Contribution au livre de Jean-Pierre Aubin & Anya Désilles

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Contents

1 Numerical schemes for the Hamiltonian approach

This section is devoted to the presentation of the most classical (and the most used) numerical scheme for solving partial differential equations in traffic flow research and engineering. We restrict our attention to the LWR model, standing for Lighthill, Whitham and Richards [30, 32]. In this model, the traffic density $q \in \mathbb{R}$ at a location p and time t satisfies to a scalar conservation law

$$\frac{\partial}{\partial t} q + \frac{\partial}{\partial p} \mathbf{h}(q) = 0, \quad \text{for any} \quad (t, p) \in [t_0, +\infty) \times [p_0, +\infty), \tag{1}$$

where $q \mapsto \varphi = \mathbf{h}(q)$ is the flow function, also called the flow-density *fundamental diagram* in the traffic flow literature. This function is also an Hamiltonian as it has been shown in the above sections. Assume moreover that the initial densities are known

$$q(t = t_0, p) = g(p), \quad \text{for any} \quad p \in [p_0, +\infty). \tag{2}$$

Existence and uniqueness of the solution of (1)-(2) can be obtained under weak assumptions on $\mathbf{h}$ and g. See [3] for instance. We simply assume that $\mathbf{h}$ is concave and C^1 differentiable and that g is Lipschitz continuous. We denote by $\mathbf{h}_\uparrow$ (resp. $\mathbf{h}_\downarrow$) the increasing (resp. decreasing) part of $\mathbf{h}$.

To solve such an equation, Daganzo and Lebacque proposed independently a numerical scheme in the mid of the nineties [8, 9, 10, 22]. This is a finite volume scheme, explicit in time, which was first published by *Serguei Godunov* in 1959 [15]. Let us introduce a time and space discretization with respectively Δt and $\Delta p > 0$ the finite steps. We denote by q_j^i, for any $(i, j) \in \mathbb{N} \times \mathbb{Z}$, a numerical approximation of the continuous solution q of the Cauchy problem (1)-(2)

$$q_j^i := \frac{1}{\Delta p} \int_{p_{j-\frac{1}{2}}}^{p_{j+\frac{1}{2}}} q(t_i, \xi) d\xi,$$

with $p_j := p_0 + \left(j - \frac{1}{2}\right) \Delta p$ and $t_i := t_0 + i \Delta t$.

Then the Godunov scheme reads as follows

$$q_j^{i+1} = q_j^i + \frac{\Delta t}{\Delta p} \left[F\left(q_{j-1}^i, q_j^i\right) - F\left(q_j^i, q_{j+1}^i\right) \right], \quad \text{for any} \quad (i, j) \in \mathbb{N} \times \mathbb{Z}, \tag{3}$$

with

$$F(q_1, q_2) := \min \left\{ \mathbf{h}_\uparrow(q_1), \mathbf{h}_\downarrow(q_2) \right\}.$$

In order to ensure the stability of the numerical scheme (3) (and thus the convergence of the scheme thanks to Lax theorem), one needs to satisfy the Courant-Friedrichs-Lewy condition [7]

$$\frac{\Delta t}{\Delta p} \geq \sup_{q \in \mathrm{Dom}(\mathbf{h})} \left| \mathbf{h}'(q) \right|. \tag{4}$$

The condition (4) teaches us that the numerical scheme has to be greater or equal to the maximal characteristic speed of the "fluid". One characteristic wave cannot go through more than one cell $[t_i, t_{i+1}] \times [p_j, p_{j+1}]$ at each time step.

Obviously, this numerical scheme can be applied for another conservation laws for instance for the LWR equation recast in Lagrangian or Lagrangian-space frameworks (see Section 2.3 p. 5).

2 A Panorama on Macroscopic Hamiltonian Models

In this section, we propose some insights coming from the traffic flow engineering world. More precisely, we aim at presenting different Hamiltonian approaches that have been developed by researchers to make the seminal LWR model (see Section **6.4 p.169**) more realistic or more easily usable.

2.1 General background

For day-to-day operations, traffic managers use macroscopic traffic flow models. These models must be simple, robust, allowing to get solutions at a low computational cost. The main macroscopic models are based on conservation laws or hyperbolic systems (see [17] or Chapter 5 in [14] for traffic aspects and [13] for mathematical aspects). The seminal LWR model (for Lighthill-Whitham and Richards) was proposed in [30, 32] as a single conservation law with unknown the vehicle density. This model based on a first order partial differential equation is very simple and robust but it fails to recapture some empirical features of traffic. Indeed, it assumes that all the vehicles are in an *equilibrium traffic state* meaning that they are never accelerating or braking. Thus, it does not allow to take into account out-of-the-equilibrium traffic states that are responsible for the set-valuedness of the flow-density fundamental diagram mainly observed in congested situations (see Figure **1.3.1 p.16** for instance). More sophisticated models referred to as *higher order* models were developed to encompass kinematic constraints of real vehicles or also the wide variety of driver behaviors, even at the macroscopic level. Some examples are given in this section where we deal with models of the Generic Second Order Modeling (GSOM) family [25, 27]. Even if these models are more complicated to deal with, they permit to reproduce traffic instabilities (such as the so-called *stop-and-go* waves, the hysteresis phenomenon or the capacity drop) which move at the traffic speed and differ from kinematic waves [33] (see also [27] and references therein). As these models combine the simplicity of the LWR model with the dynamics of driver specific attributes, we are able to recapture more specific phenomenon with a higher accuracy.

2.2 The GSOM family of Hamiltonian models

2.2.1 General GSOM formulation

Any model of the GSOM family can be stated in conservation form (and in Eulerian coordinates) as follows

$$\begin{cases} \dfrac{\partial}{\partial t} q + \dfrac{\partial}{\partial p}(qu) = 0 & \text{Conservation of vehicles,} \\[2ex] \dfrac{\partial}{\partial t}(qz) + \dfrac{\partial}{\partial p}(quz) = q\psi(z) & \text{Dynamics of the driver attribute } z, \\[2ex] u = U(q, z) & \text{Speed-density fundamental diagram,} \end{cases} \tag{5}$$

where q stands for the density of vehicles, u for the flow speed (equal to the mean spatial velocity of vehicles), p and t for position and time. The variable z is a specific driver attribute or specification which can represent for example the driver aggressiveness, the driver destination, the vehicle class or a combination of such information. The flow-density fundamental diagram is defined by

$$\mathbf{h} : (q, z) \mapsto qU(q, z).$$

The function ψ leads the dynamics of the attribute z. Its expression depends on the choice of the modeling.

2.2.2 Examples of models from the GSOM family

The GSOM family recovers a wide range of existing models:

- The LWR model [30, 32] itself is simply a GSOM model with no specific driver attribute (z is the same for any driver), expressed as follows

$$\begin{cases} \frac{\partial}{\partial t} q + \frac{\partial}{\partial p}(qu) = 0 & \text{Conservation of vehicles,} \\ u = U_e(q) & \text{Speed-density fundamental diagram.} \end{cases} \tag{6}$$

  The fundamental diagram for the LWR model $\mathbf{h} : q \mapsto \varphi = qU_e(q)$ states that traffic flow is always at an equilibrium state (no acceleration or deceleration for instance). It is commonly assumed that the flow is an increasing function of density between zero (corresponding to an empty section) and a critical density and then the flow decreases until the jam density (corresponding to a bumper-to-bumper situation). However the fundamental diagram shape is always a subject of debates (see for instance [12]) and there exists a wide variety of them in the literature encompassing concave and triangular flow functions (see Figure 1 and also Chapter 3 of [13] for additional examples).

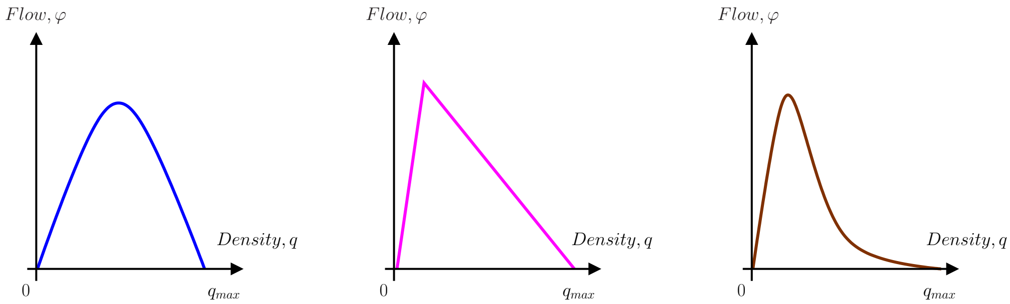

Figure 1: Illustrations of some flow functions $\mathbf{h}$ for the LWR model: Greenshields (left), triangular (center) and exponential (right).

- The LWR model with bounded acceleration proposed in [23, 24, 28] is also a GSOM model in which the propagated driver attribute is simply the speed of vehicles.

- The ARZ model (standing for Aw, Rascle [1] and Zhang [33]) for which the driver attribute is taken as the gap between the current speed and the equilibrium speed (given by the LWR model) $z = u - U_e(q)$, that gives us $U(q, z) = z + U_e(q)$.

- The Generalized ARZ model proposed in [11] that can be also seen as a particular case of the model described in [34]. These models introduce an interaction mechanism between two different fundamental diagrams for distinguish equilibrium and non-equilibrium states.

- Multi-commodity models (multi-class, multi-lanes) of Jin and Zhang [18], Bagnerini and Rascle [2] or Herty, Kirchner, Moutari and Rascle [16]. It encompasses also the model of Klar, Greenberg and Rascle [20].

- The Colombo 1-phase model deduced in [27] from the 2-phase model of Colombo [4]. In this case, the driver attribute z is a scalar which is non-trivial in congested situation. In fluid area, the model follows the classical LWR model.

- The stochastic GSOM model of Khoshyaran and Lebacque [19]. The driver attribute z is a random variable depending on the vehicle index V and on the random event ω such that $z = z(V, t, \omega)$. The random perturbations do not affect the vehicle dynamics but affect the driver perception and its behavior.

The interested reader is referred to [26] and references therein for more details on examples.

2.3 The three-dimension representation of traffic flow

Let us introduce the *Lagrangian coordinate*

$$V(t,p) := \int_p^{\infty} q(t,\xi)d\xi$$

which stands for the (continuous) label of the vehicle (or the cumulated vehicle count as presented above) at position p and at time t. Lagrangian coordinates are fixed to a given fluid particle and move with it in space-time. Note that in the continuum, $v = V(t,p)$ is not necessarily an integer.

The Lagrangian system of coordinates has been first used in the case of gas dynamics by Courant and Friedrichs [6]. It has been introduced in traffic flow theory by Leclercq and its co-authors in [29].

The term *Eulerian* refers to the "classical" framework $t - p$. Eulerian data stand for data coming from fixed equipment giving records of occupancy or flow of vehicles on a freeway section. This kind of measurements come from e.g. fixed inductive loop detectors, Radio Frequency Identification (RFID) transponders, radars or video cameras. Conversely, the term *Lagrangian* is used to characterize a moving framework. Data coming from sensors which move within the measured field of interest are called Lagrangian data. Lagrangian data are provided by on board mobile sensors such as *Global Positioning Systems* (GPS) or GPS-enabled *smartphones*.

We define the *headway* (the average time gap between vehicles), the *spacing* (the average spatial gap between vehicles) and the *pace* (the average time used to travel a unit distance) respectively as follows

$$h = \frac{1}{\varphi}, \quad s = \frac{1}{q}, \quad r = \frac{1}{u}$$

where φ, q and u denote respectively the flow, the density and the speed. Considering the different systems of coordinates for the three representations of traffic, we obtain the following systems of equations to solve (see Table 1):

The three-dimension representation of traffic flow has been firstly produced by Makigami and co-authors in [31]. The Eulerian framework is the mostly used in the traffic flow community while Lagrangian system of coordinate has been proved to provide a very good framework for specific applications like treating moving constraints (the so-called *moving bottleneck* problem). The last system of coordinates is attracting more and more attention (see for instance [5] and references therein).

	Eulerian $t-p$	Lagrangian $t-v$	Lagrangian-space $v-p$
Variables	q density φ flow	s spacing u speed	r pace h headway
First equation: Conservation law	$\frac{\partial q}{\partial t}+\frac{\partial \varphi}{\partial p}=0$	$\frac{\partial s}{\partial t}+\frac{\partial u}{\partial v}=0$	$\frac{\partial r}{\partial v}-\frac{\partial h}{\partial p}=0$
Second equation: Attribute dynamics	$\frac{\partial z}{\partial t}+u\frac{\partial z}{\partial p}=\psi(z)$	$\frac{\partial \tilde{z}}{\partial t}=\psi(\tilde{z})$	$\frac{\partial \check{z}}{\partial p}=r\psi(\check{z})$

Table 1: Coordinate systems, variable definitions and equations for the three representations.

2.4 The three kinds of Hamiltonians

It has been shown in [21] that one can define

- $V(t,p)$ the (continuous) label of the vehicle located at position p at time t
- $P(t,v)$ the position of the vehicle labeled v at time t
- $\Omega(v,p)$ the passing time of vehicle labeled v at position p (or the travel duration between a reference position and p)

such that one has

$$\begin{cases}\varphi=\dfrac{\partial V}{\partial t}, & \text{(flow)}\\[2ex] q=-\dfrac{\partial V}{\partial p}, & \text{(density)}\end{cases},\quad \begin{cases}u=\dfrac{\partial P}{\partial t}, & \text{(speed)}\\[2ex] s=-\dfrac{\partial P}{\partial v}, & \text{(spacing)}\end{cases},\quad \begin{cases}h=\dfrac{\partial \Omega}{\partial v}, & \text{(headway)}\\[2ex] r=\dfrac{\partial \Omega}{\partial p}. & \text{(pace)}\end{cases}$$

Then, it is easy to show that the conservation laws presented in Table 1 can be recast as follows (see Table 2)

	Eulerian $t-p$	Lagrangian $t-v$	Lagrangian-space $v-p$
First equation: Hamilton-Jacobi	$\frac{\partial V}{\partial t}-\mathbf{h}\left(-\frac{\partial V}{\partial p},z\right)=0$	$\frac{\partial P}{\partial t}-\mathbf{h}\left(-\frac{\partial P}{\partial v},\tilde{z}\right)=0$	$\frac{\partial \Omega}{\partial v}-\mathbf{h}\left(\frac{\partial \Omega}{\partial p},\check{z}\right)=0$
Second equation: Attribute dynamics	$\frac{\partial z}{\partial t}+u\frac{\partial z}{\partial p}=\psi(I)$	$\frac{\partial \tilde{z}}{\partial t}=\varphi(\tilde{z})$	$\frac{\partial \check{z}}{\partial p}=\frac{\partial \Omega}{\partial p}\varphi(\check{z})$

Table 2: Coordinate systems, variable definitions and equations for the three representations.

where the following Hamiltonian $\mathbf{h}$ have been designed as follows (for sake of clarity, we only consider the LWR case i.e. with no specific driver attribute – see also Figure 2):

1. in the *Eulerian case*, $\mathbf{h}$ maps densities q to flows $\varphi=\mathbf{h}(q)$. The derivative $\mathbf{h}'(q)$ is interpreted as a velocity of traffic waves (say the velocity of the characteristics such as

shock waves). The traffic mean spatial speed is given by $u = \dfrac{\varphi}{q}$ whenever $q \neq 0$. One can distinguish:

(a) The demand side associates with densities the maximal upstream flux that wishes to flow through a position. It matches the non-decreasing part of the Hamiltonian denoted by $\mathbf{h}_{\uparrow}$;

(b) The supply side associates with densities the maximal flux that can be locally accommodated downstream It matches the non-increasing part of the Hamiltonian denoted by $\mathbf{h}_{\downarrow}$.

2. in the *Lagrangian case*, $\mathbf{h}$ maps the *spacing* (or *interdistance* between vehicles) $s := \dfrac{1}{q}$ to speed $u := \mathbf{h}(s)$. The derivative $\mathbf{h}'(s)$ is homogeneous to a flux. The traffic flux is computed as $\varphi = \dfrac{u}{s}$. One can distinguish:

   (a) The demand side associates with the interdistance the wished maximal speed allowed to the vehicle. The demand function reduces to the horizontal asymptote $u = u_{max}$;

   (b) The supply side associates with the interdistance the actual maximum speed allowed to the vehicle by downstream traffic conditions. It exactly matches the Hamiltonian.

3. in the the *spatial Lagrangian case*, $\mathbf{h}$ maps rhythm or (pace, frequency) $r := \dfrac{1}{u}$ to the *headway* $h := \mathbf{h}(r)$ corresponding to the inverse of flow φ. The derivative $\mathbf{h}'(r)$ is homogeneous to a spacing. The traffic spacing is given by $s = \dfrac{h}{r}$. One can distinguish:

   (a) The demand side associates with the pace the minimal headway at which the driver wishes to circulate. It matches the non-increasing part of the Hamiltonian denoted by $\mathbf{h}_{\downarrow}$;

   (b) The supply side associates with the pace the minimal headway allowed by downstream traffic conditions. It matches the non-decreasing part of the Hamiltonian denoted by $\mathbf{h}_{\uparrow}$.

Naturally, the Hamiltonians can depend also on time, duration and position, on one hand, as well as specification or attributes z and their velocities z'.

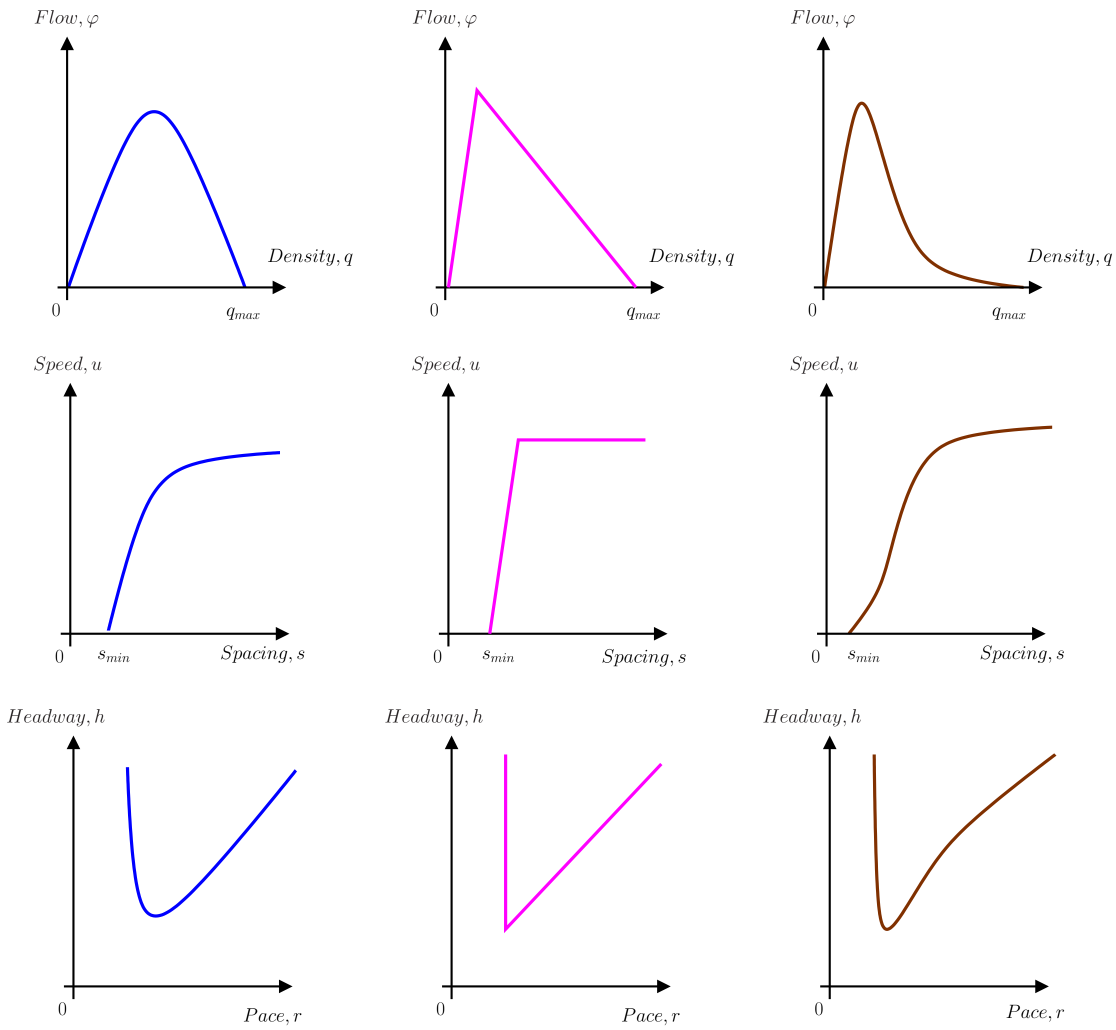

Figure 2: Illustrations of some Hamiltonians **h**: Greenshields (left), triangular (center) and exponential (right).

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