Micro-to-macro passage in traffic models including multi-anticipation effect

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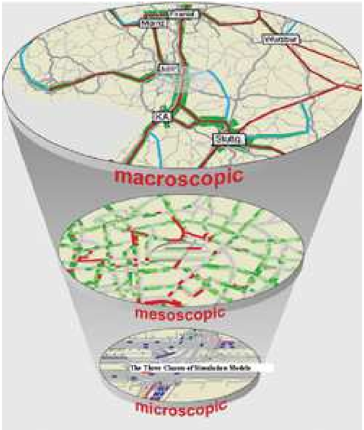
Traffic scales



- Microscopic scale
  - Individual vehicle behaviors
  - Car-following (or Follow-The-Leader)
  - Lane changing, gap acceptance...
- Macroscopic scale
  - Aggregate variables
  - Hydrodynamics
- Passage from micro to macro?
  - Mathematical sound basis and consistency
  - Well-established for classical first order models
  - Non-local models?

Outline

Outline

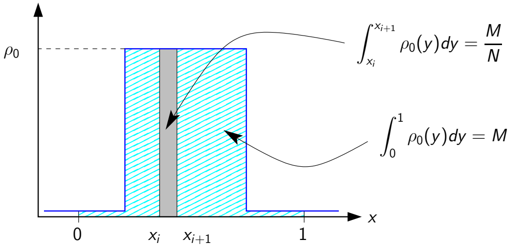
General setting

- N vehicles ($N \in \mathbb{N}$)
- initial density $\rho_0 \in BV_c(\mathbb{R}, [0,1])$

$$\int_{\mathbb{R}} \rho_0(y)dy = 1$$

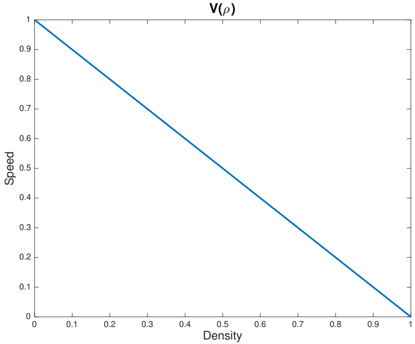
- linear mass attributed to each vehicle

$$l_N := \frac{1}{N} \int_{\mathbb{R}} \rho_0(y)dy = \frac{1}{N}$$



Speed function

(V) Speed function $v : \rho \mapsto v(\rho)$ strictly decreasing on $[0, 1]$ and $v(0) = v_{max}$ with $0 < v_{max} < +\infty$ and $v(1) = 0$



Hyperbolic approach (Di Francesco, Rosini, 2015)

Micro-to-macro for the LWR model [3]

- Micro car-following model

$$\begin{cases} \dot{x}_N = v_{\max} \\ \dot{x}_i = v\left(\dfrac{l_N}{x_{i+1}-x_i}\right), & \forall i = 1,\dots,N-1, \\ x_i(t=0) = x_{i,0}, & \forall i = 1,\dots,N. \end{cases} \tag{1}$$

- Macro LWR model

$$\begin{cases} \rho_t + (\rho v(\rho))_x = 0, & \text{on} \quad (0,+\infty)\times\mathbb{R}, \\ \rho(0,x) = \rho_0(x), & \text{on} \quad \mathbb{R} \end{cases} \tag{2}$$

Some theoretical results (Di Francesco, Rosini)

Theorem (Convergence in $\mathbf{L}^1_{loc}$)

Set the empirical density $\hat{\rho}_N$ as follows

$$\hat{\rho}_N(t,x) := \sum_{i=0}^{N-1} \frac{l_N}{x_{i+1}(t) - x_i(t)} \chi_{[x_i(t), x_{i+1}(t)[}(x).$$

It converges to the unique entropy solution ρ of the Cauchy problem (2) almost everywhere and in $\mathbf{L}^1_{loc}([0,+\infty) \times \mathbb{R})$.

Some theoretical results (Di Francesco, Rosini) (continued)

Theorem (Convergence w.r.t. the 1-Wasserstein distance)

The empirical measure $\tilde{\rho}_N$ *defined as*

$$\tilde{\rho}(t,x) := l_N \sum_{i=0}^{N-1} \delta_{x_i(t)}(x)$$

converges to the unique entropy solution ρ *of the Cauchy problem* (2) *in the topology of* $\mathbf{L}^1_{loc}([0,+\infty); d_{L,1})$ *where* $d_{L,1}$ *defines a scaled 1-Wasserstein distance between two measures.*

sketch of the proof

1. **Definitions:** Introduce the cumulative distribution of $\hat{\rho}$

$$\hat{X}(x) := \inf \{ y \in \mathbb{R} \mid \hat{\rho}((-\infty, y]) > x \}$$

and the one of $\tilde{\rho}$

$$\tilde{X}(x) := \inf \{ y \in \mathbb{R} \mid \tilde{\rho}((-\infty, y]) > x \} .$$

Introduce finally the discrete Lagrangian density

$$\check{\rho} := \hat{\rho} \circ \hat{X}.$$

2. **Convergence:**
   - proof of $\left(\tilde{X}\right) \to X$ in $\mathbf{L}^1_{loc}([0, +\infty) \times [0, L]; \mathbb{R})$
   - equivalent to prove that $(\tilde{\rho}) \to \rho$ in $\mathbf{L}^1_{loc}([0, +\infty); d_{L,1})$
   - And proof of $\left(\hat{X}\right) \to X$ in $\mathbf{L}^1_{loc}([0, +\infty) \times [0, L]; \mathbb{R})$
   - equivalent to prove that $(\hat{\rho}) \to \rho$ in $\mathbf{L}^1_{loc}([0, +\infty); d_{L,1})$

sketch of the proof
(continued)

3. **Bounds:**
   - X has difference quotients bounded below by $1/R$
   - ρ is in $\mathbf{L}^\infty$ and bounded by R
4. **Weak-* convergence:** $(\check{\rho}) \rightharpoonup \check{\rho}$ in $\mathbf{L}^\infty$
5. **Uniform BV estimates:**
   - if $\bar{\rho} \in \mathcal{M}_L \cap \mathbf{BV}$, then direct for $\hat{\rho}$
   - if $\bar{\rho} \in \mathcal{M}_L \cap \mathbf{L}^\infty$, then discrete version of the Oleinik condition for $\check{\rho}$ that implies $\mathbf{BV}$ estimates for $\check{\rho}$ and thus for $\hat{\rho}$

sketch of the proof
(continued)

6. **Passing to the limit in the system of ODEs:** Formulation of the system of ODEs as a PDE

$$\tilde{X}_t = v\left(\check{\rho}\right).$$

7. **Discrete entropy condition + limit using the $\mathbf{L}^1$ compactness:**

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left[|\rho(t,x) - k| \, \varphi_t(t,x) + \operatorname{sgn}\left(\rho(t,x) - k\right) \left[f\left(\rho(t,x)\right) - f(k) \right] \varphi_x(t,x) \right] dtdx$$
$$+ \int_{\mathbb{R}} |\bar{\rho}(x) - k| \, \varphi(0,x) dx \geq 0$$

Some extensions

- ARZ model [(Aw, Rascle, 2000), (Zhang, 2002)]

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho w) + \partial_x(\rho v w) = A \dfrac{\rho}{T_R}(V(\rho) - v) \end{cases}$$

with $w := v + P(\rho)$ (P called a "pseudo-pressure")

  - Lagrangian approach (Aw et al., 2002) [1]
  - Eulerian approach (Di Francesco et al., 2015) [2]

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  - Lagrangian approach (Aw et al., 2002) [1]
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- Non-local model (Goatin, Blandin, 2015) (Goatin, Rossi, 2015) [4]

Goatin-Rossi

(macro)

$$\begin{cases} \partial_t \rho + \partial_x \left[\rho v \left(\int_x^{x+\eta} \rho(t,y) w(y-x) dy \right) \right] = 0, & \text{for} \quad x \in \mathbb{R}, \; t > 0, \\ \rho(0,x) = \rho_0(x), & \text{for} \quad x \in \mathbb{R} \end{cases} \tag{3}$$

(w) $\eta > 0$ is a given parameter and $w : [0, \eta] \to \mathbb{R}^+$ is a non-increasing Lipschitz weight satisfying

$$\int_0^\eta w(x) dx = 1$$

Goatin-Rossi
(micro)

$$
\begin{cases}
\dot{x}_N(t) = v(0), & \text{for any} \quad t > 0, \\
\dot{x}_i(t) = v\left(\frac{1}{N}\sum_{j=1}^{N} w_{l_N}(x_j(t) - x_i(t))\right), & \text{for} \quad i = 1, \dots, N-1, \quad \text{and} \\
x_i(0) = x_{i,0}, & \text{for} \quad i = 1, \dots, N
\end{cases}
\tag{4}
$$

with

$$
w_{l_N} := \begin{cases}
w(0)\dfrac{l_N + 2x}{l_N}, & \text{if} \quad x \in \left[-\dfrac{l_N}{2}, 0\right], \\
w(x), & \text{if} \quad x \in [0, \eta], \\
w(\eta)\dfrac{2\eta + l_N - 2x}{l_N}, & \text{if} \quad x \in \left[\eta, \eta + \dfrac{l_N}{2}\right], \\
0, & \text{elsewhere.}
\end{cases}
$$

Goatin-Rossi

(convergence result)

Theorem (Convergence of a micro model to the unique solution of (3))

Fix any $0 < T < +\infty$.
If $(x_i)_i$ *is a solution of the system of coupled ODEs* (4)*, then, for any* $N \in \mathbb{N}$*, we have*

$$[\rho(t)] = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)} \rightharpoonup \rho(t)$$

where $\rho \in C^0\left([0, T], \mathcal{P}_c(\mathbb{R})\right)$ *is the unique solution of* (3).

Outline

Multi-anticipative model

- N vehicles
- $1 \leq k < N$ considered leaders

$$
\begin{cases}
\dot{x}_N = v_{\max} \\
\dot{x}_i = v\left(\displaystyle\sum_{j=1}^{N-i} \tilde{w}_j \left(j \frac{l_N}{x_{i+j} - x_i}\right)\right), & \forall i = N-k+1, \dots, N-1, \\
\dot{x}_i = v\left(\displaystyle\sum_{j=1}^{k} w_j \left(j \frac{l_N}{x_{i+j} - x_i}\right)\right), & \forall i = 1, \dots, N-k, \\
x_i(t=0) = x_{i,0}, & \forall i = 1, \dots, N.
\end{cases}
\tag{5}
$$

Assumptions

(V) Speed function $v : \rho \mapsto v(\rho)$ strictly decreasing on $[0, 1]$ and $v(0) = v_{max}$ with $0 < v_{max} < +\infty$ and $v(1) = 0$

(W) The map $w_j : \rho \mapsto w_j(\rho)$ is Lipschitz continuous, non-decreasing from $[0, 1]$ to $[0, 1]$ for any $j = 1, \dots, k$ and satisfies moreover

$$\sum_{j=1}^{k} w_j(\rho) = \rho, \quad \text{for any} \quad \rho \in [0, 1].$$

Simplified model

- Only two leaders $k = 2$

$$\begin{cases} \dot{x}_N = v_{\max} \\ \dot{x}_{N-1} = v\left(\dfrac{l_N}{x_N - x_{N-1}}\right), \\ \dot{x}_i = v\left(w_1\left(\dfrac{l_N}{x_{i+1} - x_i}\right) + w_2\left(\dfrac{2l_N}{x_{i+2} - x_i}\right)\right), & \forall i = 1, \dots, N-2, \\ x_i(t=0) = x_{i,0}, & \forall i = 1, \dots, N. \end{cases} \tag{6}$$

Simplified model

- Only two leaders $k = 2$
- Linear weights: $\exists \theta \in [0,1]$ such that

$$w_1(\rho) = \theta\rho \quad \text{and} \quad w_2(\rho) = (1-\theta)\rho$$

$$\begin{cases} \dot{x}_N = v_{\max} \\ \dot{x}_{N-1} = v\left(\dfrac{l_N}{x_N - x_{N-1}}\right), \\ \dot{x}_i = v\left(\theta \dfrac{l_N}{x_{i+1} - x_i} + (1-\theta)\dfrac{2l_N}{x_{i+2} - x_i}\right), & \forall i = 1, \dots, N-2, \\ x_i(t=0) = x_{i,0}, & \forall i = 1, \dots, N. \end{cases} \tag{6}$$

Maximum principle

Proposition (Discrete maximum principle)

Assume that **(V)** *and* **(W)** *hold true.*
Consider $(x_i)_{i=1,\dots,N}$ *the unique solution of* (6).
If there exists $l \geq l_N > 0$ *such that*

$$x_{i+1,0} - x_{i,0} \geq l, \quad \text{for any} \quad i = 1, \dots, N-1,$$

Then, it follows

$$x_{i+1}(t) - x_i(t) \geq l, \quad \text{for any} \quad i = 1, \dots, N-1, \quad t > 0.$$

Equivalently, if there exists a $i_0 \in [\![1, N-1]\!]$ *such that* $x_{i_0+1}(\bar{t}) - x_{i_0}(\bar{t}) = l \geq l_N$ *for some time* $\bar{t} \geq 0$*, then*

$$\dot{x}_{i_0}(\bar{t}) \leq \dot{x}_{i_0+1}(\bar{t}).$$

Convergence (still ongoing)

Theorem (Convergence)

Assume **(V)-(W)**. *Assume also that* $\rho_0 \in \mathbf{BV}_c(\mathbb{R}) \cap \mathbf{L}^1(\mathbb{R})$ *and that*

$$\int_{x_{i,0}}^{x_{i+1,0}} \rho_0(y)dy = l_N, \quad \textit{for any} \quad i = 1, \dots, N-1.$$

Consider the unique solution $(x_i)_{i=1,\dots,N}$ *of* (6).
Then, the empirical density

$$\rho_N(t,x) := \sum_{i=1}^{N-1} \frac{l_N}{x_{i+1}(t) - x_i(t)} \chi_{[x_i(t), x_{i+1}(t)[}(x) \tag{7}$$

converges in $\mathbf{L}^1$ *towards the unique weak entropy solution* ρ *of* (2), *when* N *goes to* $+\infty$.

BV estimates (still ongoing)

Proposition (BV estimates)

Assume that **(V)-(W)** *hold true. Assume also that* $\rho_0 \in \mathbf{BV}_c(\mathbb{R}) \cap \mathbf{L}^1(\mathbb{R})$. *Then the total variation* $t \mapsto TV[\rho_N(t)]$ *of* ρ_N *for any* $N \in \mathbb{N}$ *is bounded as follows*

$$TV[\rho_N(t)] \leq C(t) + \mathcal{K}_N \left(2 \sum_{i=1}^{N} |\varepsilon_i(t)| + TV[\rho_N(0)] \right), \quad \text{for any} \quad t \geq 0, \tag{8}$$

where

$$\begin{cases} C(t) := |y_1(t)| + |y_{N-1}(t)| \\ \mathcal{K}_N := 2N. \end{cases} \tag{9}$$

BV estimates (still ongoing)

Proposition (Estimation on the Total Variation)

If **(W)** *holds true and if* $\theta > \dfrac{2}{3}$*, then we have*

$$\frac{1}{2-\theta}\sum_{i=1}^{N-3}|L_i| \leq TV[\rho_N] \leq \frac{1}{3\theta-2}\sum_{i=1}^{N-3}|L_i| + 4, \tag{10}$$

from which we deduce that

$$\lim_{N\to+\infty}\frac{TV[\rho_N]}{N} = 0.$$

Some references I

A. Aw, A. Klar, M. Rascle, and T. Materne, *Derivation of continuum traffic flow models from microscopic follow-the-leader models*, SIAM Journal on Applied Mathematics, 63 (2002), pp. 259–278.

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P. Goatin and F. Rossi, *A traffic flow model with non-smooth metric interaction: well-posedness and micro-macro limit*, arXiv preprint arXiv:1510.04461, (2015).

Thanks for your attention

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