Some recent developments in the traffic flow variational formulation

Guillaume Costeseque
collaborations with J-P. Lebacque and J. Laval

Inria Sophia-Antipolis Méditerranée

Séminaire Modélisation des Réseaux de Transports
– IFSTTAR Marne-la-Vallée –
May 12, 2016

HJ & Lax-Hopf formula

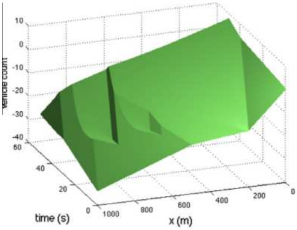
Hamilton-Jacobi equations: why and what for?

- Smoothness of the solution (no shocks)
- Physically meaningful quantity

HJ & Lax-Hopf formula

Hamilton-Jacobi equations: why and what for?

- Smoothness of the solution (no shocks)
- Physically meaningful quantity
- Analytical expression of the solution
- Efficient computational methods
- Easy integration of GPS data

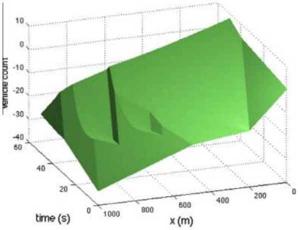


[Mazaré et al, 2012]

HJ & Lax-Hopf formula

Hamilton-Jacobi equations: why and what for?

- Smoothness of the solution (no shocks)
- Physically meaningful quantity
- Analytical expression of the solution
- Efficient computational methods
- Easy integration of GPS data

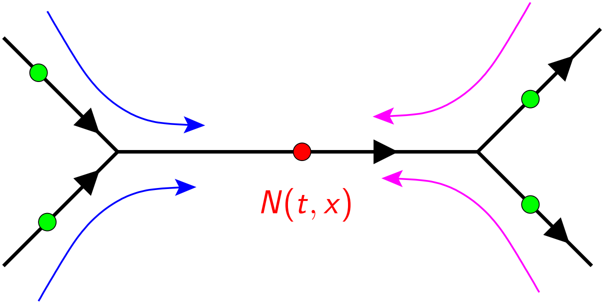


[Mazaré et al, 2012]

Everything broken for network applications?

Network model

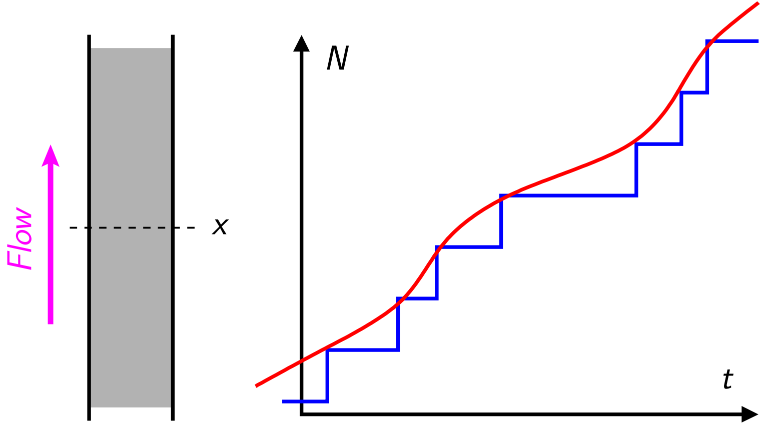
Simple case study: generalized three-detector problem (Newell (1993))



Outline

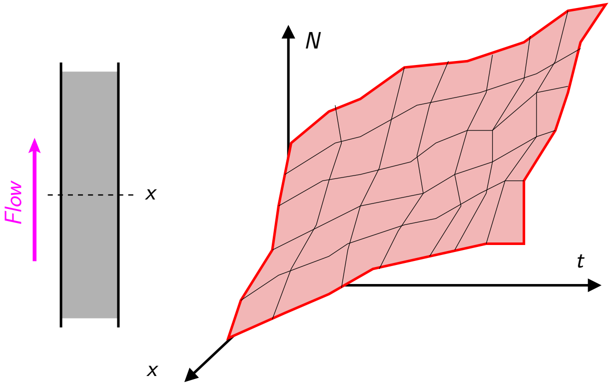
Outline

Convention for vehicle labeling



Three representations of traffic flow

Moskowitz' surface



See also [Makigami et al, 1971], [Laval and Leclercq, 2013]

Overview: conservation laws (CL) / Hamilton-Jacobi (HJ)

		Eulerian $t - x$	Lagrangian $t - n$
CL	Variable	Density ρ	Spacing r
	Equation	$\boxed{\partial_t \rho + \partial_x Q(\rho) = 0}$	$\boxed{\partial_t r + \partial_x V(r) = 0}$
HJ	Variable	Label N	Position $\mathcal{X}$
		$N(t,x) = \int_x^{+\infty} \rho(t,\xi) d\xi$	$\mathcal{X}(t,n) = \int_n^{+\infty} r(t,\eta) d\eta$
	Equation	$\boxed{\partial_t N + H(\partial_x N) = 0}$	$\boxed{\partial_t \mathcal{X} + \mathcal{V}(\partial_x \mathcal{X}) = 0}$
	Hamiltonian	$H(p) = -Q(-p)$	$\mathcal{V}(p) = -V(-p)$

Outline

Setting

Consider Cauchy problem

$$\begin{cases} u_t + H(Du) = 0, & \text{in} \quad \mathbb{R}^n \times (0, +\infty), \\ u(., 0) = u_0(.), & \text{on} \quad \mathbb{R}^n. \end{cases} \tag{1}$$

Two formulas according to the smoothness of

- the Hamiltonian H
- the initial data u_0

Lax-Hopf formulæ

Assumptions: **case 1**

(A1) $H : \mathbb{R}^n \to \mathbb{R}$ is convex

(A2) $u_0 : \mathbb{R}^n \to \mathbb{R}$ is uniformly Lipschitz

Theorem (First Lax-Hopf formula)

If (A1)-(A2) hold true, then

$$u(x,t) := \inf_{z \in \mathbb{R}^n} \sup_{y \in \mathbb{R}^n} \left[u_0(z) + y.(x-z) - tH(y) \right] \tag{2}$$

is the unique uniformly continuous viscosity solution of (1).

Legendre-Fenchel transform

First Lax-Hopf formula (2) can be recast as

$$u(x,t) := \inf_{z \in \mathbb{R}^n} \left[u_0(z) - tH^* \left(\frac{x-z}{t} \right) \right]$$

thanks to Legendre-Fenchel transform

$$L(z) = H^*(z) := \sup_{y \in \mathbb{R}^n} \left(y.z - H(y) \right).$$

Proposition (Bi-conjugate)

If H is strictly convex, 1-coercive i.e. $\lim_{|p| \to \infty} \frac{H(p)}{|p|} = +\infty$,
then H^* is also convex and

$$(H^*)^* = H.$$

Legendre-Fenchel transform

First Lax-Hopf formula (2) can be recast as

$$u(x,t) := \inf_{z\in\mathbb{R}^n} \left[u_0(z) - tH^*\left(\frac{x-z}{t}\right)\right]$$

thanks to Legendre-Fenchel transform

$$L(z) = H^*(z) := \sup_{y\in\mathbb{R}^n} \left(y.z - H(y) \right).$$

Proposition (Bi-conjugate)

If H is strictly convex, 1-coercive i.e. $\lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty$,

then H^* is also convex and

$$(H^*)^* = H.$$

LWR in Eulerian (t, x)

- Cumulative vehicles count (CVC) or Moskowitz surface $N(t, x)$

$$q = \partial_t N \quad \text{and} \quad \rho = -\partial_x N$$

- If density ρ satisfies the scalar (LWR) conservation law

$$\partial_t \rho + \partial_x Q(\rho) = 0$$

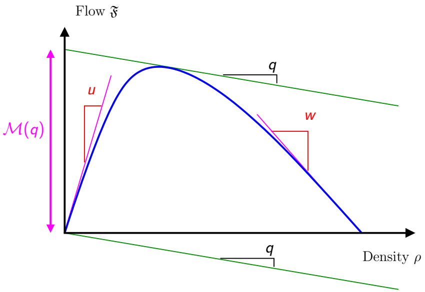
- Then N satisfies the first order Hamilton-Jacobi equation

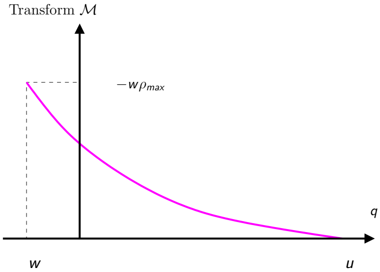
$$\partial_t N - Q(-\partial_x N) = 0 \tag{3}$$

LWR in Eulerian (t, x)

- Legendre-Fenchel transform with Q concave (*relative capacity*)

$$\mathcal{M}(q) = \sup_{\rho} \; [Q(\rho) - \rho q]$$



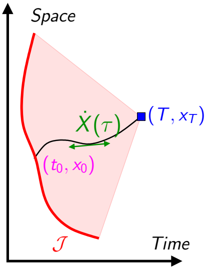


LWR in Eulerian (t, x)
(continued)

- Lax-Hopf formula

$$N(T, x_T) = \min_{u(.),(t_0,x_0)} \int_{t_0}^{T} \mathcal{M}(u(\tau))d\tau + N(t_0, x_0),$$

$$\left| \begin{array}{l} \dot{X} = u \\ u \in \mathcal{U} \\ X(t_0) = x_0, \quad X(T) = x_T \\ (t_0, x_0) \in \mathcal{J} \end{array} \right. \tag{4}$$

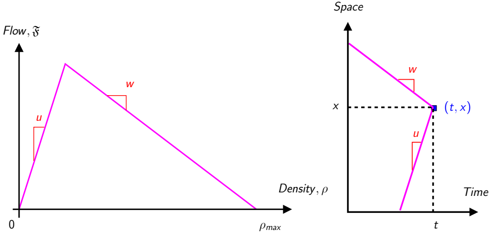


- Viability theory [Claudel and Bayen, 2010]

LWR in Eulerian (t, x)

(Historical note)

- Dynamic programming [Daganzo, 2006] for triangular FD (u and w free and congested speeds)

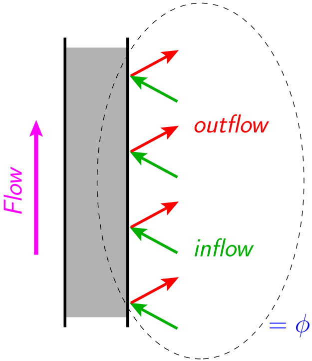


- Minimum principle [Newell, 1993]

$$N(t,x) = \min \left[N\left(t - \frac{x - x_u}{u}, x_u\right), \right. \\ \left. N\left(t - \frac{x - x_w}{w}, x_w\right) + \rho_{max}(x_w - x) \right], \tag{5}$$

Outline

- Long homogeneous corridor with numerous entrances and exits
- Net lateral freeway "inflow" rate ϕ



$$\boxed{\begin{aligned} \partial_t \rho + \partial_x H(\rho) &= \phi, \\ k &= g \quad \text{on} \quad \Gamma \end{aligned}} \tag{6}$$

$$\boxed{\begin{aligned} \partial_t N - H\left(-\partial_x N\right) &= \Phi, \\ N &= G \quad \text{on} \quad \Gamma, \end{aligned}} \tag{7}$$

where

$$\Phi(t,x) = -\int_0^x \phi(t,y)dy$$

$$G(t,x) = \oint_\Gamma g(t,x)d\Gamma, \qquad (t,x) \in \Gamma$$

Some remarks

- The flow reads $q = N_t - \Phi$ and the cumulative count curves are

$$N(t,x) = \underbrace{\int_0^t q(s,x)ds}_{\text{usual } N\text{-curve}} + \underbrace{\int_0^t \int_0^x \phi(s,y)dsdy}_{\text{net number of vehicles ``entering''}}$$

- If $\phi = \phi(k)$ then

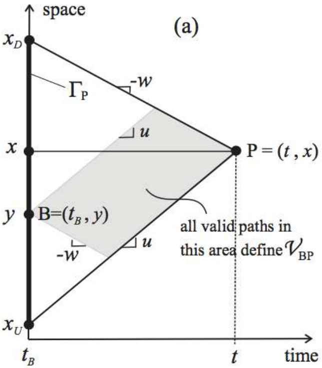
$$\Phi(t,x) = \tilde{\Phi}(t,x,-N_x) = -\int_0^x \phi(-N_x(t,y))dy, \qquad (8)$$

- This means that (7) becomes the more general HJ equation

$$N_t - \tilde{H}(t,x,-N_x) = 0$$

where $\tilde{H}(t,x,k) = H(k) + \tilde{\Phi}(t,x,k)$.

Variational problem



Lax-Hopf formula:

$$N(P) = \min_{B \in \Gamma^P,\ \xi \in \mathcal{V}_{BP}} f(B, \xi) \tag{9}$$

$$f(B, \xi) := G(B) + \int_{t_B}^{t} R(s, \xi(s), \xi'(s))\, ds$$

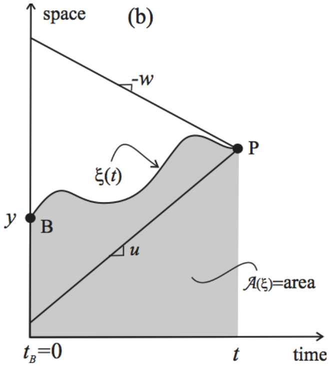
- $P \equiv (t, x)$ "target" point
- $B \equiv (t_B, y)$ on the boundary Γ^P
- $\xi \in \mathcal{V}_{BP}$ set of valid paths $B \to P$
- $R(\cdot)$ Legendre transform of $\tilde{H}$

$$R(t, x, v) = \sup_k \left\{ \tilde{H}(t, x, k) - vk \right\}.$$

- Assume a triangular flow-density diagram
- The function $f(B,\xi)$ to be minimized reads

$$f(B,\xi) = G(B) + (t-t_B)Q - (x-y)K + \underbrace{\int_{t_B}^{t} \Phi(s,\xi(s))\, ds}_{=:J} \qquad (10)$$

where Q = capacity, K = critical density.



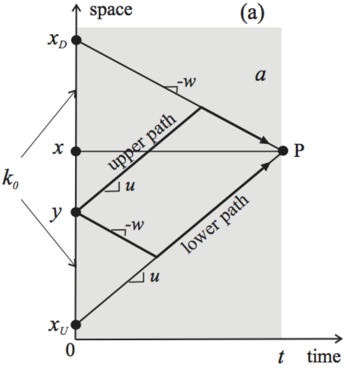
J = net number of vehicles *leaving* the area $A(\xi)$ below the curve $x = \xi(t)$

$$J = -\int_{t_B}^{t} \int_{y}^{\xi(s)} \phi(s,x)\, dxds, \qquad (11)$$

Initial value problems with constant density

Assume

$$\begin{aligned} N(0,x) &= G(x) = -k_0 x \quad (g(x) = k_0)\,, \\ \phi(t,x) &= a, \end{aligned} \tag{12}$$



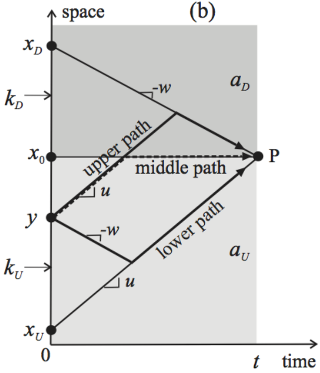
- $\min f\,(B \equiv (y,0), \xi)$ reached for a path
  - (i) maximizing $A(\xi)$ when $a > 0$
  - (ii) or minimizing $A(\xi)$ when $a < 0$
- $f(y) = c_0 + c_1 y + c_2 y^2$ with $c_2 > 0$
- Explicit solution:

$$N(t,x) = \begin{cases} f(y^*), & t > \frac{K - k_0}{a} > 0 \\ \min\{f(x_U), f(x_D)\}, & \text{otherwise} \end{cases}$$

Extended Riemann problem (ERP)

Consider

$$(g(x), \phi(x)) = \begin{cases} (k_U, a_U), & x \leq x_0 \\ (k_D, a_D), & x > x_0, \end{cases} \tag{13}$$



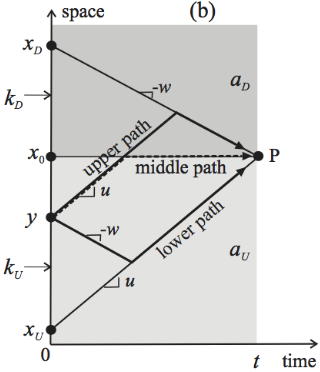
Assuming $G(x_0) = 0$

$$G(x) = \begin{cases} (x_0 - x)k_U, & x \leq x_0 \\ (x_0 - x)k_D, & x > x_0 \end{cases} \tag{14}$$

J-integral = weighted average of the portion of $A(\xi)$ upstream and downstream of $x = x_0$ with weighs a_U and a_D

Extended Riemann problem (ERP)
(continued)

Same minimization of $J(\xi)$



- $f(y) = G(y) + tQ - (x_0 - y)K + J(y)$ with

$$J(y) = \begin{cases} \min\{j_1(y), j_2(y), j_3(y)\}, & y > x_0 \\ \min\{j_4(y), j_5(y), j_6(y)\}, & y \leq x_0 \end{cases}$$

- Possible minima for the components of $f(y)$

$$y = y_i \in \Gamma^P, \ \ i = 1, \dots 6$$

- Semi-explicit solution (9 candidates):

$$N(t, x_0) = \min_{y \in \mathcal{Y}} f(y)$$

$$\mathcal{Y} = \{x_U, x_0, x_D, y_1^*, \dots y_6^*\}$$

Godunov's method

- Basis of the well known Cell Transmission (CT) model
- Time and space increments Δt and $\Delta x = u\Delta t$
- Numerical approximation of the density

$$k_i^j = k(j\Delta t, i\Delta x) \tag{15}$$

- Discrete approximation of the conservation law (6):

$$\frac{k_i^{j+1} - k_i^j}{\Delta t} + \frac{q_{i+1}^j - q_i^j}{\Delta x} = \phi(k_i^j) \tag{16}$$

with (CT rule)

$$q_i^j = \min\{Q, uk_i^j, (\kappa - k_{i+1}^j)w\} \tag{17}$$

Example

- Consider an empty freeway at $t = 0$ with

$$g(x) = 0, \tag{18a}$$
$$\phi(k) = ax - buk, \qquad a, b > 0. \tag{18b}$$

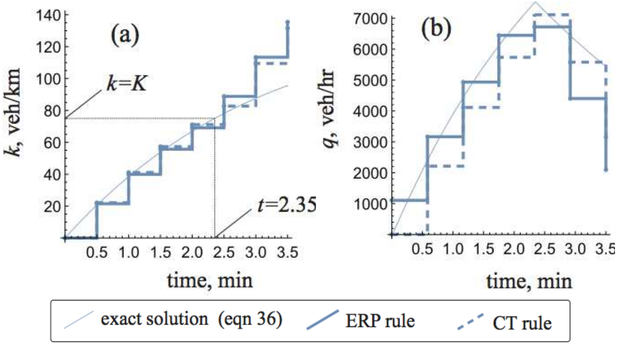
- Exact solution (method of characteristics):

$$k(t, x) = \frac{a}{b^2 u} \left(bx - 1 + (1 - b(x - tu))e^{-btu} \right) \tag{19}$$

provided $k(t, x) \leq K$ (Laval, Leclercq (2010))

Example
(continued)

Comparison of numerical solutions ($\Delta t = 40$ s) and the exact solution

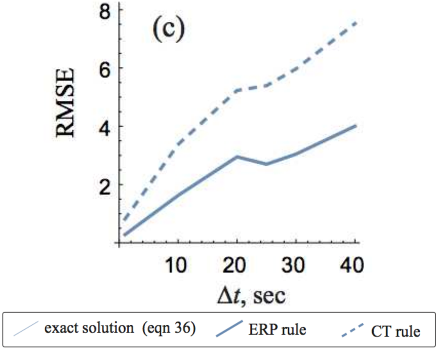


Main difference = the flow estimates

Example

(continued)

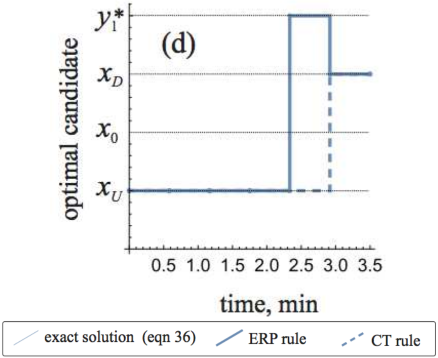
Density RMSE (numerical VS exact solution) for varying Δt:



- Both converge as $\Delta t \to 0$
- Accuracy of ERP $>$ CT rule

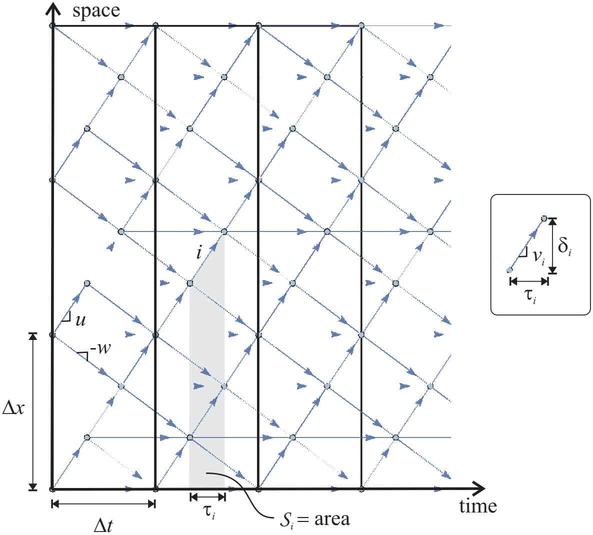
Example
(continued)

Optimal candidate that minimizes $f(y)$ at each time step

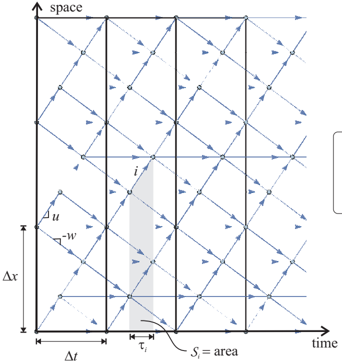


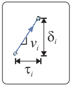
- Difference when $k \to K$
- Most accurate optimal candidate y_1^*

Variational networks



Variational networks





- Only three wave speeds with costs

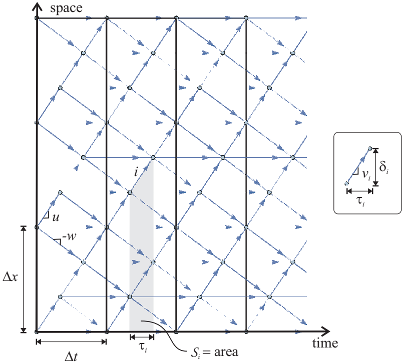
$$\mathcal{L}(v_i) = \begin{cases} w\kappa, & v_i = -w \\ Q, & v_i = 0 \\ 0, & v_i = u \end{cases} \tag{20}$$

- The cost on each link:

$$c_i = \mathcal{L}(v_i)\tau_i + J_i.$$

- J_i = contribution of the J-integral in the cost of each link i

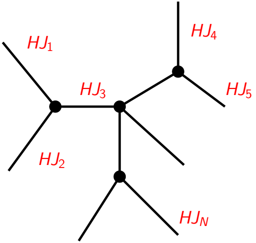
Variational networks



- Advantage: free of numerical errors (when inflows are exogenous)
- Drawback:
  - cumbersome to implement unless $\frac{u}{w}$ is an integer
  - merge models expressed in terms of flows or densities rather than N values: additional computational layer needed

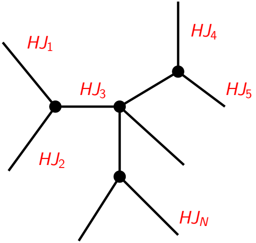
Outline

A special network = junction

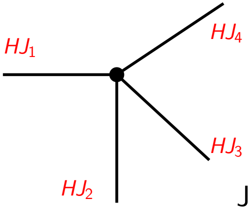


Network

A special network = junction



Network



Junction J

Space dependent Hamiltonian

Consider HJ equation posed on a junction J

$$\begin{cases} u_t + H(x, u_x) = 0, & \text{on} \quad J \times (0, +\infty), \\ u(t = 0, x) = g(x), & \text{on} \quad J \end{cases} \tag{21}$$

Extension of Lax-Hopf formula(s)?

- No simple linear solutions for (21)
- No definition of convexity for discontinuous functions

Junction models

Classical approaches for CL:

- Macroscopic modeling on (homogeneous) sections
- Coupling conditions at (pointwise) junction

For instance, consider

$$\begin{cases} \rho_t + (Q(\rho))_x = 0, & \text{scalar conservation law,} \\ \rho(., t=0) = \rho_0(.), & \text{initial conditions,} \\ \psi(\rho(x=0^-, t), \rho(x=0^+, t)) = 0, & \text{coupling condition.} \end{cases} \quad (22)$$

See Garavello, Piccoli [4], Lebacque, Khoshyaran [6] and Bressan et al. [1]

Examples of junction models

- Model with internal state (= buffer(s))
  Bressan & Nguyen (NHM 2015) [2]
    - $\rho \mapsto Q(\rho)$ strictly concave
    - advection of $\gamma_{ij}(t, x)$ turning ratios from (i) to (j) (GSOM model with passive attribute)
    - internal dynamics of the buffers (ODEs): queue lengths
- Extended Link Transmission Model
  Jin (TR-B 2015) [5]
    - Link Transmission Model (LTM) Yperman (2005, 2007)
    - Triangular diagram

$$Q(\rho) = \min\{u\rho,\ w(\rho_{max} - \rho)\} \quad \text{for any} \quad \rho \in [0, \rho_{max}]$$

    - Commodity = turning ratios $\gamma_{ij}(t)$
    - Definition of boundary supply and demand functions

First remarks

If N solves

$$N_t + H(N_x) = 0$$

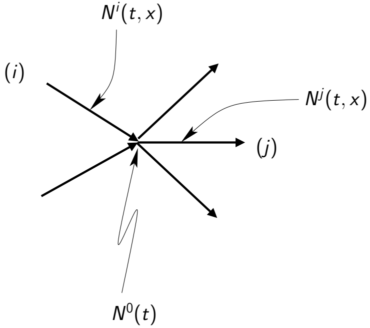
then $\bar{N} = N + c$ for any $c \in \mathbb{R}$ is also a solution

First remarks

If N solves

$$N_t + H(N_x) = 0$$

then $\bar{N} = N + c$ for any $c \in \mathbb{R}$ is also a solution



- No a priori relationship between initial conditions
- $N^k(t,x)$ consistent along the same branch J_k and

$$\partial_t N^0(t) = \sum_i \partial_t N^i\left(t, x = 0^-\right) = \sum_j \partial_t N^j\left(t, x = 0^+\right)$$

Key idea

Assume that H is piecewise linear (triangular FD)

$$N_t + H\left(N_x\right) = 0$$

with

$$H(p) = \max\{\underbrace{H^+(p)}_{\text{supply}}, \underbrace{H^-(p)}_{\text{demand}}\}$$

Key idea

Assume that H is piecewise linear (triangular FD)

$$N_t + H\left(N_x\right) = 0$$

with

$$H(p) = \max\{\underbrace{H^+(p)}_{\text{supply}}, \underbrace{H^-(p)}_{\text{demand}}\}$$

Partial solutions N^+ and N^- that solve resp.

$$\begin{cases} N_t^+ + H^+\left(N_x^+\right) = 0, \\ N_t^- + H^-\left(N_x^-\right) = 0 \end{cases} \quad \text{such that} \quad N = \min\left\{N^-, N^+\right\}$$

- Upstream demand advected by waves moving forward
- Downstream supply transported by waves moving backward

Junction model

Optimization junction model (Lebacque's talk)
Lebacque, Khoshyaran (2005) [6]

$$\max \left[\sum_i \phi_i(q_i) + \sum_j \psi_j(r_j) \right]$$
$$\text{s.t.} \left| \begin{array}{ll} 0 \leq q_i & \forall i \\ q_i \leq \delta_i & \forall i \\ 0 \leq r_j & \forall j \\ r_j \leq \sigma_j & \forall j \\ 0 = r_j - \sum_i \gamma_{ij} q_i & \forall j \end{array} \right. \tag{23}$$

where ϕ_i, ψ_j are concave, non-decreasing

Example of optimization junction models

- Herty and Klar (2003)
- Holden and Risebro (1995)
- Coclite, Garavello, Piccoli (2005)
- Daganzo's merge model (1995) [3]

$$\begin{cases} \phi_i(q_i) = N_{max} \left(q_i - \dfrac{q_i^2}{2 p_i q_{i,max}} \right) \\ \psi = 0 \end{cases}$$

where p_i is the priority of flow coming from road i and $N_{max} = \phi_i'(0)$

Solution of the optimization model

LEBACQUE, KHOSHYARAN (2005)

Karush-Kuhn-Tucker optimality conditions:

- For any incoming road i

$$\phi_i'(q_i) + \sum_k s_k \gamma_{ik} - \lambda_i = 0, \quad \lambda_i \geq 0, \quad q_i \leq \delta_i \quad \text{and} \quad \lambda_i(q_i - \delta_i) = 0,$$

- and for any outgoing road j

$$\psi_j'(r_j) - s_j - \lambda_j = 0, \quad \lambda_j \geq 0, \quad r_i \leq \sigma_j \quad \text{and} \quad \lambda_j(r_j - \sigma_j) = 0,$$

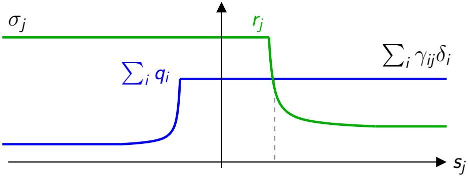
where (s_j, λ_j) = Karush-Kuhn-Tucker coefficients (or Lagrange multipliers)

Solution of the optimization model

LEBACQUE, KHOSHYARAN (2005)

$$\begin{cases} q_i = \Gamma_{[0,\delta_i]}\left((\phi_i')^{-1} \left(-\sum_k \gamma_{ik} s_k \right) \right), & \text{for any} \quad i, \\ r_j = \Gamma_{[0,\sigma_j]}\left((\psi_j')^{-1}(s_j) \right), & \text{for any} \quad j, \end{cases} \tag{24}$$

$\Gamma_{\mathcal{K}}$ is the projection operator on the set $\mathcal{K}$



Model equations

$$
\begin{cases}
N^i_t + H_i(N^i_x) = 0, & \text{for any} \quad x \neq 0, \\
\\
\begin{cases} \partial_t N^i(t, x^-) = q_i(t), \\ \partial_t N^j(t, x^+) = r_j(t), \end{cases} & \text{at} \quad x = 0, \\
\\
N^i(t = 0, x) = N^i_0(x), \\
\\
\partial_t N^i(t, x = \xi_i) = \Delta_i(t), \\
\partial_t N^j(t, x = \chi_j) = \Sigma_j(t)
\end{cases}
$$

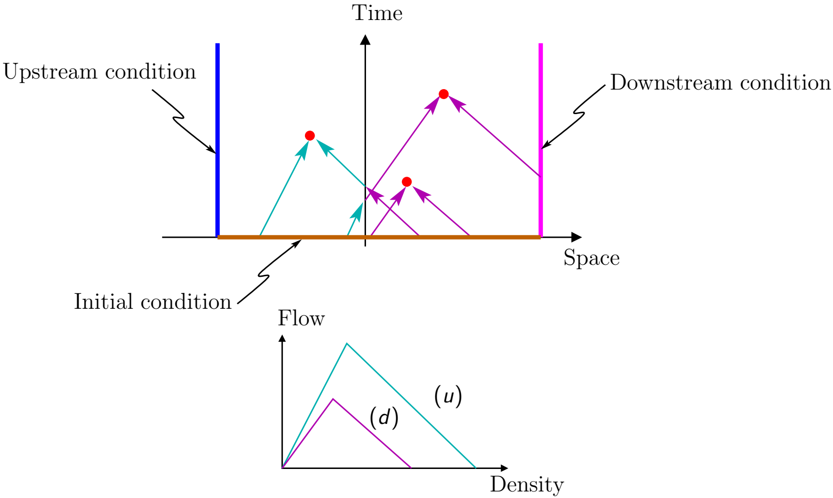
Algorithm

Inf-morphism property: compute partial solutions for

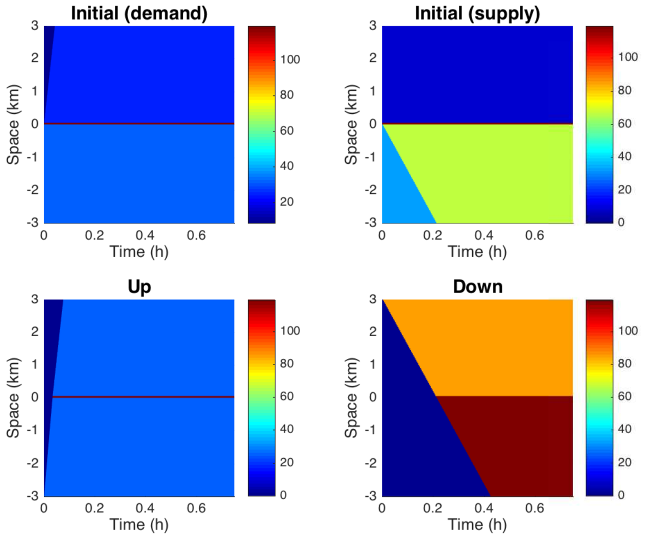
- initial conditions
- upstream boundary conditions
- downstream boundary conditions
- internal boundary conditions

1. Propagate demands forward
   - through a junction, assume that the downstream supplies are maximal
2. Propagate supplies backward
   - through a junction, assume that the upstream demands are maximal

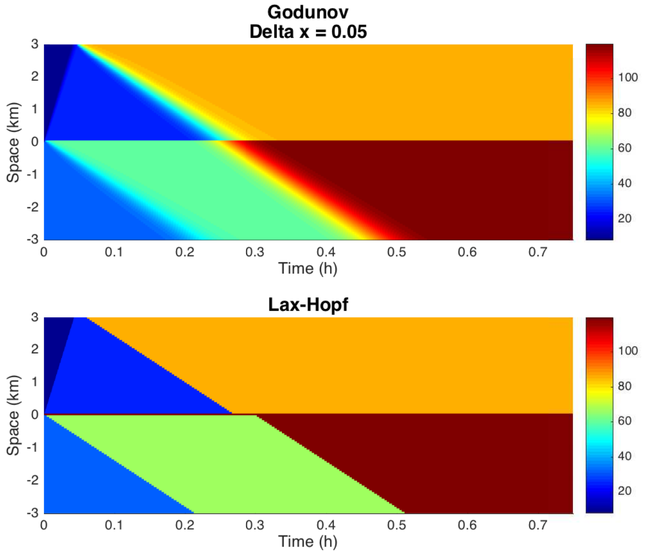
Spatial discontinuity



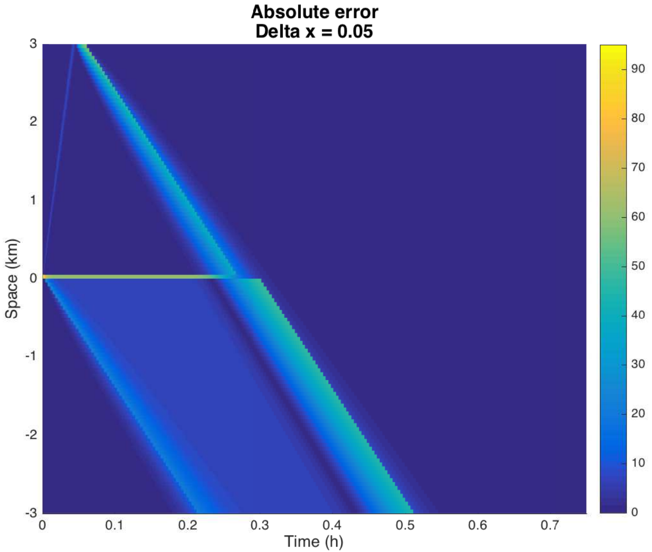
Numerical results



Numerical results



Numerical results



Final remarks

In a nutshell:

- No explicit solution right now
- Only specific cases
- Importance of the supply/demand functions
- General optimization problem at the junction

Perspectives:

- Lane changing behaviors
- Estimation on networks
- Stationary states

Some references I

- A. Bressan, S. Canic, M. Garavello, M. Herty, and B. Piccoli, *Flows on networks: recent results and perspectives*, EMS Surveys in Mathematical Sciences, (2014).
- A. Bressan and K. T. Nguyen, *Conservation law models for traffic flow on a network of roads*, to appear, (2014).
- C. F. Daganzo, *The cell transmission model, part ii: network traffic*, Transportation Research Part B: Methodological, 29 (1995), pp. 79–93.
- M. Garavello and B. Piccoli, *Traffic flow on networks*, American institute of mathematical sciences Springfield, MO, USA, 2006.
- W.-L. Jin, *Continuous formulations and analytical properties of the link transmission model*, Transportation Research Part B: Methodological, 74 (2015), pp. 88–103.

Some references II

J.-P. Lebacque and M. M. Khoshyaran, *First-order macroscopic traffic flow models: Intersection modeling, network modeling*, in Transportation and Traffic Theory. Flow, Dynamics and Human Interaction. 16th International Symposium on Transportation and Traffic Theory, 2005.

Thanks for your attention

guillaume.costeseque@inria.fr