The moving bottleneck problem: a Hamilton-Jacobi approach

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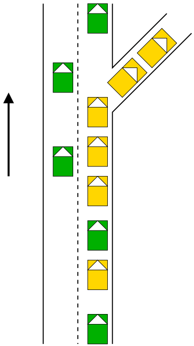
TRAM3 Terminus Meeting 2016, Sophia-Antipolis
January 07, 2016

Example

- Congested off-ramp



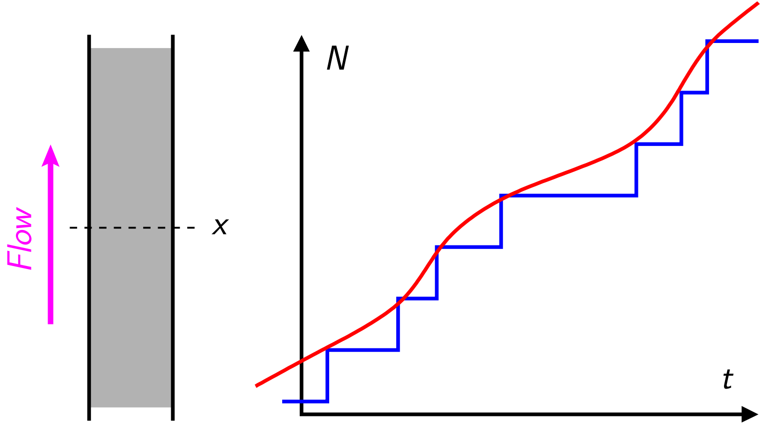
[Richmond Bridge, ©Bay Area Council]

- Traffic flow model for the upstream section?

Outline

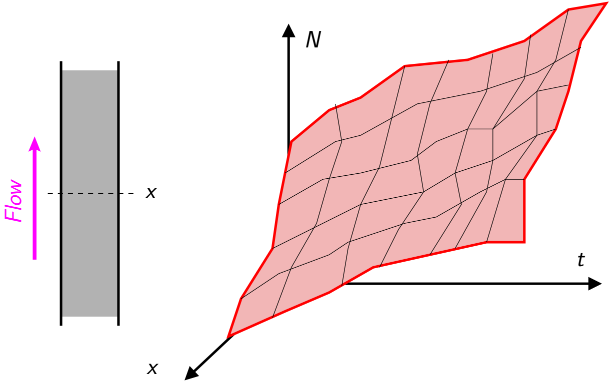
Outline

Convention for vehicle labeling



Three representations of traffic flow

Moskowitz' surface

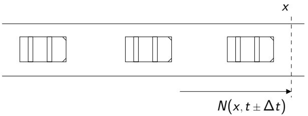


See also [Moskowitz and Newman(1963), Makigami et al(1971), Laval and Leclercq(2013)]

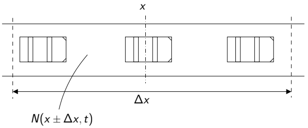
Notations

(macroscopic)

- $N(t,x)$ vehicle label at (t,x)
- the flow $q(t,x) = \lim_{\Delta t \to 0} \frac{N(t+\Delta t, x) - N(t,x)}{\Delta t}$,



- the density $\rho(t,x) = \lim_{\Delta x \to 0} \frac{N(t,x) - N(t, x+\Delta x)}{\Delta x}$,



- the stream speed (mean spatial speed) $V(t,x)$.

First order: the LWR model

LWR model [Lighthill and Whitham, 1955], [Richards, 1956]

Scalar one dimensional conservation law

$$\partial_t \rho + \partial_x Q(\rho) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R} \tag{1}$$

with

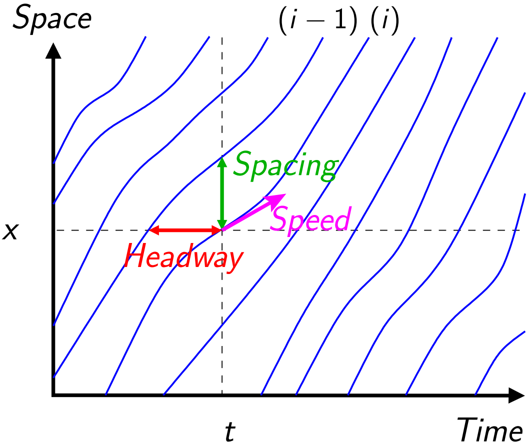
$$Q : \rho \mapsto q =: Q(\rho)$$

Overview: conservation laws (CL) / Hamilton-Jacobi (HJ)

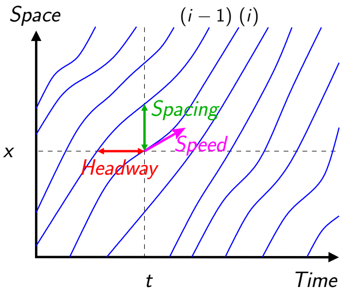
		Eulerian $t - x$	Lagrangian $t - n$
CL	Variable	Density ρ	Spacing $r := \frac{1}{\rho}$
	Equation	$\partial_t \rho + \partial_x Q(\rho) = 0$	$\partial_t r + \partial_n V(r) = 0$
HJ	Variable	Label N	Position $\mathcal{X}$
		$N(t,x) = \int_x^{+\infty} \rho(t,\xi) d\xi$	$\mathcal{X}(t,n) = \int_n^{+\infty} r(t,\eta) d\eta$
	Equation	$\partial_t N - Q(-\partial_x N) = 0$	$\partial_t \mathcal{X} - \mathcal{V}(-\partial_n \mathcal{X}) = 0$

Notations

(micro and meso)



Mesoscopic resolution of the LWR model



Introduce

- the pace $p := \frac{1}{v}$
- the headway $h = H(p)$
- the passing time

$$T(n, x) := \int_{-\infty}^{x} p(n, \xi) d\xi.$$

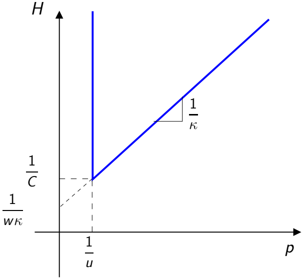
$$\begin{cases} \partial_n T = h, & \text{(headway)} \\ \partial_x T = p, & \text{(pace)} \end{cases}$$

[Leclercq and Bécarie(2012), Laval and Leclercq(2013)]

Mesoscopic resolution of the LWR model
(Continued)

		Lagrangian-space $n - x$
CL	Variable	Pace p
	Equation	$\partial_n p - \partial_x H(p) = 0$
HJ	Variable	Passing time T $T(n,x) = \int_{-\infty}^{x} p(n,\xi)d\xi$
	Equation	$\partial_n T - H(\partial_x T) = 0$

Lax-Hopf formula



Assume

$$H(r) = \begin{cases} \frac{1}{\kappa} r + \frac{1}{w\kappa}, & \text{if} \quad r \geq \frac{1}{u}, \\ +\infty, & \text{otherwise}, \end{cases}$$

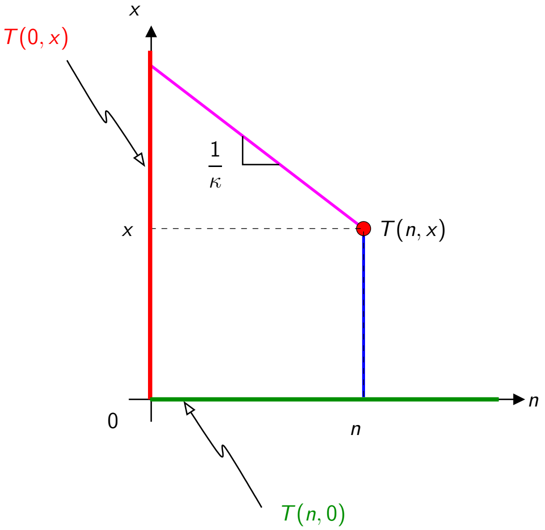
Proposition (Representation formula (Lax-Hopf))

The solution under smooth boundary conditions is given by

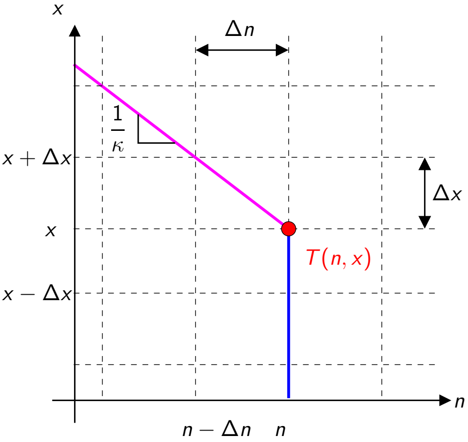
$$T(n,x) = \max \Big\{ \underbrace{T(n,0) + \frac{x}{u}}_{= \text{ free flow}}, \underbrace{T\left(0, x + \frac{n}{\kappa}\right) + \frac{n}{w\kappa}}_{= \text{ congested}} \Big\}. \quad (2)$$

See [Laval and Leclercq(2013)]

Lax-Hopf formula



Dynamic programming



Introduce a grid with steps $(\Delta n, \Delta x)$ satisfying

$$\Delta n = \kappa \Delta x$$

Mesoscopic: what for?

- Strengths
  1. Consistent with micro and macro representations
  2. Large scale networks // spatial discontinuities OK
  3. Data assimilation (from Eulerian and Lagrangian sensors)
- Weakness
  1. Single pipe
  2. Mono class
  3. No capacity drop at junctions
- Developments
  1. Multilane and multiclass approach
  2. Friction term

$\longrightarrow$ Moving bottleneck theory

Outline

Some definitions

Following [Gazis and Herman(1992), Newell(1998), Laval and Leclercq(2013)],

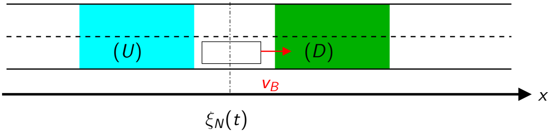
Definition

A moving bottleneck (MB) is defined as a vehicle with a maximal free flow speed strictly lower than the free flow speed of its immediate following vehicle.

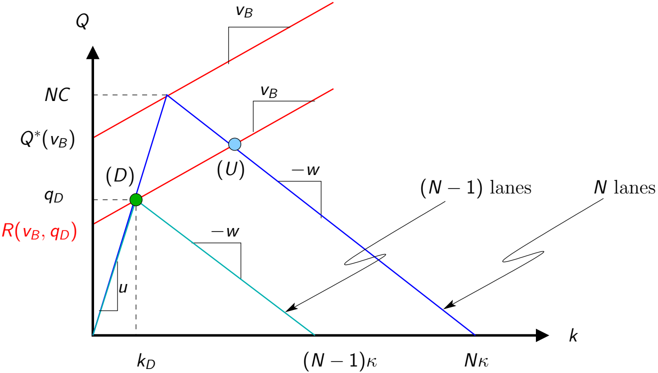
A moving bottleneck is said to be active if and only if it generates queues at upstream positions with respect to the moving bottleneck and if the upstream state is different from the downstream state.

The passing rate is the maximal flow that can overtake the moving bottleneck in the Eulerian framework.

Notations

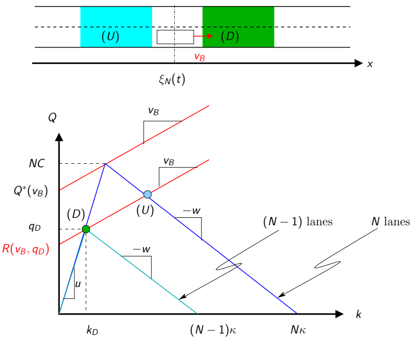


(U)
(D)
v_B
x
$\xi_N(t)$



Q
v_B
NC
v_B
$Q^*(v_B)$
(D)
(U)
$-w$
$(N-1)$ lanes
N lanes
q_D
$R(v_B, q_D)$
$-w$
u
k
k_D
$(N-1)\kappa$
$N\kappa$

Notations



- $\xi_N(t)$ = trajectory of the MB in $t - x$
- $\xi_T(n)$ = location where the vehicle n crosses the MB
- $v_B(t) := \dot{\xi}_N(t)$ MB speed
- $k_D := k\left(t, \xi_N(t)^+\right)$ downstream density,
- $q_D := Q(k_D)$ downstream flow,
- $R(v_B, q_D)$ = passing rate

$$R(v_B, q_D) := q_D - k_D v_B$$

Math problem in Eulerian framework

Coupled ODE-PDE problem on $[0, +\infty) \times \mathbb{R}$

$$\begin{cases} \partial_t k + \partial_x \left(Q(k)\right) = 0, \\ Q(k(t, \xi_N(t))) - \dot{\xi}_N(t) k(t, \xi_N(t)) \leq \dfrac{N-1}{N} Q^* \left(\dot{\xi}_N(t)\right), \\ \dot{\xi}_N(t) = \min\left\{v_b, \ V\left(k(t, \xi_N(t)^+)\right)\right\}, \end{cases} \tag{3}$$

with

$$\begin{cases} k(0, x) = k_0(x), & \text{on} \quad \mathbb{R}, \\ \xi_N(0) = \xi_0. \end{cases} \tag{4}$$

Theorem (Existence result [Delle Monache and Goatin(2014)])

For BV initial data, there exist weak solutions of (3)-(4).
Uniqueness is still an open problem.

Interpretation

Remark

Upper bound on the passing rate in (3) depends on the moving bottleneck speed

$$v_B(t) \mapsto Q^*(v_B(t))$$

and it is the Legendre-Fenchel transform of the flow-density FD Q

$$Q^*(v) := \sup_{\rho \in Dom(Q)} \{Q(\rho) - v\rho\}.$$

Outline

- Stretch of road $[x_0, x_1]$ composed by N separate lanes
- Two classes of users: “rabbits” ($I = 1$) and “slugs” ($I = 2$).
- Class-dependent Hamiltonian

$$H : (r, I) \mapsto H(r, I)$$

for a given class attribute $I \in \{1, 2\}$

- Stretch of road $[x_0, x_1]$ composed by N separate lanes
- Two classes of users: “rabbits” ($I = 1$) and “slugs” ($I = 2$).
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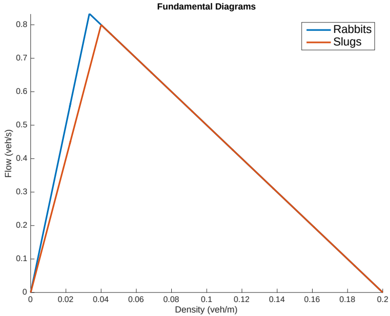
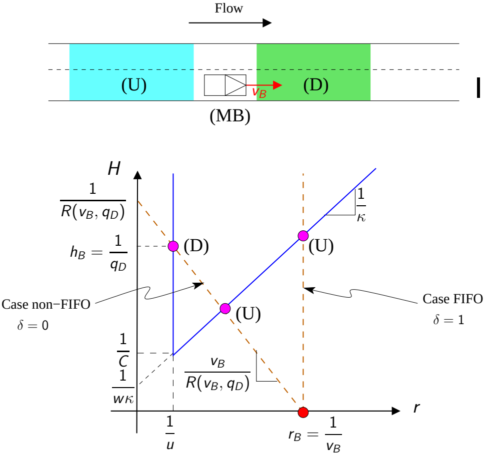
$$H : (r, I) \mapsto H(r, I)$$

for a given class attribute $I \in \{1, 2\}$

- If triangular, then $u_1 > u_2$



Capacity drop parameter

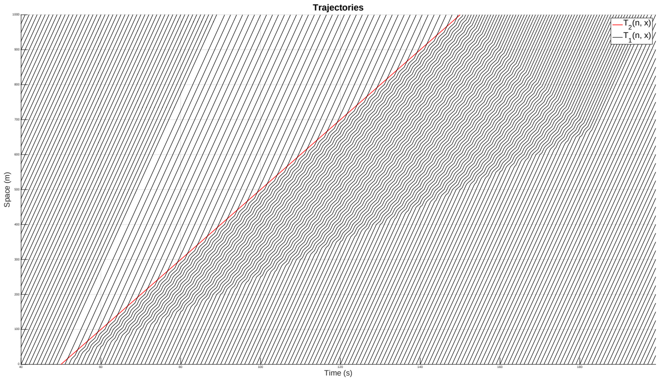


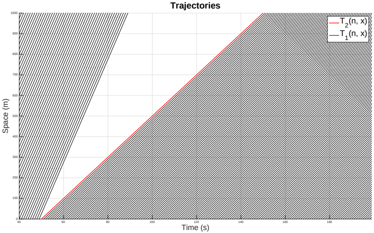
Introduce parameter $\delta \in [0, 1]$

- If $\delta = 0$, strictly non-FIFO
- If $0 < \delta < 1$, reduction of the residual headway on the passing lane(s)
- If $\delta = 1$, strictly FIFO

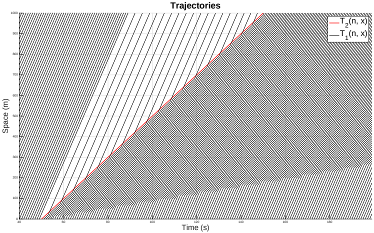
Illustration of the capacity drop effect

- Link composed of two lanes
- A single vehicle from class 2 (MB)
- No capacity drop occurs $\delta = 0$

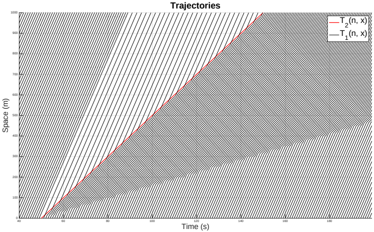




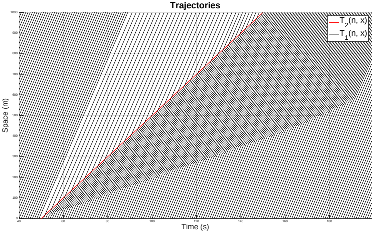
$\delta = 1$



$\delta = 0.6$



$\delta = 0.4$



$\delta = 0.2$

System of coupled PDEs

$$\begin{cases} \partial_n T_1 - H\left(\partial_x T_1, I_1\right) = 0, \\ \partial_n T_2 - H\left(\partial_x T_2, I_2\right) = 0, \\ H\left(\partial_x T_1(n, \xi(n)), I_1\right) - (1-\delta)\dot{\xi}\left(n_2^*\right)\partial_x T_1(n, \xi(n)) \\ \qquad \geq \dfrac{N}{N-1} H^{\boxtimes}\left(\left(1-\delta\right)\dot{\xi}\left(n_2^*\right), I_1\right), \\ T_2\left(n, \xi(n)\right) \geq T_1(n_1^*, \xi(n)) + H\left(\partial_x T_1(n_1^*, \xi(n)), I_2\right), \end{cases} \tag{5}$$

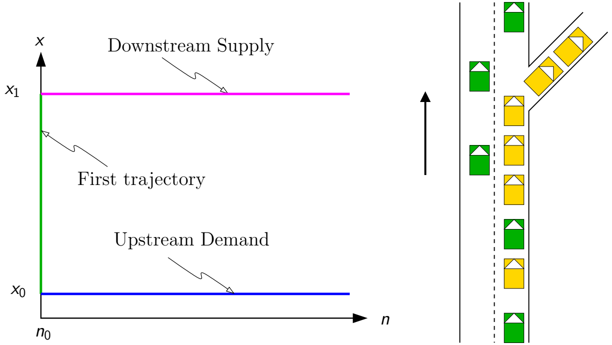
where

$$H^{\boxtimes}(s, I) = \inf_{r \in \mathrm{Dom}(H(\cdot, I))} \{H(r, I) - sr\}$$

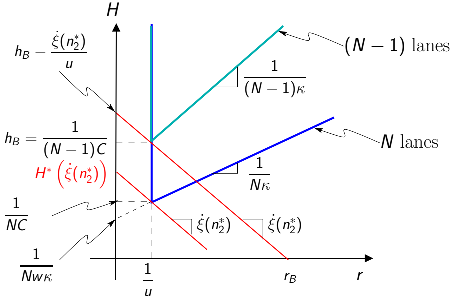
$$\begin{cases} n_2^* := \operatorname{argmax}\ T_2(\cdot, \xi(n)) \quad \text{and} \quad \xi(n) = \xi(n_2^*), & \text{for} \quad n \in [n_0, +\infty), \\ n_1^* := \operatorname{argmax}\ T_1(\cdot, \xi(n)) & \text{for} \quad n \in [n_0, +\infty), \end{cases}$$

+ mixed Neumann-Dirichlet boundary conditions

$$\begin{cases} \partial_n T_i(n, x_0) = \check{g}_i(n), & \text{on} \quad [n_0, +\infty), \\ \partial_n T_i(n, x_1) = \hat{g}_i(n), & \text{on} \quad [n_0, +\infty), \\ T_i(n_0, x) = G_i(x), & \text{on} \quad [x_0, x_1], \end{cases} \quad \text{for} \quad i \in \{1, 2\}. \qquad (6)$$



Assumptions



- n_i^* ($i \in \{1, 2\}$: the nearest leader from class i for vehicle n of class $j \neq i$
- Modified passing headway $\tilde{H}$

$$\tilde{H} = \frac{1}{1-\delta} h_B - \frac{\dot{\xi}_T\left(n_2^*\right)}{u_1}$$

where $h_B = \dfrac{1}{(N-1)C}$ is the residual headway

Outline

- Finite steps $(\Delta n, \Delta x) > 0$

$$\Delta n = \kappa \Delta x.$$

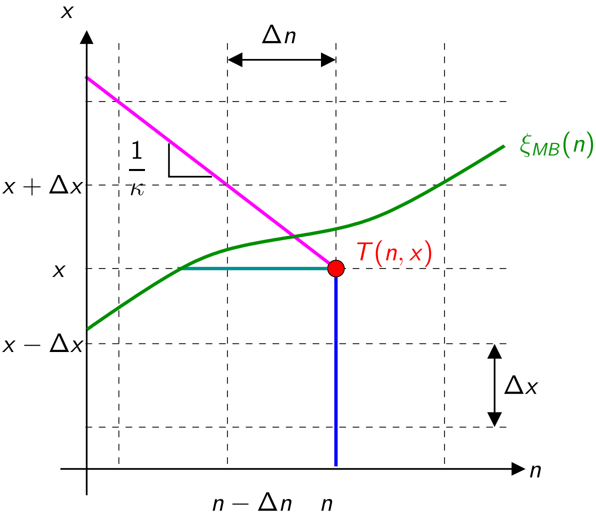
- Class-specific Hamiltonians H are triangular

$$H(r, I_i) = \begin{cases} \dfrac{1}{\kappa N(I_i)} \left(r + \dfrac{1}{w} \right), & \text{if} \quad r \geq \dfrac{1}{u_i}, \\ +\infty, & \text{otherwise}, \end{cases} \tag{7}$$

- $N(I_i)$ stands for the number of lanes that are accessible for the class i

$$N(I_i) := \begin{cases} N, & \text{for} \quad i = 1, \\ 1, & \text{for} \quad i = 2. \end{cases}$$

Dynamic programming



Representation formulæ

For any $(n, x) \in [n_0, n_{max}] \times [x_0, x_1]$,

$$\begin{cases} T_1(n,x) = \max\left\{ T_1(n, x - \Delta x) + \dfrac{\Delta x}{u_1}, T_1(n - \Delta n, x + \Delta x) + \dfrac{\Delta x}{w} \right\}, \\ T_2(n,x) = \max\left\{ T_2(n, x - \Delta x) + \dfrac{\Delta x}{u_2}, \ T_2(n - \Delta n, x + \Delta x) + \dfrac{\Delta x}{w} \right\} \end{cases} \tag{8}$$

Representation formulæ

For any $(n, x) \in [n_0, n_{max}] \times [x_0, x_1]$,

$$\begin{cases} T_1(n,x) = \max \Big\{ T_1(n, x - \Delta x) + \dfrac{\Delta x}{u_1}, T_1(n - \Delta n, x + \Delta x) + \dfrac{\Delta x}{w}, \\ \qquad\qquad T_2(n_2^*, x) + \dfrac{1}{1-\delta} h_B \Big\} \\ T_2(n,x) = \max \Big\{ T_2(n, x - \Delta x) + \dfrac{\Delta x}{u_2}, \; T_2(n - \Delta n, x + \Delta x) + \dfrac{\Delta x}{w}, \\ \qquad\qquad T_1(n_1^*, x) + H\left(\dfrac{T_1(n_1^*, x) - T_1(n_1^*, x - \Delta x)}{\Delta x}, I_2 \right) \Big\} \end{cases} \tag{9}$$

with

$$\begin{cases} n_1^* := \operatorname{argmax} \; T_1(\cdot, x) \\ n_2^* := \operatorname{argmax} \; T_2(\cdot, x) \end{cases}$$

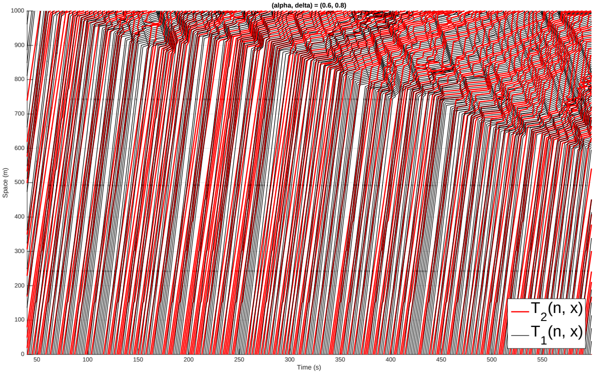
Proposition (Existence of a continuous solution to (5)-(6))

Assume that $r \mapsto H(r, l_i)$ *is convex for any* $i \in \{1, 2\}$. *Then, the IBV problem* (5)-(6) *admits a solution* (T_1, T_2) *continuous on* $[n_0, +\infty] \times [x_0, x_1]$.

Proposition (Convergence)

Assume that $r \mapsto H(r, l_i)$ *is convex for any* $i \in \{1, 2\}$. *Then the numerical solution* $(T_1^\varepsilon, T_2^\varepsilon)$ *of the scheme* (9) *converges towards a continuous solution* (T_1, T_2) *of the IBV problem* (5)-(6) *when* $\varepsilon := (\Delta n, \Delta x)$ *goes toward zero.*

- Distribution per class: class 1 $= 60\%$ and class 2 $= 40\%$
- Capacity drop: $\delta = 0.8$




Vehicle label
20
40
60
80
100
120
140
160
Class 2
Class 1
Travel time (s)
180
160
140
120
100
80
60
40
Travel time (s)
20
40
60
80
100
120
140
160
180
200
220
240
Vehicle label

Outline

- A new event-based mesoscopic model for multi-class traffic flow modeling on multi-lane sections
- Theory of moving bottlenecks
- Wide networks

Among the perspectives:

- Well posedness
- Sensitivity analysis ($\rightarrow$ Régis' presentation)
- Validation with real traffic data
- Data assimilation for real-time applications ($\rightarrow$ Francesca's presentation)

Some references I

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THANKS FOR YOUR ATTENTION

Any question?

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