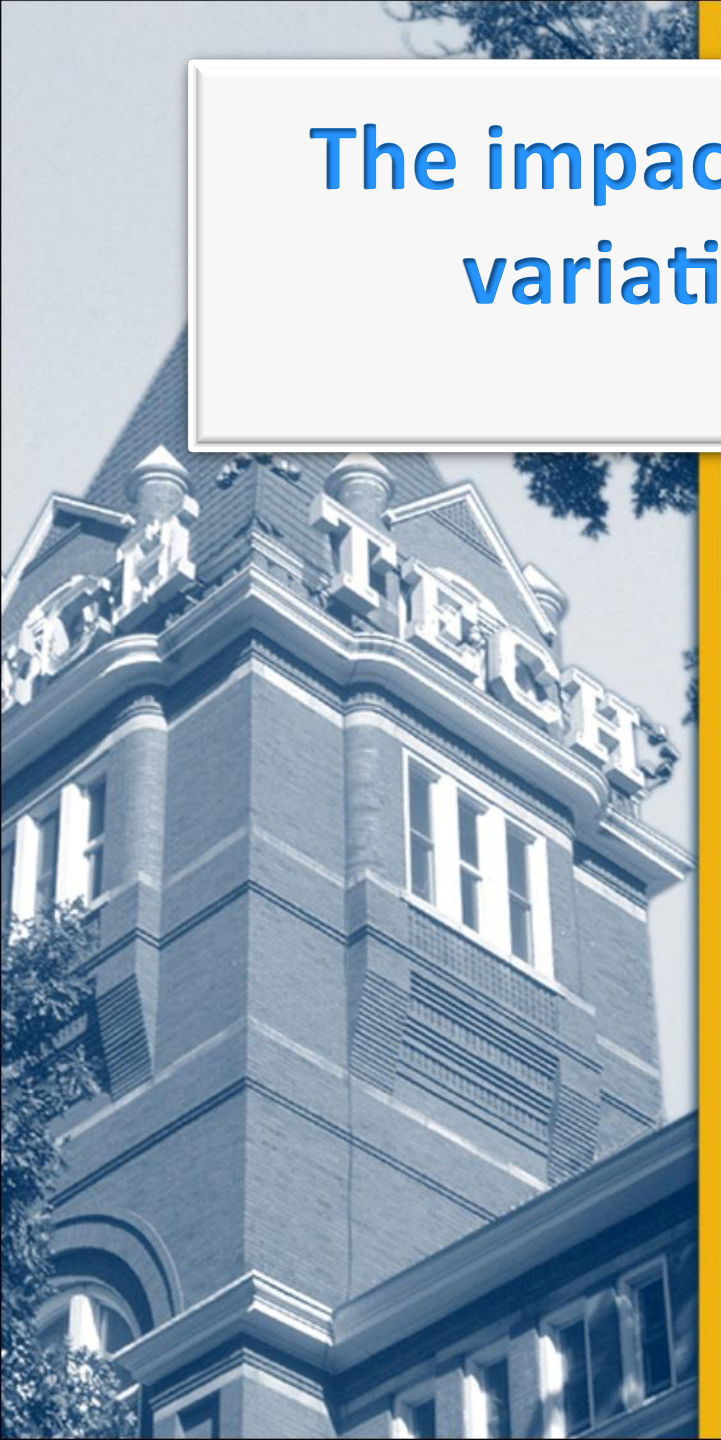
The impact of source terms in the variational representation of traffic flow



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Motivation

- Variational theory (VT) was an important milestone to solve the kinematic wave model:
  – Allows analytical global solutions
  – No need of shocks and/or entropy conditions (as in the method of characteristics)
- Current approximation methods for the Macroscopic Fundamental Diagram (MFD) of urban networks rely on VT
- ➔Here we revisit VT now including Eulerian lateral inflows and outflows.

Main result

- VT solutions:
  - OK in Eulerian coordinates for exogenous source terms
  - but not when they are a function of traffic conditions (merge model e.g.)
- In discrete time source terms become exogenous
  - → improved numerical solution methods
- In Lagrangian-Space and Lagrangian-Time coordinates, VT solutions **may** not exist even if source terms are exogenous.

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Problem formulation

- Long homogeneous freeway with many entrances and exits.
- Let ϕ = net lateral freeway inflow units of veh/time-distance.
- The Kinematic Wave model reads:

$$k_t + H(k)_x = \phi, \tag{1a}$$
$$k = g \quad \text{on} \quad \Gamma \tag{1b}$$

where H is the fundamental diagram, g is the data defined on a boundary Γ

- integrate (1) with respect to x to obtain its Hamilton-Jacobi form:

$$N_t - H(-N_x) = \Phi, \tag{2a}$$
$$N = G \quad \text{on} \quad \Gamma, \tag{2b}$$

where $N_x = -\,k$ and :

$$G(t,x) = \oint_\Gamma g(t,x)d\Gamma, \qquad (t,x) \in \Gamma, \qquad \text{and} \tag{3a}$$
$$\Phi(t,x) = -\int_0^x \phi(t,y)dy \qquad \text{(potential function)} \tag{3b}$$

Exogenous inflow

- If $\Phi(t,x)$ is given: $\tilde{H}(t,x,k) = H(k) + \Phi(t,x)$
- Triangular FD: passing rate is $Q - Kv + \Phi(t,x)$ and the variational problem reads:

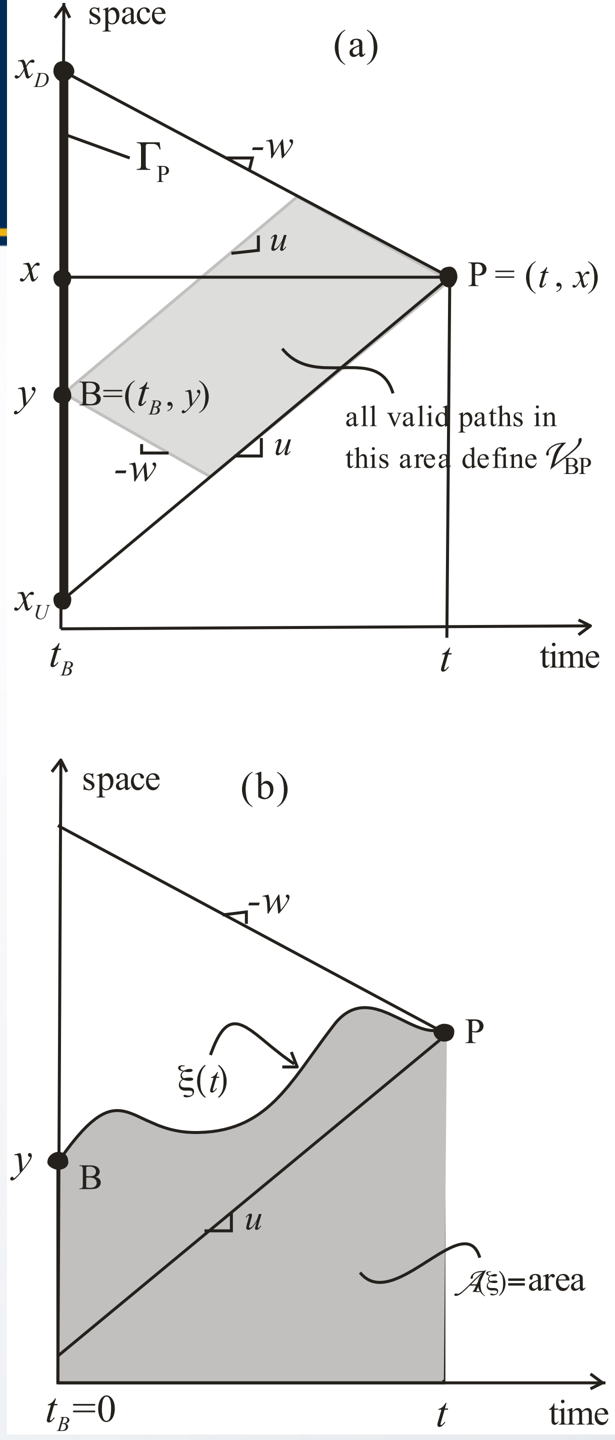
$$N(P) = \min_{B \in \Gamma_P,\ \xi \in \mathcal{V}_{BP}} f(B,\xi), \qquad \text{with}$$

$$f(B,\xi) = G(B) + (t - t_B)Q - (x - y)K + J$$

$$J = -\int_{t_B}^{t} \int_{y}^{\xi(s)} \phi(s,x)\, dx ds,$$

where Q=capacity K=critical density.

- J= net # of vehs *leaving* area $A(\xi)$
- min f : find $\xi(t)$ that maximizes the net number of vehicles *entering* $A(\xi)$.



Extended Riemann problems (ERP)

- building block for Godunov-type methods.
- initial data:

$$(g(x), \phi(x)) = \begin{cases} (k_U, a_U), & x \le x_0 \\ (k_D, a_D), & x > x_0, \end{cases}$$

- We show that:

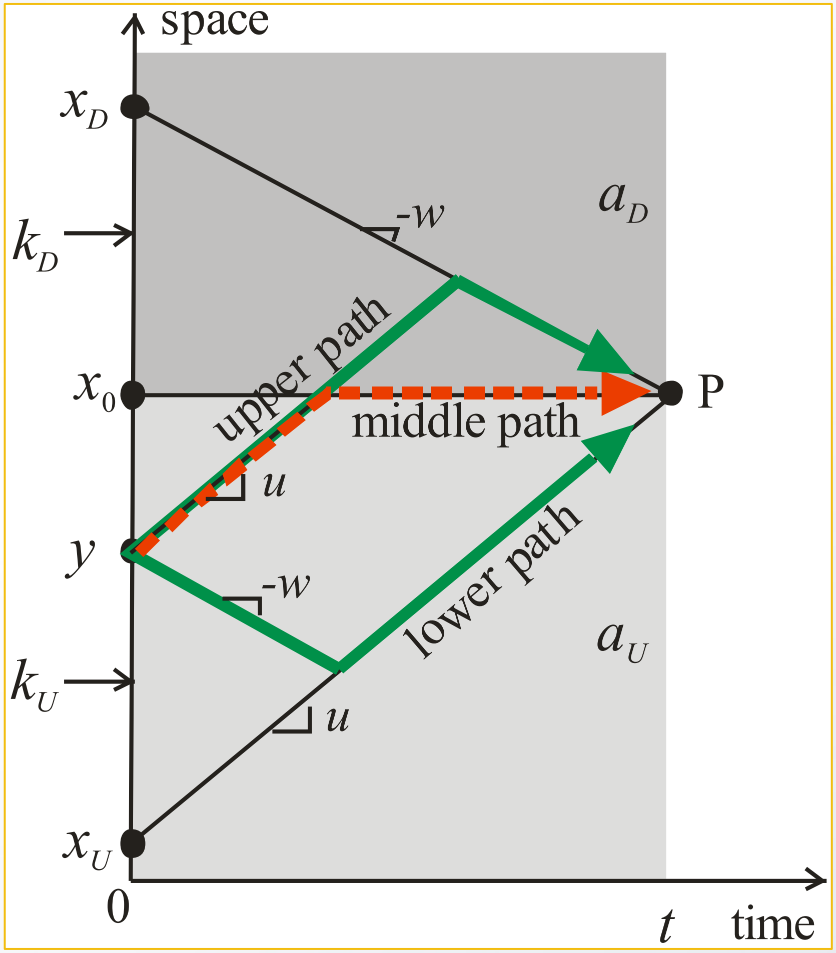
$$N(t, x_0) = \min_{y \in \mathcal{Y}} f(y)$$

$$\mathcal{Y} = \{x_U, x_0, x_D, y_1^*, \ldots y_6^*\}$$

$$f(y) = G(y) + tQ - (x_0 - y)K + J(y)$$

$$J(y) = \begin{cases} \min\{j_1(y), j_2(y), j_3(y)\}, & y > x_0 \\ \min\{j_4(y), j_5(y), j_6(y)\}, & y \le x_0 \end{cases}$$

- $j_1 \ldots j_6$ = value of $J(\xi)$ along upper, lower and middle paths.



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Endogenous inflow

- If $\phi = \phi(k)$ then the potential function Φ depends non-locally on N:

$$\Phi(t,x) = \tilde{\Phi}(N,x) = -\int_0^x \phi(-N_x(t,y))dy, \qquad (1)$$

- Our problem becomes the more general HJ equation

$$N_t - \tilde{H}(x, N, -N_x) = 0$$

The Hamiltonian's N-dependency is what complicates matters. Barron (1996 , 2015) show that a Variational solution do not exist in general.

Numerical solution methods

- Idea: in discrete time stepping methods, endogenous inflows computed from the previous time step become exogenous for the current time step and therefore the VT solution may be applied.

- 2 methods are presented:
  - Godunov's method (Kinematic Wave model)
  - Variational networks (Hamilton-Jacobi model)

Godunov's method

• increments Δt and $\Delta x = u\Delta t$, and:

$$k_i^j = k(j\Delta t, i\Delta x) \tag{1}$$

is the numerical approximation of the density.

• update scheme:

$$\frac{k_{i+1}^j - k_i^j}{\Delta t} + \frac{q_i^{j+1} - q_i^j}{\Delta x} = \phi(k_i^j) \tag{2}$$

• the flow into cell i, q_i^j are obtained by solving Riemann problems.

• Traditionally, inflows not considered in Riemann problems; e.g. the Cell Transmission (CT) rule:

$$q_i^j = \min\{Q, uk_i^j, (\kappa - k_i^{j+1})w\}, \qquad \text{(CT rule)} \tag{3}$$

• we now compare the CT rule with the extended Riemann problems (ERP) presented earlier.

Godunov's method - Example

- empty freeway at $t = 0$ subject to an inflow linear in both x and k; i.e.:

$$g(x) = 0, \tag{1a}$$

$$\phi(k) = ax - buk, \qquad a, b > 0. \tag{1b}$$

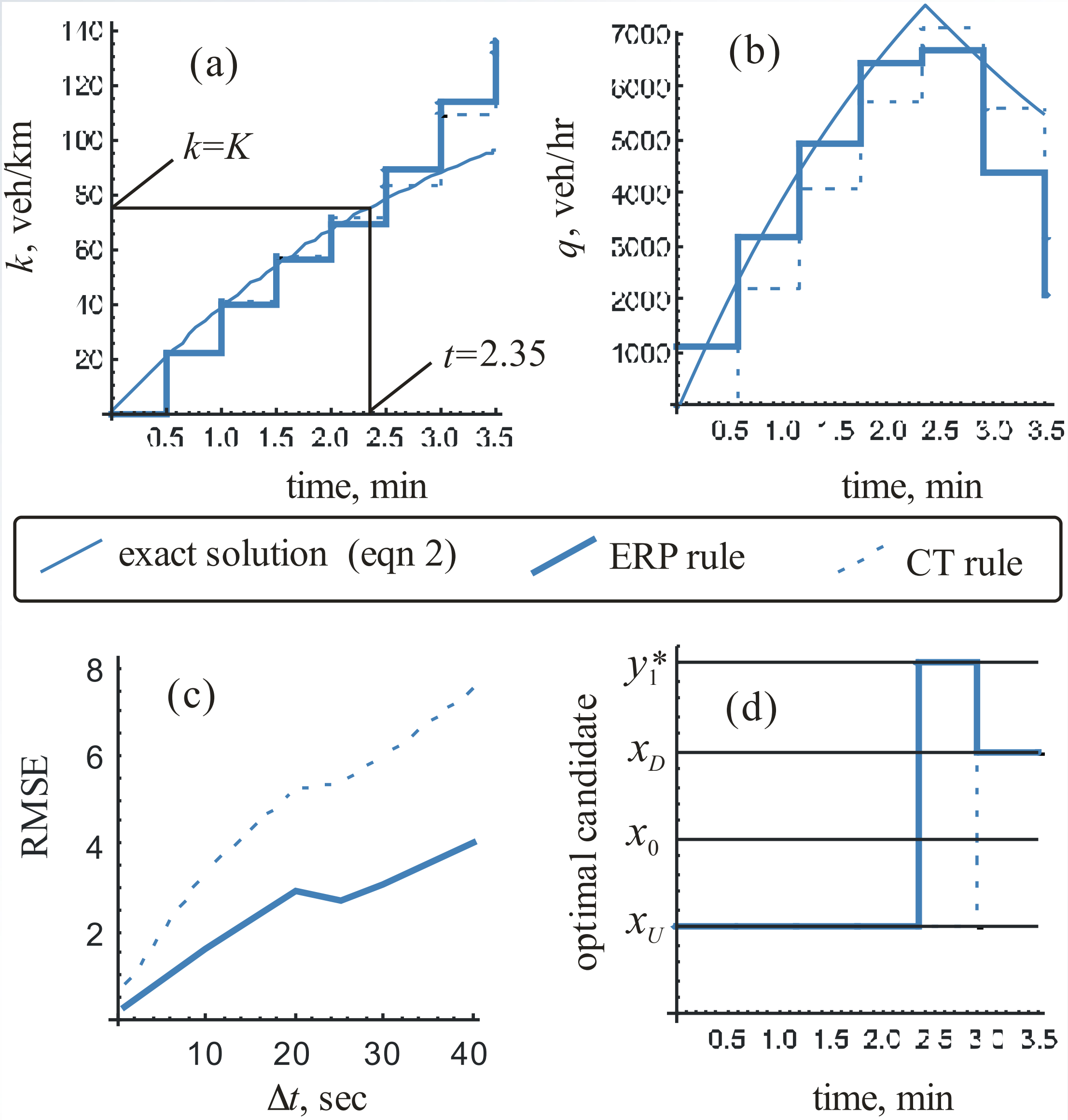
- linear inflows arise in the continuum approximation of the Newell-Daganzo merge model
- analytical solution is:

$$k(t, x) = \frac{a}{b^2 u}\left(bx - 1 + (1 - b(x - tu))e^{-btu}\right) \tag{2}$$

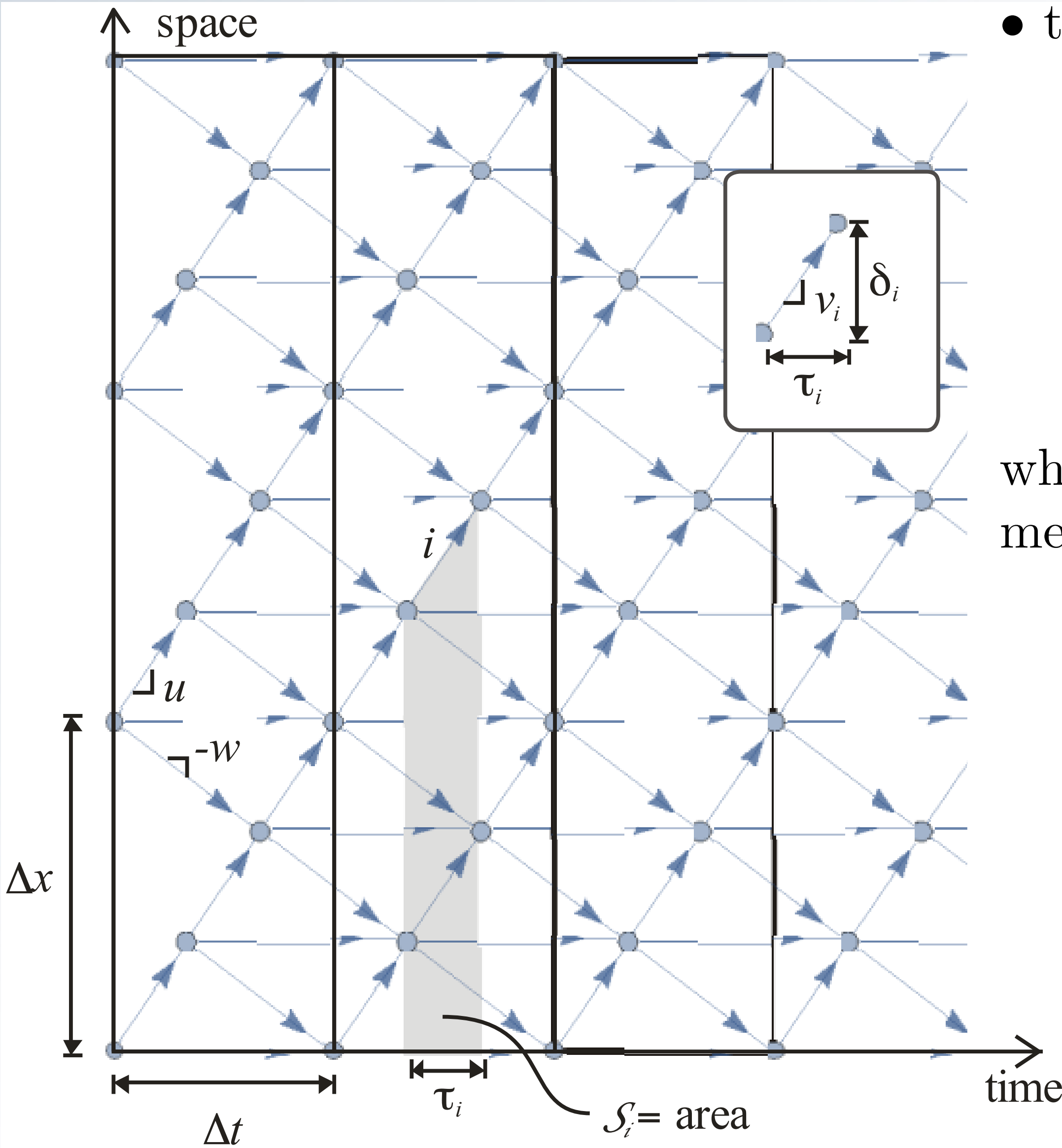
provided $k(t, x) \leq K$.

- we now compare (2) with the CT rule and ERP rule.

Godunov's method - Example



Variational networks



- the cost in each link becomes:

$$c_i = \mathcal{L}(v_i)\tau_i + J_i. \tag{1}$$

$$J_i = -\tau_i \sum_{j \in \mathcal{S}_i} \delta_j a_j, \tag{2a}$$

where a_j = inflow in link j and $j \in \mathcal{S}_i$ means all links that "touch" area $\mathcal{S}_i$.

$$\mathcal{L}(v_i) = \begin{cases} w\kappa, & v_i = -w \quad (3a) \\ Q, & v_i = 0 \quad (3b) \\ 0, & v_i = u \quad (3c) \end{cases}$$

Other coordinates

- Space-Lagrangian coordinates:

Let $X(t,n)$ be the position of vehicle n at time t. We showed that the corresponding HJ equation reads:

$$X_t - V(-X_n) = -X_n \Phi(t, X), \tag{1}$$

where $V(s)$ is the spacing-speed FD. We conclude that (1) does not admit a VT solution due to the term involving X.

- Time-Lagrangian coordinates:

Let $T(n,x)$ be the time vehicle n crosses location x. Now:

$$T_n - \frac{F(T_x)}{1 + \Phi(T,x)F(T_x)} = 0, \tag{2}$$

where $F(r)$ is the headway-pace relationship. Again, (2) does not admit a VT solution due to the term involving T.

Main result

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Q & A

THANK YOU !