-540.

[LW55] M. J. Lighthill and G. B. Whitham, On kinetic waves. II. Theory of Traffic Flows on Long Crowded Roads, Proc. Roy. Soc. London Ser. A, 229 (1955), pp. 317-345.

[LE93] J.P. Lebacque, Les modèles macroscopiques du trafic, Annales des Ponts 67, (1993), pp. 28-45.

[LE96] J.P. Lebacque, The Godunov scheme and what it means for first order traffic flow models. In J. B. Lesort, editor, 13th ISTTT Symposium, Elsevier, New York, (1996), pp. 647–678.

[LE05] J.P. Lebacque, Intersection Modeling, Application to Macroscopic Network Traffic Flow Models and Traffic Management, Traffic and Granular Flow '03, Part 3, (2005), pp. 261-278

[LK05] J.P. Lebacque, M.M. Koshyaran, First-order macroscopic traffic flow models: intersection modeling, network modeling, In H.S. Mahmassani, editor, *Proceedings of the 16th International Symposium on the Transportation and Traffic Theory*, College Park, Maryland, USA, Elsevier, Oxford, (2005), pp. 365-386.

[LMH07] J.P. Lebacque, S. Mammar, H. Haj-Salem, Generic second-order traffic flow modeling. In: (R.E. Allsop, M.G.H. Bell, B.G. Heydecker Eds) Proceedings of the 17th International Symposium on Transportation and Traffic Theory, London, (2007), pp. 749–770.

[LLC07] L. Leclercq, J.A. Laval, E. Chevallier, The Lagrangian coordinates and what it means for first order traffic flow models. In: Allsop, R.E., Bell, M.G.H., Heydecker, B.G. (Eds), 17th ISTTT, Elsevier, London, (2007), 735-753.

[NL05] D. Ni and J. Leonard, A simplified kinematic wave model at a merge bottleneck. Applied Mathematical Modelling, 29 (11), 2005, pp. 1054–1072.

[NE93a] G.F. Newell, A simplified theory of kinematic waves in highway traffic, part I: General theory, Transportation Research B 27B (4), (1993), pp. 281–287.

[NE93b] G.F. Newell, A simplified theory of kinematic waves in highway traffic, part II: Queueing at freeway bottlenecks, Transportation Research B 27B (4), (1993), pp. 289–303.

[NE93c] G.F. Newell, A simplified theory of kinematic waves in highway traffic, part III: Multi-destination flows, Transportation Research B 27B (4), (1993), pp. 303–313.

[RI56] P. I. Richards, Shock Waves on the Highway. Oper. Res., 4 (1956), pp. 42-51.

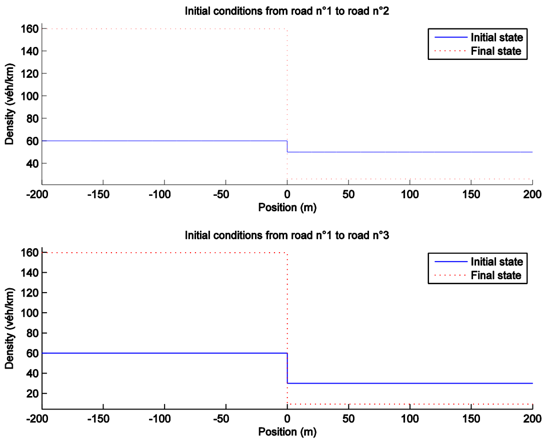
[TCCI11] C. Tampere, R. Corthout, D. Cattrysse and L. Immers, A generic class of first order node models for dynamic macroscopic simulations of traffic flows, Transportation Research Part B, 45 (1) (2011), pp. 289–309.

Appendix A. Simulations for a diverge junction

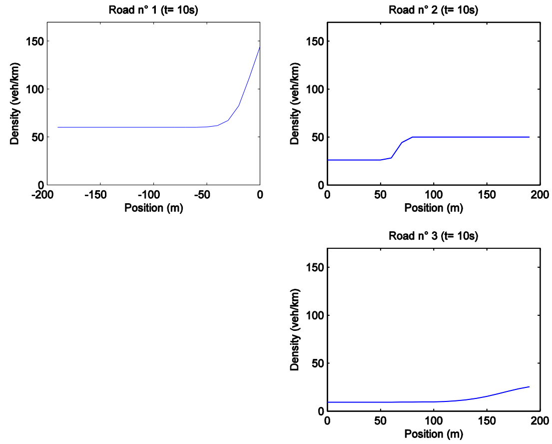
Let us consider a divergent for which we have one single incoming road and two outgoing roads. This case should represent an exciting road on a beltway. The incoming road (number 1) and one of the outgoing (number 2) have a maximal speed supposed to be taken at $v_{max}^{1,2} = 90\ km/h$. However road 1 has 3 lanes and road 2 has 2 lanes. The second outgoing road (number 3) has a single lane and its maximal speed is selected at $v_{max}^{3} = 50\ km/h$. Let us consider the following initial conditions:

$$\begin{cases} \rho^1(t=0) = 60\ veh/km \\ \rho^2(t=0) = 50\ veh/km \\ \rho^3(t=0) = 40\ veh/km \end{cases}$$

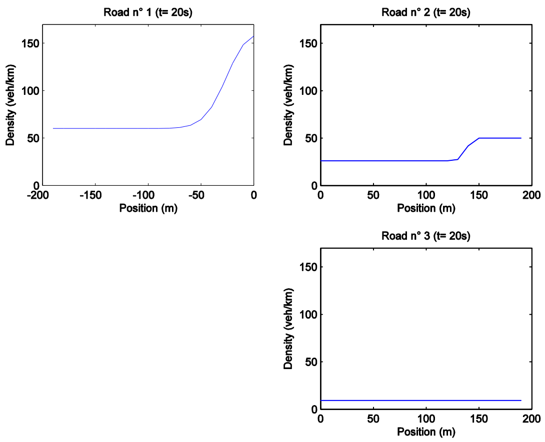
The turning fraction coefficients are defined such that $\gamma^2 = 1 - \gamma^3 = 0{,}8$. Moreover let us suppose that the flow relationship is piecewise parabolic.



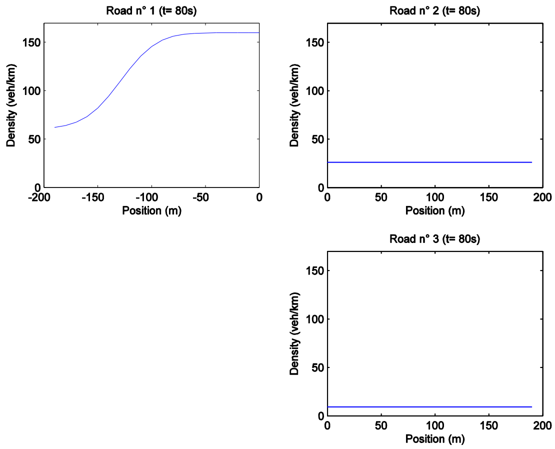
(a) Initial conditions



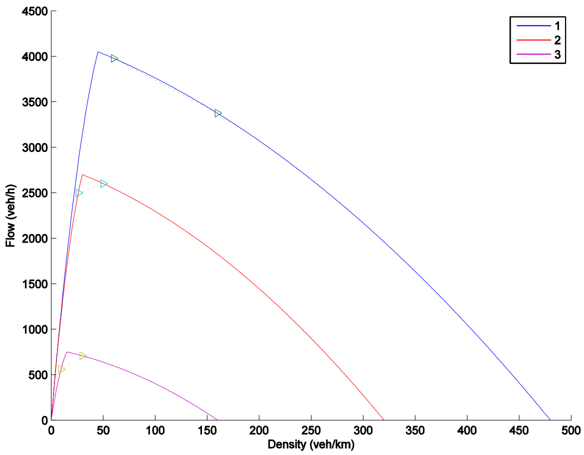
(b) Densities at time $t = 10$ s



(c) Densities at time $t = 20$ s



(d) Densities at time $t = 80$ s



(e) Fundamental Diagram