A coupled PDE-ODE model for bounded acceleration in macroscopic traffic flow models

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Abstract: In this paper, we propose a new mathematical model accounting for the boundedness of traffic acceleration at a macroscopic scale. Our model is built on a first order macroscopic PDE model coupled with an ODE describing the trajectory of the leader of a platoon accelerating at a given constant rate. We use Wave Front Tracking techniques to construct approximate solutions to the Initial Value Problem. We present some numerical examples including the case of successive traffic signals on an arterial road and we compare the solution to our model with the solution given by the classical LWR equation in order to evaluate the impact of bounded acceleration.

Keywords: Bounded acceleration; macroscopic traffic flow model; PDE-ODE coupling; wave front tracking scheme.

1. INTRODUCTION

1.1 Motivation and review of the literature

In modern societies, traffic flows on road networks represent a major concern for public authorities. In order to design efficient control strategies, macroscopic traffic flow models seem better suited than microscopic models in most cases since they are robust, computationally efficient and allow for near-to-real-time computations. Among those models, the first order LWR model (Lighthill and Whitham (1955); Richards (1956)) has been deeply investigated. It reads as a scalar conservation law in one space dimension

$$\partial_t \rho + \partial_x q(\rho) = 0, \quad x \in \mathbb{R}, \ t > 0, \tag{1}$$

where $\rho \in [0, \rho_{\max}]$ is the density of cars and $q : \rho \mapsto q(\rho) = \rho v(\rho)$ is the flow-density fundamental diagram.

However, the LWR model has the drawback to only represent equilibrium states and it fails to reproduce some physical properties of traffic. In particular, if the traffic density upstream is higher than downstream, like for instance when a traffic light turns green, then the LWR model (1) solution displays an unbounded traffic acceleration, since the traffic velocity jumps instantaneously to a higher value. This poses a severe limitation to the coupling of the LWR model (and also second order models like Payne (1971); Whitham (1974) and Aw and Rascle (2000); Zhang (2002)) with consumption and pollution models (see for example Treiber and Kesting (2013, Chapter 20) and references therein), in which the acceleration component plays a crucial role. Also, bounded acceleration of vehicles may have an impact on management procedures, such as traffic signals optimization, which require a precise estimation of queue dissipation times (see Anderson (2015) and references therein).

The problem of bounded acceleration has been previously addressed by Lebacque (2002, 2003); Leclercq (2007) starting from the LWR model. In particular, the mathematical approach by Lebacque leads to a two phase model in which the bounded acceleration phase is described by a non-strictly hyperbolic system of balance laws, while the latter applies to rather restrictive cases limited to triangular fundamental diagrams. The above descriptions can hardly be extended to general situations and implemented rigorously in field situations. In this paper, we propose a general approach to account for bounded acceleration of vehicles from a macroscopic point of view, and we provide a rigorous constructive algorithm to compute solutions to our model, which can be applied to a large class of fundamental diagrams.

The LWR model provides solutions with unbounded accelerations only in specific situations. Upward jumps in velocity may happen at downward discontinuities of the initial density datum or can be generated by boundary conditions, for instance downstream a merge or a bottleneck whose positions are known. To handle these situations, we adapt the framework of *moving bottlenecks* proposed in the engineering literature by Lebacque et al. (1998). The reader interested by the mathematical analysis of the moving bottleneck problem is referred to Delle Monache and Goatin (2014) and references therein. In our case, we assume that all vehicles accelerate at the same constant rate, and that overtaking is not possible. Thus, the first vehicle will accelerate and regulate upstream traffic. Hence it will behave as a moving bottleneck. The *passing rate* is thus set to zero.

1.2 The PDE-ODE model

We consider the following assumptions:

(A0) The map $q : \rho \mapsto q(\rho)$ is strictly concave, differentiable, satisfies $q(0) = q(\rho_{\max}) = 0$ and there exists a unique $\rho_{\text{crit}} \in [0, \rho_{\max}]$ such that q is maximal.

(A1) The map $v : \rho \mapsto v(\rho)$ is decreasing, Lipschitz continuous and satisfies $v(0) = V_{\max}$ and $v(\rho_{\max}) = 0$.

(A2) The initial datum $\rho^0 \in \mathbb{BV}(\mathbb{R}; [0, \rho_{\max}])$ is a piecewise constant function with a finite number of jumps (any function in $\mathbb{BV}(\mathbb{R}; [0, \rho_{\max}])$ can be approximated by a piecewise constant function, see Bressan (2000)[Lemma 2.2]) .

We are now ready to introduce our coupled ODE-PDE model:

$$
\begin{aligned}
&\partial_t \rho + \partial_x q(\rho) = 0, && x \in \mathbb{R},\ t > 0, && \text{(2a)}\\
&\rho(0, x) = \rho^0(x), && x \in \mathbb{R}, && \text{(2b)}\\
&q\left(\rho(t, y_i(t))\right) - \rho(t, y_i(t))\dot{y}_i(t) \le 0, && t > 0, && \text{(2c)}\\
&\dot{y}_i(t) = \omega_i\left(t, \rho(t, y_i(t)+)\right) && t > 0, && \text{(2d)}\\
&y_i(0) = y_i^0, && && \text{(2e)}
\end{aligned}
$$

for every y_i^0, $i = 1, \dots, I$, such that the initial datum has a downward jump: $\rho^0(y_i^0-) > \rho^0(y_i^0+)$. In the model above, we have defined

$$\omega_i(t, \rho) := \min\left\{At + v_i^0, v(\rho)\right\}$$

where $v_i^0 = v(\rho^0(y_i^0-))$ stands for the initial speed of the moving bottleneck starting at position y_i^0 and $A > 0$ is the constant acceleration rate. y_i denotes the trajectory of the i-th moving bottleneck.

In the model above, the traffic density evolution is described by the scalar conservation law (2a) and the corresponding initial condition (2b). The equation (2c) accounts for the assumption that no overtaking is possible. It enforces that the flux directly upstream of the bottlenecks created by the leader vehicles is bounded by the flux along their trajectory. The ODE (2d) gives the trajectory of each accelerating leader starting from the position given by the initial condition (2e), i.e. any point of downward jump in the density. We note that in this case the LWR model would provide a solution in which the leader vehicles change their speed instantaneously, *i.e.* with an infinite acceleration.

Following Delle Monache and Goatin (2014), solutions to (2) are intended in the weak sense specified below.

Definition 1.1. (Definition of solutions to (2)). We call $(\rho, y) \in \mathcal{C}^0(\mathbb{R}^+; \mathbb{L}^1 \cap \mathbb{BV}(\mathbb{R}, [0, \rho_{max}])) \times \mathbb{W}^{1,1}((\mathbb{R}^+; \mathbb{R})$ a solution to (2) if and only if

(i) ρ is a weak solution of (2a), (2b), i.e. $\forall \phi \in \mathcal{C}_c^1(\mathbb{R}^2; \mathbb{R})$,
$$\int_0^T \int_{-\infty}^{\infty} (\rho\phi_t + f(\rho)\phi_x) \mathrm{d}x\mathrm{d}t + \int_{-\infty}^{\infty} \rho^0(x)\phi(0, x)\mathrm{d}x = 0;$$

(ii) $\forall i \in I, y_i$ is a Carathéodory solution of (2d), (2e), *i.e.* $y_i(t) = y_i^0 + \int_0^t \omega_i\left(t, \rho(s, y_i(s)+)\right) \mathrm{d}s$;

(iii) the constraint (2c) is satisfied for almost every $t \ge 0$.

The main result of this paper is the following.

Theorem 1. (Existence of solution). Assume **(A0)**, **(A1)** and **(A2)** hold, with only one jump discontinuity in the initial datum ρ^0. Then the Cauchy problem (2) admits a solution in the sense of Definition 1.1.

The rest of the paper is organized as follows: Theorem 1 is proved in Section 2. We then provide some numerical examples in Section 3. Finally, Section 4 gives a wrap-up discussion and details for some future research directions.

2. ANALYSIS OF THE PROPOSED MODEL

Below, we present the main technical elements for proving the existence of a solution to the PDE-ODE model (2).

2.1 The Riemann problem

In this paper, we focus on the Riemann problem for (2), i.e. the problem with initial density given by a piecewise constant initial datum with only one discontinuity:

$$\rho^0(x) = \begin{cases} \rho_L & \text{if } \quad x < y_0, \\ \rho_R & \text{if } \quad x > y_0. \end{cases} \tag{3}$$

Note that for sake of simplicity, since $I = 1$, we drop the lower-script notation and set $y_0 := y_1^0$ and $y := y_1$.

Solutions with unbounded acceleration only appear in the case $\rho_L > \rho_R$, when a platoon of vehicles has to accelerate from an initial velocity $v(\rho_L)$ to a higher one $v(\rho_R)$. Indeed, from **(A1)**, if $\rho_L > \rho_R$, then $v(\rho_L) < v(\rho_R)$. The case $\rho_L \le \rho_R$ gives the classical solution to the LWR model i.e. a shock wave.

Definition 2.1. (Classical solutions to (1), (3)). Consider the simple Riemann problem

$$
\begin{aligned}
&\partial_t \rho + \partial_x q(\rho) = 0, \qquad x \in \mathbb{R},\ t > 0,\\
&\rho(0, x) = \begin{cases} \rho_L & \text{if } \quad x < y_0, \\ \rho_R & \text{if } \quad x > y_0. \end{cases}
\end{aligned}
$$

The unique weak entropic solution (see Garavello and Piccoli (2006)) to the above Cauchy problem is given by

$$
\begin{aligned}
\rho(t, x) &= \mathcal{R}(\rho_L, \rho_R)(\xi) \text{ with } \xi := \frac{x - y_0}{t}\\
&= \begin{cases}
\rho_L & \text{if } \rho_L < \rho_R,\ \xi < \dfrac{q(\rho_L) - q(\rho_R)}{\rho_L - \rho_R} \\
& \text{or } \rho_L \ge \rho_R,\ \xi < q'(\rho_L), \\
\rho_R & \text{if } \rho_L < \rho_R,\ \xi > \dfrac{q(\rho_L) - q(\rho_R)}{\rho_L - \rho_R} \\
& \text{or } \rho_L \ge \rho_R,\ \xi > q'(\rho_R), \\
(q')^{-1}(\xi) & \text{if } \rho_L \ge \rho_R,\ q'(\rho_L) < \xi < q'(\rho_R).
\end{cases}
\end{aligned}
$$

We are now ready to give a constructive definition of solutions to (2) in the case of a Riemann initial datum (3):

Definition 2.2. (Constrained solutions to (2), (3)). For a fixed speed of the moving bottleneck $\dot{y}$, we denote by $\hat{\rho}$ the unique solution to the equation $v(\rho) = \dot{y}$. We define a solution $\rho(t, x) = \mathcal{R}_{\dot{y}}(\rho_L, \rho_R)(\xi)$ to (2), (3) as follows:

- If $q(\mathcal{R}(\rho_L, \rho_R)(\dot{y})) - \dot{y}\,\mathcal{R}(\rho_L, \rho_R)(\dot{y}) > 0$, then the bottleneck is active and
$$\mathcal{R}_{\dot{y}}(\rho_L, \rho_R)(\xi) = \begin{cases} \mathcal{R}(\rho_L, \hat{\rho})(\xi), & \text{if } \xi < \dot{y}, \\ \mathcal{R}(0, \rho_R)(\xi), & \text{if } \xi \ge \dot{y}. \end{cases}$$
- If $q(\mathcal{R}(\rho_L, \rho_R)(\dot{y})) - \dot{y}\,\mathcal{R}(\rho_L, \rho_R)(\dot{y}) \le 0$, then the bottleneck is inactive and
$$\mathcal{R}_{\dot{y}}(\rho_L, \rho_R)(\xi) = \mathcal{R}(\rho_L, \rho_R)(\xi) \quad \text{for all } \xi.$$

2.2 Construction of approximate solutions by Wave Front Tracking (WFT) algorithm

We construct a sequence of approximate solutions $(\rho_N)_{N\in\mathbb{N}}$ with the Wave Front Tracking algorithm (see for instance Holden and Risebro (2015)) and we show in the next subsection that they converge to a solution of (2) in the sense of Definition 1.1.

Definition of the mesh: Consider a given $N \in \mathbb{N}$. We set a mesh of densities as follows

$$\mathcal{M}_N := 2^{-N}\rho_{\max}\mathbb{N} \cap [0, \rho_{\max}]$$

such that for any distinct $(\rho_1, \rho_2) \in \mathcal{M}_N$, we have $\|\rho_1 - \rho_2\| \geq 2^{-N}\rho_{\max}$. The mesh has exactly $2^N + 1$ elements. We denote by $v_{j,N}$ the speed associated with any density $\rho_{j,N} \in \mathcal{M}_N$, say $v_{j,N} := v(\rho_{j,N}) = \dfrac{q(\rho_{j,N})}{\rho_{j,N}}$. We consider an approximation of the initial datum taking values on the grid:

$$\rho_N^0(x) = \begin{cases} \rho_{L,N} & \text{if} \quad x < y_0, \\ \rho_{R,N} & \text{if} \quad x \geq y_0, \end{cases}$$

such that $\rho_{L,N}$, $\rho_{R,N} \in \mathcal{M}_N$. We then define the two indices i_0 and $i_0 - K$ such that $\rho_{i_0} = \rho_{L,N}$ and $\rho_{i_0-K} = \rho_{R,N}$ with $K = \frac{\rho_{L,N}-\rho_{R,N}}{\rho_{\max}}2^N$.

In the remaining, when the index N is not necessary for the computations, we will elude it for sake of clarity.

Approximated trajectory of the moving bottleneck: According to (2d), while the accelerating vehicle (say the moving bottleneck) has not reached its maximal speed $V_{\max}$ or caught up with the last of the downstream vehicles, its trajectory is given by the parabola

$$y(t) = \frac{A}{2}t^2 + v_{i_0}t + y_i^0, \quad t > 0.$$

We will approximate this parabola by a piecewise affine trajectory, such that the initial slope is equal to $v(\rho_{L,N}) = v_{i_0,N}$, and then increases along the grid, taking values $v_{i_0-1,N}$, $v_{i_0-2,N}$ and so on. The parabola exists on a time horizon included in $[t_0 = 0, t_{i_0}]$, since the velocity cannot exceed $v_0 = V_{\max}$. We partition the time-horizon with intervals $[t_n, t_{n+1}[$ where for any $n < i_0$, t_n is defined for each grid-parameter N such that

$$\begin{cases} t_0 = 0, \quad t_{n+1} = t_n + \Delta t_n, \\ \Delta t_n := \dfrac{v_{i_0-(n+1)} - v_{i_0-n}}{A}, \end{cases} \quad \forall n \in \mathbb{N}. \tag{4}$$

The interval Δt_n corresponds to the time necessary to accelerate between two consecutive velocities v_{i_0-n} and $v_{i_0-(n+1)}$ at a constant acceleration rate A.

The approximate parabola is then defined for each grid-parameter N and $t \in [t_n, t_{n+1}]$ as

$$\tilde{y}_N(t) := y_0 + \sum_{i=0}^{n-1} v_{i_0-i}\Delta t_i + v_{i_0-n}(t - t_n).$$

After $t = t_{i_0}$, the leader vehicle trajectory is a straight line, with slope v_0. For $t > t_{i_0}$, the curve is defined by $y(t) = y(t_{i_0}) + v_0(t - t_{i_0})$. If necessary, we define t_{i_0+1} the time at which the curve crosses the shock wave emitted by the initial datum.

Algorithm: Let $T > 0$ given. Following (4), we can partition the time interval $[0, T]$ in intervals $[t_n, t_{n+1}[$. We denote by ρ_N the approximate solution constructed via WFT method (see Fig. 1):

(i) Using the constrained Riemann solution $\mathcal{R}_{v_{i_0}}$ given by Definition 2.2, solve the constrained Riemann problem at $x = y_0$. The solution consists in a jump discontinuity between $\rho_{L,N}$ and 0, moving at speed v_{i_0}, eventually followed by a shock between 0 and $\rho_{R,N}$, moving at speed $v_{i_0-K} = v(\rho_{R,N})$, and it is defined for $t < t_1$.

(ii) At $t = t_1$, the speed of the moving bottleneck is set equal to v_{i_0-1}. We then solve the constrained Riemann problem with $\mathcal{R}_{v_{i_0-1}}$: the solution consists in an approximate rarefaction jump of size $2^{-N}\rho_{\max}$ between $\rho_{i_0} = \rho_{L,N}$ and ρ_{i_0-1}, moving at speed

$$\lambda_{i_0-1/2} = \frac{q(\rho_{i_0}) - q(\rho_{i_0-1})}{\rho_{i_0} - \rho_{i_0-1}},$$

followed by a jump discontinuity between ρ_{i_0-1} and 0, moving at speed v_{i_0-1}. The solution is defined for $t \in [t_1, \bar{t}_2[$, where $\bar{t}_2 = \min\{t_2, t_{\text{int}}\}$, t_{int} being the interaction time between the jump discontinuity $\tilde{y}_N$ and the shock between 0 and $\rho_{R,N}$ originated at $t = 0$.

(iii) We repeat step (2) until $t = t_{\text{int}} \in [t_1, +\infty]$.

(iv) If $t_{int} < +\infty$, the solution to the Riemann problem at $t = t_{int}$ and $x = y_0 + v_{i_0-K}\, t_{int}$ consists in a classical shock between the left state before the interaction $\bar{\rho_L}$ and $\rho_{R,N}$. Since there is an interaction, the bottleneck propagates faster than the initial shock, and thus $\bar{\rho_L} < \rho_{R,N}$. The solution is determined thanks to Definition 2.1.

(v) For $t > t_{int}$ only interactions between classical fronts can occur since we restrict ourselves to a Riemann problem, giving rise to classical solutions.

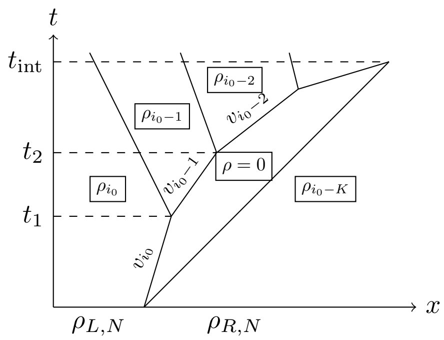

Fig. 1. Representation of the WFT approximate solutions and notation used in the algorithm.

2.3 Proof of Theorem 1

We first show the convergence of the piecewise linear trajectory $\tilde{y}_N$ towards the parabola described by the accelerating vehicle.

Lemma 2. For all $T > 0$, $\|\tilde{y}_N - y\|_{L^\infty([0,T])} \underset{N\to+\infty}{\longrightarrow} 0$.

Proof. Since the sequence $\{\tilde{y}_N\}_{N\in\mathbb{N}}$ is uniformly bounded on any interval $[0, T]$ and equicontinous, Ascoli-Arzela theorem guarantees that it is uniformly convergent.
To show that the limit is y, let now $n \in \mathbb{N}$. For any $t \in [t_n, t_{n+1}[$, we have

$$
\begin{aligned}
&\dot{y}(t) - \dot{\tilde{y}}_N(t)\\
&\quad = v_{i_0} - v_{i_0-n} + At_n + A(t-t_n)\\
&\quad = v_{i_0} - v_{i_0-n} + A\sum_{i=0}^{n-1}\frac{v_{i_0-i-1}-v_{i_0-i}}{A} + A(t-t_n)\\
&\quad = A(t-t_n) \geq 0.
\end{aligned}
$$

Let us fix $T \in [t_0, t_{i_0}]$ and $n \in \mathbb{N}$ such that $T \in [t_n, t_{n+1}[$. Since y_N is differentiable almost everywhere, we can write:

$$
\begin{aligned}
y(T) - \tilde{y}_N(T) &= \int_0^T (\dot{y}(t) - \dot{\tilde{y}}_N(t))dt\\
&= \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} (\dot{y}(t) - \dot{\tilde{y}}_N(t))dt\\
&\quad + \int_{t_n}^{T} (\dot{y}(t) - \dot{\tilde{y}}_N(t))dt\\
&= \frac{A}{2}\sum_{i=0}^{n-1}\Delta t_i^{\,2} + \frac{A}{2}(T-t_n)^2 \geq 0
\end{aligned}
$$

We also compute

$$
\begin{aligned}
\sum_{i=0}^{n-1}\Delta t_i^{\,2} &= \frac{1}{A^2}\sum_{i=0}^{n-1}(v_{i_0-i-1} - v_{i_0-i})^2\\
&\leq \frac{\mathrm{Lip}(v)^2}{A^2}\sum_{i=0}^{n-1}|\rho_{i_0-i-1} - \rho_{i_0-i}|^2\\
&= \frac{L^2}{A^2}\sum_{i=0}^{n-1}2^{-2N}\\
&\leq \frac{L^2}{A^2}(2^N+1)2^{-2N} \underset{N\to+\infty}{\longrightarrow} 0
\end{aligned}
$$

Thus $\{\tilde{y}_N(t)\}_{N\in\mathbb{N}}$ converges to $y(t)$ pointwise almost everywhere.

We define for each solution ρ_N constructed via Wave Front Tracking the following Glimm type functional:

$$
\Upsilon(t) = \Upsilon(\rho_N(t,\cdot)) = \begin{cases} \mathrm{TV}(\rho_N^0(\cdot)) + 2\rho_{R,N} & \text{if } t = 0\\ \mathrm{TV}(\rho_N(t,\cdot)) & \text{if } t > 0\end{cases}
$$

It easy to check that the approximate solutions constructed with the WFT algorithm satisfy the following lemma.

Lemma 3. The map $t \to \Upsilon(t)$ is non-increasing.

A standard application of Helly's theorem provides the convergence of the sequence of approximate solutions $\{\rho_N\}_N$ to some function $\rho \in \mathcal{C}^0(\mathbb{R}^+; \mathbb{L}^1 \cap \mathbb{BV}(\mathbb{R}, [0, \rho_{max}]))$. In addition, the approximate parabola $\dot{\tilde{y}}_N$ converges to $\dot{y}$. Following Delle Monache and Goatin (2014), one can prove that the limit functions (ρ, y) provide a solution to (2) in the sense of Definition 1.1. This concludes the proof of Theorem 1.

3. NUMERICAL EXAMPLES

In this section, we provide two numerical examples to illustrate the construction of the approximate solutions we described in the previous section. We solve Cauchy problems both for the LWR model (1) and for the LWR model with bounded acceleration (2). To do that, our algorithm is based on the Wave Front Tracking algorithm (Holden and Risebro, 2015), using the Riemann solvers defined in Definition 2.1 for the LWR model and in Definition 2.2 for the LWR model with bounded acceleration.

The Wave Front Tracking algorithm is known to be very accurate but computationally cumbersome to implement. It is noteworthy that other schemes could be used to compute a numerical solution to the problem (2). For instance, an adapted finite volume scheme has been proposed in Chalons et al. (2018) for a slightly different model of a classical moving bottleneck which is not accelerating. A numerical method based on a semi-analytical Lax-Hopf formula for Hamilton-Jacobi equations has been also recently presented in Simoni and Claudel (2017) for "classical" fixed and moving bottlenecks.

In the two following numerical illustrations, we consider the speed-density function

$$
v(\rho) = V_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right), \qquad \rho \in [0, \rho_{\max}].
$$

This leads to a quadratic flow-density fundamental diagram of Greenshield's type, with $\rho_{\text{crit}} = \frac{\rho_{\max}}{2}$. Even if this choice is arguable for traffic flow applications, different fundamental diagrams can be implemented exactly in the same way.

The maximal density corresponds to a bumper-to-bumper situation, so we select $\rho_{\max} = 200$ veh/km. The acceleration rate is fixed to $A = 2$ m/sec^2, which is a standard value. Finally, the parameter for our density grid is set to $N = 10$. These values are common to both examples. Density values in the interval $[0, 2^{-N}\rho_{\max}[$, corresponding to the void, are depicted in white color in the figures.

3.1 Case of a Riemann problem

In this first example, we consider a Riemann problem (3) initially located at $y_0 = 400$ m, with $\rho_L = 180$ veh/km and $\rho_R = 80$ veh/km. We set $V_{\max} = 30$ m/sec.

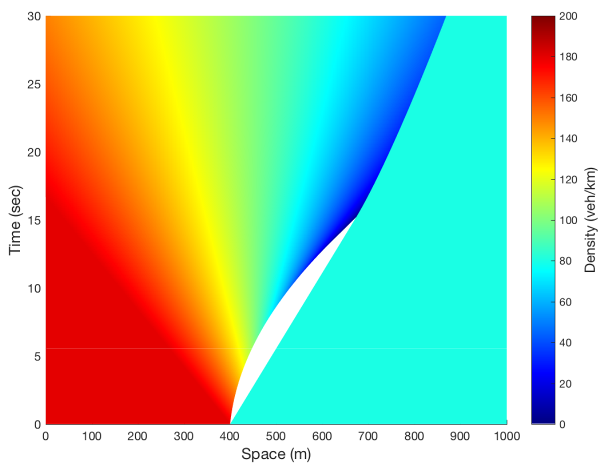

Fig. 2. Solution to our PDE-ODE model (2) for a Riemann initial datum (3) with $\rho_L = 180$ veh/km and $\rho_R = 80$ veh/km.

The classical solution to the LWR model with such an initial condition consists in a rarefaction wave. The solution to the LWR model with bounded acceleration is displayed on Fig. 2, while the difference of densities between the classical LWR model and the bounded acceleration model is shown on Fig. 3. One can easily observe the vacuum created in front of the first accelerating vehicle which is specific to the PDE-ODE model. This vacuum lasts for approximately 15 seconds and covers around 300 meters, which is not negligible. Moreover, it is noteworthy that the differences in terms of density spread over a wider area. Indeed, the bounded acceleration of the first vehicle affects all the following vehicles. From this simple example, one can see that the bounded acceleration has not just a local effect. From Fig. 3, we observe that the traffic is first condensed (green and blue zones) and then it is relaxed (yellow to pink zones) in comparison to the classical solution with gaps up to around 80 veh/km in some regions.

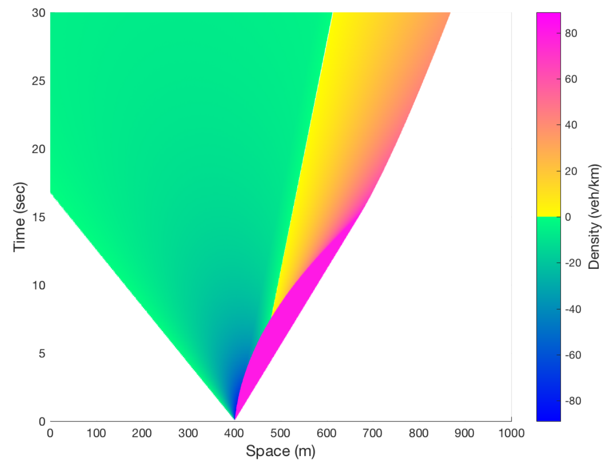

Fig. 3. Density difference between the solution to the LWR model and the solution to the PDE-ODE model.

We can also compare the solutions of both models by plotting the queues generated along the time. We define a queue length as the distance between extremal points for which the density is higher or equal to $\frac{3}{4}\rho_{\max}$. With our choice of a linear speed-density function, we know that this density is above the critical density $\rho_{\text{crit}} = \frac{\rho_{\max}}{2}$. The result is given on Fig.4. While the queue length is also non-increasing during the considered time period for the LWR model, there is an increase for the PDE-ODE model that corresponds to the very beginning of the bounded acceleration. Interestingly, this gap of more or less 45 meters is conserved for the whole simulated period.

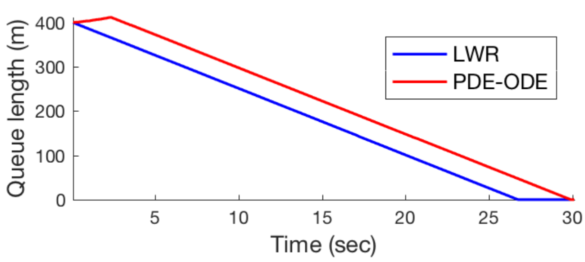

Fig. 4. Queue lengths obtained for the LWR model and the PDE-ODE model.

3.2 Example of successive traffic lights

In this numerical study, our aim is twofold. First, we want to show that our algorithm can be easily extended to more general initial conditions than a Riemann problem. Second, we highlight the impact of bounded acceleration for a realistic situation of successive traffic lights. We select $V_{\max} = 15$ m/sec to match an arterial road behaviour and we consider three traffic signals located respectively at positions $x = 300$, 700 and 1000 meters. We assume that initially, there are queues standing at each traffic signal with local Riemann problems such that $\rho_L = \rho_{\max}$ and $\rho_R = 0$. The initial datum contains additional Riemann problems upstream of each traffic light, accounting for the cars that did not reach the initial queues at time $t = 0$. We assume that the traffic lights turn simultaneously to green at time $t = 0$.

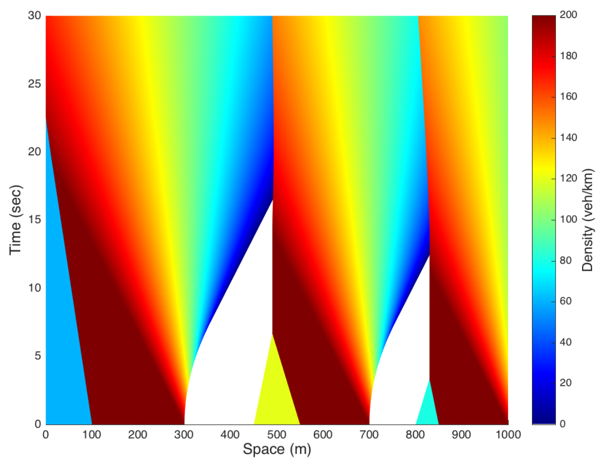

Fig. 5. Solution to our PDE-ODE model for the series of traffic lights.

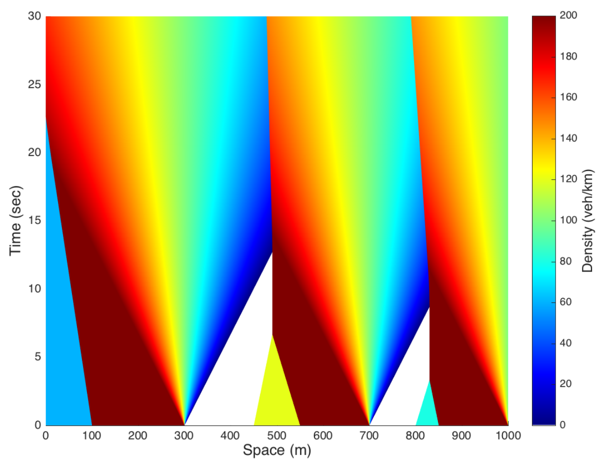

Fig. 6. Solution to the LWR model for the series of traffic lights.

The solution to the LWR model with bounded acceleration is displayed on Fig. 5 while the solution to the classical LWR model appears on Fig. 6. The reader can note the formation of rarefactions (dissipation of queues) and shockwaves as well as the interactions between all these waves when vehicles catch downstream traffic. The difference of densities due to the bounded acceleration shown on Fig. 7 reveals high discrepancies along interacting shockwaves and rarefaction fans (in pink). We recover the same phenomenon than in the previous example with condensed (in green) and relaxed (in yellow) traffic due to the bounded acceleration.

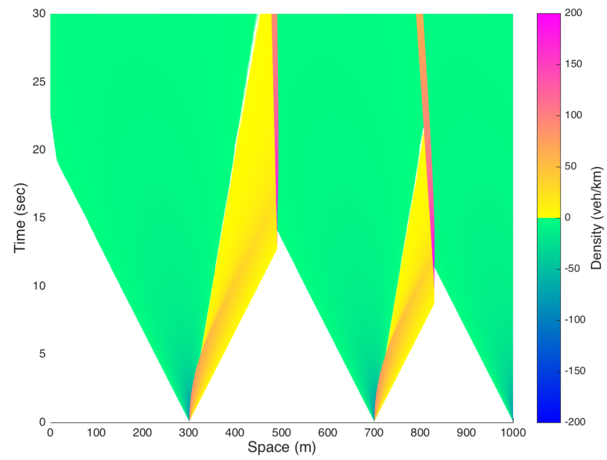

Fig. 7. Density difference between the solution to the LWR model and the solution to the PDE-ODE model for the series of traffic lights.

The study of the cumulative queues carried over on Fig. 8 with the same threshold $\rho \geq \frac{3}{4}\rho_{\max}$ exhibits a significant difference with and without bounded acceleration. For instance, the model with bounded acceleration captures up to 50 additional meters of queues over a 1000 meters long road, which is significant if one thinks to responsive traffic signals based on an estimation of queue lengths.

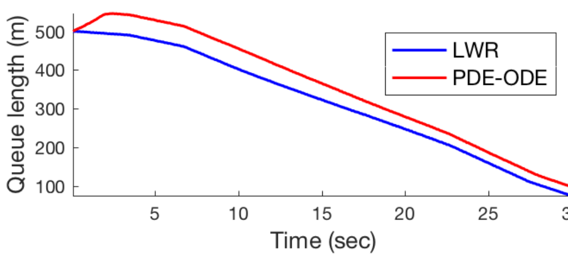

Fig. 8. Queue lengths obtained for the LWR model and the PDE-ODE model.

4. DISCUSSION AND FUTURE WORK

This paper proposes a new traffic flow model based on a coupled PDE-ODE approach. This model is designed in order to realistically reproduce vehicles acceleration at the macroscopic scale. We compare it to the seminal LWR model on some numerical cases and we observe a significant increase of queue lengths which should be closer to reality due to the fact that cars have finite acceleration. This intuition should nonetheless be verified by confronting the PDE-ODE model to traffic data in future research.

Further work will aim at extending the analytical proof to more general initial data. This model could also be used to analyze the impact of bounded acceleration on congestion in the case of different, potentially desynchronized, traffic light cycles.

We might also investigate the opportunity of using finite volume schemes, based on the approach of Chalons et al. (2018), to model several moving bottlenecks and compare the numerical results to the previous ones.

Finally, we could extend this framework to take into account the deceleration of traffic as well.

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