

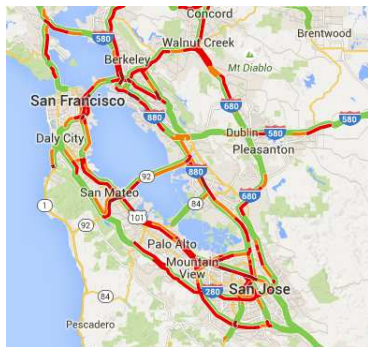
Second Order Traffic Flow Models on Networks

Guillaume Costeseque
in collaboration with J-P. Lebacque (IFSTTAR)

Inria Sophia-Antipolis Méditerranée

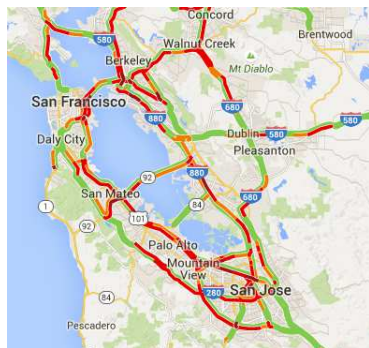
LJAD seminar, UNS
January 12, 2017

Traffic flows on a network

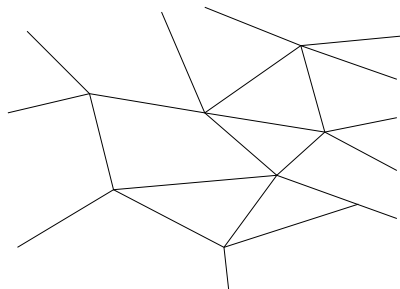


[Caltrans, Oct. 7, 2015]

Traffic flows on a network



[Caltrans, Oct. 7, 2015]



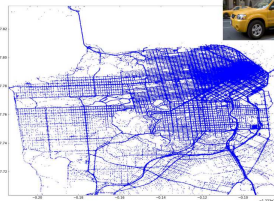
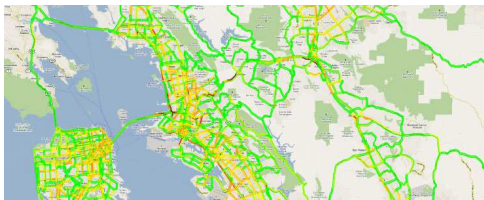
Road network \equiv graph made of edges and vertices

Breakthrough in traffic monitoring

Traffic monitoring

- “old”: **loop detectors** at **fixed** locations (Eulerian)
- “new”: **GPS** devices **moving** within the traffic (Lagrangian)

Data assimilation of Floating Car Data



[Mobile Millenium, 2008]

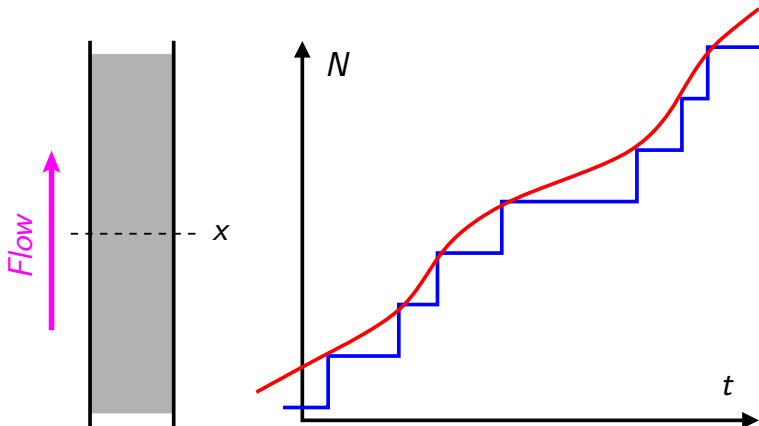
Outline

- 1 Introduction to traffic
- 2 Variational principle applied to GSOM models
- 3 GSOM models on a junction

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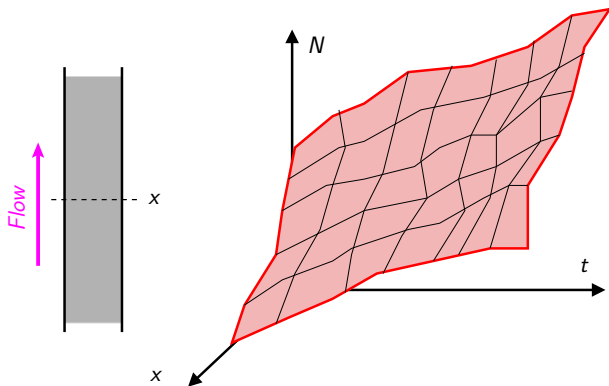
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Convention for vehicle labeling



Three representations of traffic flow

Moskowitz' surface

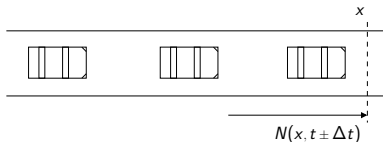


See also [Makigami et al, 1971], [Laval and Leclercq, 2013]

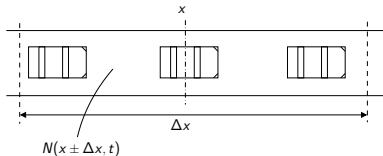
Notations: macroscopic

- $N(t, x)$ vehicle **label** at (t, x)

- the **flow** $Q(t, x) = \lim_{\Delta t \rightarrow 0} \frac{N(t + \Delta t, x) - N(t, x)}{\Delta t} = \partial_t N(t, x)$



- the **density** $\rho(t, x) = \lim_{\Delta x \rightarrow 0} \frac{N(t, x) - N(t, x + \Delta x)}{\Delta x} = -\partial_x N(t, x)$

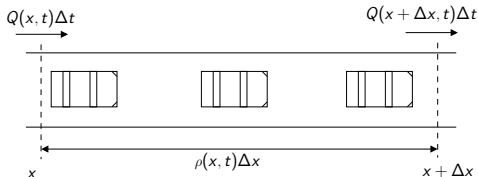


- the stream **speed** (mean spatial speed) $V(t, x)$.

Macroscopic models

- **Hydrodynamics** analogy
- Two main categories: first and second order models
- Two common equations:

$$\begin{cases} \partial_t \rho(t, x) + \partial_x Q(t, x) = 0 & \text{conservation equation} \\ Q(t, x) = \rho(t, x) V(t, x) & \text{definition of flow speed} \end{cases} \quad (1)$$



First order: the LWR model

LWR model [Lighthill and Whitham, 1955], [Richards, 1956] [6, 7]

Scalar one dimensional **conservation law**

$$\partial_t \rho(t, x) + \partial_x \mathfrak{F}(\rho(t, x)) = 0 \quad (2)$$

with

$$\mathfrak{F} : \rho(t, x) \mapsto \mathfrak{F}(\rho(t, x)) := Q(t, x)$$

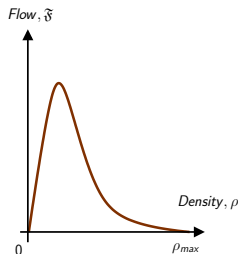
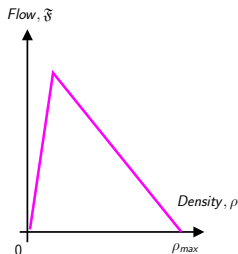
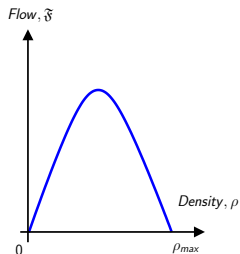
Overview: conservation laws (CL) / Hamilton-Jacobi (HJ)

		Eulerian $t - x$	Lagrangian $t - n$
CL	Variable	Density ρ	Spacing r
	Equation	$\partial_t \rho + \partial_x \mathfrak{F}(\rho) = 0$	$\partial_t r + \partial_n V(r) = 0$
HJ	Variable	Label N $N(t, x) = \int_x^{+\infty} \rho(t, \xi) d\xi$	Position \mathcal{X} $\mathcal{X}(t, n) = \int_n^{+\infty} r(t, \eta) d\eta$
	Equation	$\partial_t N + H(\partial_x N) = 0$	$\partial_t \mathcal{X} + \mathcal{V}(\partial_n \mathcal{X}) = 0$
	Hamiltonian	$H(p) = -\mathfrak{F}(-p)$	$\mathcal{V}(p) = -V(-p)$

Fundamental diagram (FD)

Flow-density fundamental diagram \mathfrak{F}

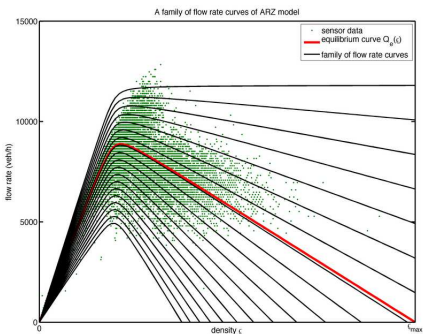
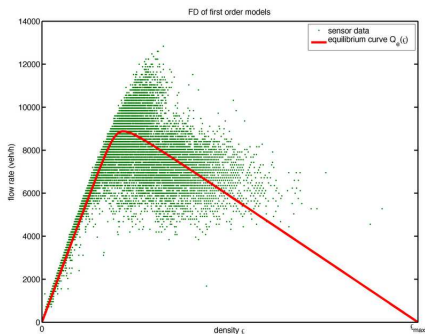
- Empirical function with
 - ρ_{max} the maximal or jam density,
 - ρ_c the critical density
- Flux is increasing for $\rho \leq \rho_c$: **free-flow** phase
- Flux is decreasing for $\rho \geq \rho_c$: **congestion** phase



[Garavello and Piccoli, 2006]

Motivation for higher order models

- Experimental evidences
 - fundamental diagram: **multi-valued** in congested case



[S. Fan, U. Illinois], NGSIM dataset

Motivation for higher order models

- **Experimental** evidences
 - fundamental diagram: **multi-valued** in congested case
 - phenomena not accounted for: bounded acceleration, capacity drop...
- Need for models able to **integrate** measurements of **different traffic quantities** (acceleration, fuel consumption, noise)

GSOM family [Lebacque, Mammar, Haj-Salem 2007] [5]

- Generic Second Order Models (GSOM) family

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 & \text{Conservation of vehicles,} \\ \partial_t(\rho l) + \partial_x(\rho v l) = \rho \varphi(l) & \text{Dynamics of the driver attribute } l, \\ v = \mathfrak{I}(\rho, l) & \text{Speed-density fundamental diagram,} \end{cases} \quad (3)$$

- Specific driver attribute l

- the driver aggressiveness,
- the driver origin/destination or path,
- the vehicle class,
- ...

- Flow-density fundamental diagram

$$\mathfrak{F} : (\rho, l) \mapsto \rho \mathfrak{I}(\rho, l).$$

GSOM family [Lebacque, Mammari, Haj-Salem 2007] [5]

- Generic Second Order Models (GSOM) family

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 & \text{Conservation of vehicles,} \\ \partial_t I + v \partial_x I = \varphi(I) & \text{Dynamics of the driver attribute } I, \\ v = \mathfrak{J}(\rho, I) & \text{Speed-density fundamental diagram,} \end{cases} \quad (3)$$

- Specific driver attribute I
 - the driver aggressiveness,
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GSOM family [Lebacque, Mammar, Haj-Salem 2007] [5]

(continued)

- **Kinematic** waves or 1-waves:
 - similar to the seminal LWR model
 - density variations at speed $\nu = \partial_{\rho} \mathfrak{J}(\rho, l)$
 - driver attribute l is **continuous**

- **Contact discontinuities** or 2-waves:
 - variations of driver attribute l at speed $\nu = \mathfrak{J}(\rho, l)$
 - the flow speed v is **constant**.

Examples of GSOM models

- LWR model = GSOM model with no specific driver attribute
- LWR model with bounded acceleration = GSOM model with $l := v$
- ARZ model = GSOM with $l := v + p(\rho)$

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho w) + \partial_x(\rho v w) = 0, \\ w = v + p(\rho) \end{cases}$$

- Generalized ARZ model [Fan, Herty, Seibold]
- Multi-commodity models (multi-class, multi-lanes) of [Jin and Zhang], [Bagnerini and Rascle] or [Herty, Kirchner, Moutari and Rascle], [Klar, Greenberg and Rascle]
- Colombo 1-phase model
- Stochastic GSOM model [Khoshyaran and Lebacque]

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LWR in Eulerian (t, x)

- Cumulative vehicles count (CVC) or Moskowitz surface $N(t, x)$

$$Q = \partial_t N \quad \text{and} \quad \rho = -\partial_x N$$

- If density ρ satisfies the scalar (LWR) **conservation law**

$$\partial_t \rho + \partial_x \mathfrak{F}(\rho) = 0$$

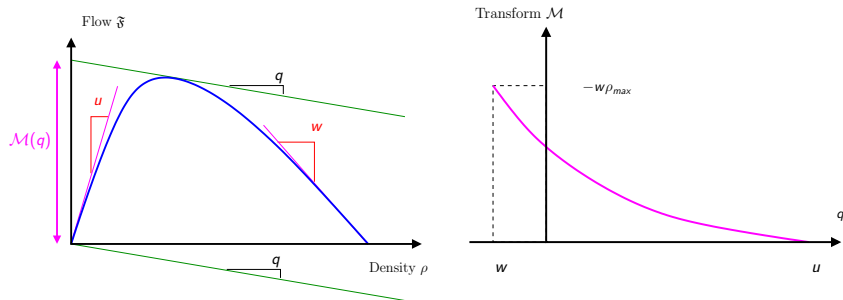
- Then N satisfies the first order **Hamilton-Jacobi equation**

$$\partial_t N - \mathfrak{F}(-\partial_x N) = 0 \tag{4}$$

LWR in Eulerian (t, x)

- Legendre-Fenchel transform with \mathfrak{F} **concave** (*relative capacity*)

$$\mathcal{M}(q) = \sup_{\rho} [\mathfrak{F}(\rho) - \rho q]$$



LWR in Eulerian (t, x)

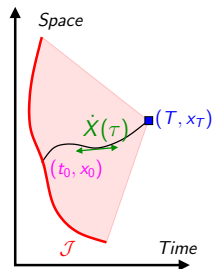
(continued)

- **Lax-Hopf formula** (representation formula) [Daganzo, 2006]

$$N(T, x_T) = \min_{u(\cdot), (t_0, x_0)} \int_{t_0}^T \mathcal{M}(u(\tau)) d\tau + N(t_0, x_0),$$

$$\left| \begin{array}{l} \dot{X} = u \\ u \in \mathcal{U} \\ X(t_0) = x_0, \quad X(T) = x_T \\ (t_0, x_0) \in \mathcal{J} \end{array} \right.$$

(5)

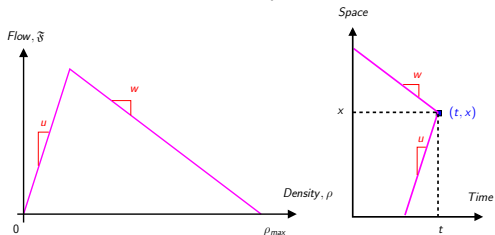


- **Viability theory** [Claudel and Bayen, 2010]

LWR in Eulerian (t, x)

(Historical note)

- **Dynamic programming** [Daganzo, 2006] for **triangular FD** (u and w free and congested speeds)



- **Minimum principle** [Newell, 1993]

$$N(t, x) = \min \left[N \left(t - \frac{x - x_u}{u}, x_u \right), \right. \\ \left. N \left(t - \frac{x - x_w}{w}, x_w \right) + \rho_{max}(x_w - x) \right], \quad (6)$$

LWR in Lagrangian (n, t)

- Consider $X(t, n)$ the location of vehicle n at time $t \geq 0$

$$v = \partial_t X \quad \text{and} \quad r = -\partial_n X$$

- If the spacing $r := 1/\rho$ satisfies the LWR model (Lagrangian coord.)

$$\partial_t r + \partial_n \mathcal{V}(r) = 0$$

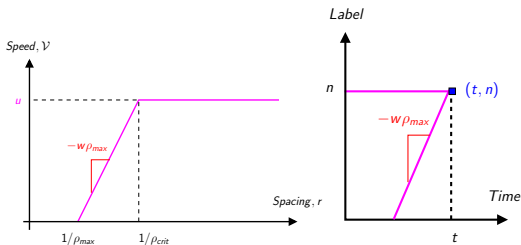
- Then X satisfies the first order Hamilton-Jacobi equation

$$\partial_t X - \mathcal{V}(-\partial_n X) = 0. \tag{7}$$

LWR in Lagrangian (n, t)

(continued)

- Dynamic programming for triangular FD



- Minimum principle \Rightarrow car following model [Newell, 2002]

$$X(t, n) = \min \left[X(t_0, n) + u(t - t_0), \right. \\ \left. X(t_0, n + w\rho_{max}(t - t_0)) + w(t - t_0) \right]. \quad (8)$$

GSOM in Lagrangian (n, t)

- From [Lebacque and Khoshyaran, 2013], GSOM in Lagrangian

$$\begin{cases} \partial_t r + \partial_N v = 0 & \text{Conservation of vehicles,} \\ \partial_t I = 0 & \text{Dynamics of } I, \\ v = \mathcal{W}(N, r, t) := \mathcal{V}(r, I(N, t)) & \text{Fundamental diagram.} \end{cases} \quad (9)$$

- Position $\mathcal{X}(N, t) := \int_{-\infty}^t v(N, \tau) d\tau$ satisfies the **HJ equation**

$$\partial_t \mathcal{X} - \mathcal{W}(N, -\partial_N \mathcal{X}, t) = 0, \quad (10)$$

- And $I(N, t)$ solves the **ODE**

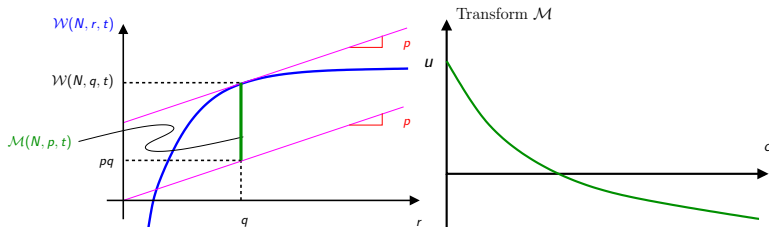
$$\begin{cases} \partial_t I(N, t) = 0, \\ I(N, 0) = i_0(N), \end{cases} \quad \text{for any } N.$$

GSOM in Lagrangian (n, t)

(continued)

- Legendre-Fenchel transform of \mathcal{W} according to r

$$\mathcal{M}(N, c, t) = \sup_{r \in \mathbb{R}} \{ \mathcal{W}(N, r, t) - cr \}$$



GSOM in Lagrangian (n, t)

(continued)

- **Lax-Hopf** formula

$$\mathcal{X}(N_T, T) = \min_{u(\cdot), (N_0, t_0)} \int_{t_0}^T \mathcal{M}(N, u, t) dt + \mathbf{c}(N_0, t_0),$$

$$\left| \begin{array}{l} \dot{N} = u \\ u \in \mathcal{U} \\ N(t_0) = N_0, \quad N(T) = N_T \\ (N_0, t_0) \in \mathcal{K} \end{array} \right. \quad (11)$$

GSOM in Lagrangian (n, t)

(continued)

- Optimal trajectories = **characteristics**

$$\begin{cases} \dot{N} = \partial_r \mathcal{W}(N, r, t), \\ \dot{r} = -\partial_N \mathcal{W}(N, r, t), \end{cases} \quad (12)$$

- System of ODEs to solve
- Difficulty: **not straight lines** in the general case

General ideas

First key element: **Lax-Hopf formula**

- Computations only for the **characteristics**

$$\mathcal{X}(N_T, T) = \min_{(N_0, r_0, t_0)} \int_{t_0}^T \mathcal{M}(N, \partial_r \mathcal{W}(N, r, t), t) dt + \mathbf{c}(N_0, r_0, t_0),$$

$$\left| \begin{array}{l} \dot{N}(t) = \partial_r \mathcal{W}(N, r, t) \\ \dot{r}(t) = -\partial_N \mathcal{W}(N, r, t) \\ N(t_0) = N_0, \quad r(t_0) = r_0, \quad N(T) = N_T \\ (N_0, r_0, t_0) \in \mathcal{K} \end{array} \right.$$
(13)

- $\mathcal{K} := \text{Dom}(\mathbf{c})$ is the set of initial/boundary values

General ideas

(continued)

Second key element: **inf-morphism** prop. [Aubin et al, 2011]

- Consider a union of sets (initial and boundary conditions)

$$\mathcal{K} = \bigcup_I \mathcal{K}_I,$$

- then the global minimum is

$$\mathcal{X}(N_T, T) = \min_I \mathcal{X}_I(N_T, T), \quad (14)$$

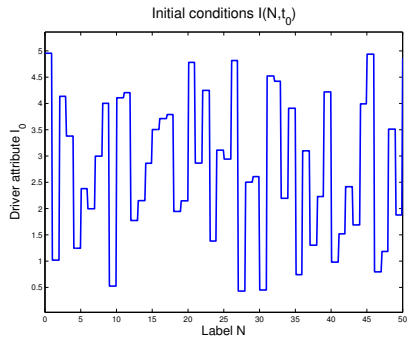
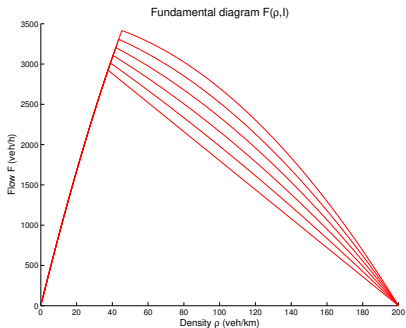
- with \mathcal{X}_I **partial solution** to sub-problem \mathcal{K}_I .

IBVP

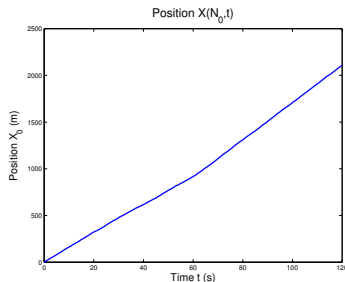
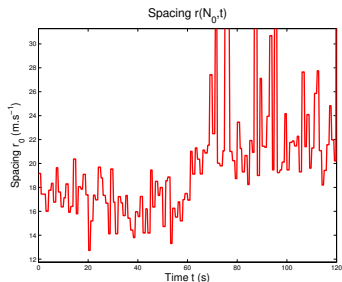
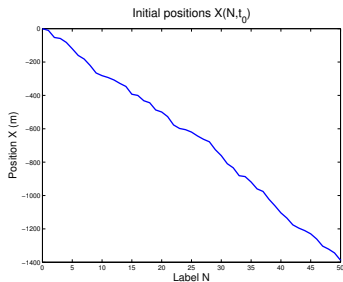
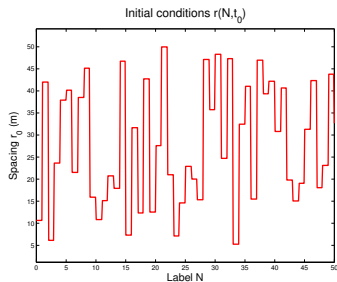
Consider **piecewise affine** initial and boundary conditions:

- **initial condition** at time $t = t_0 =$ initial position of vehicles $\xi(\cdot, t_0)$
- “**upstream**” **boundary condition** = trajectory $\xi(N_0, \cdot)$ of the first vehicle,
- and **internal boundary conditions** given for instance by cumulative vehicle counts at fixed location $\mathcal{X} = x_0$.

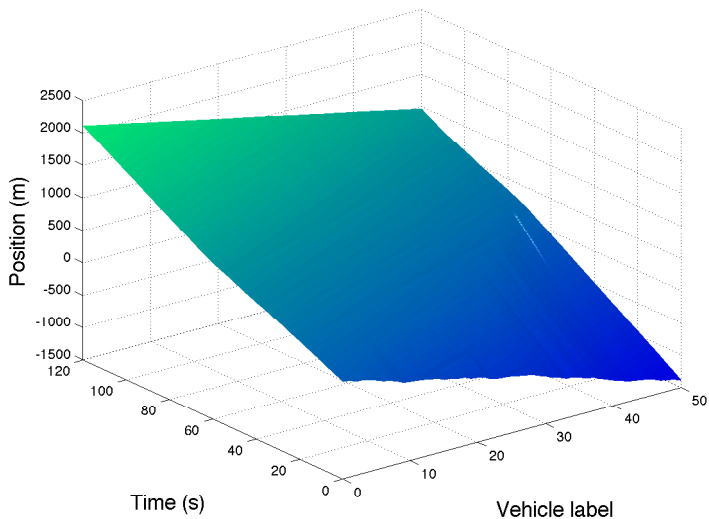
Fundamental Diagram and Driver Attribute



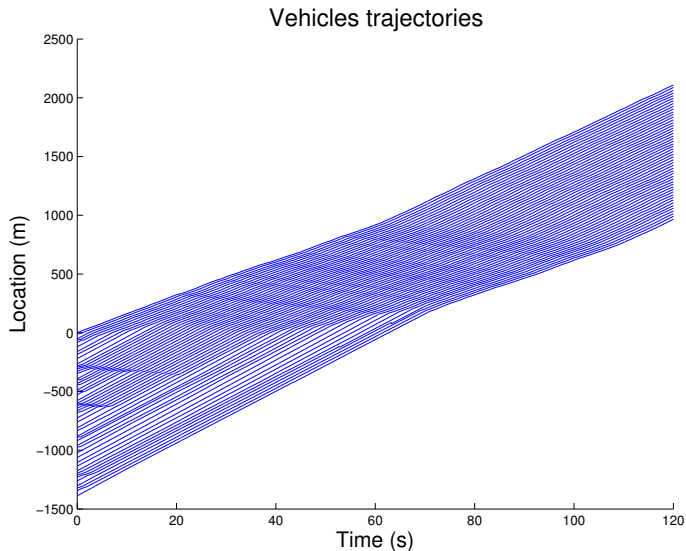
Initial and Boundaries Conditions



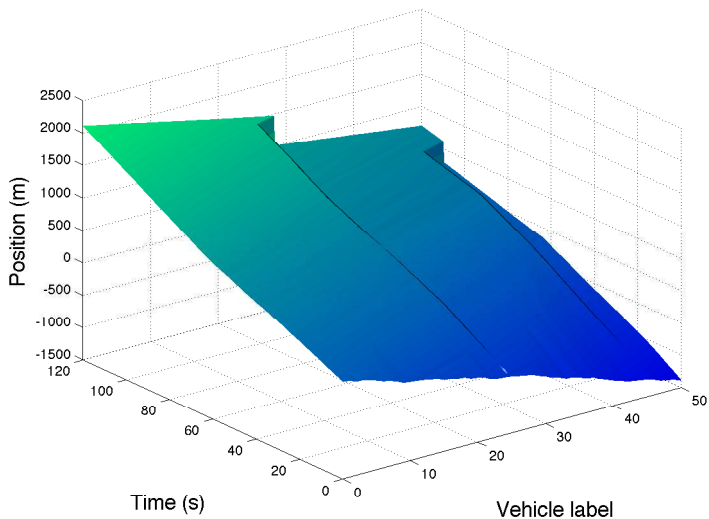
Numerical result (Initial cond. + first traj.)



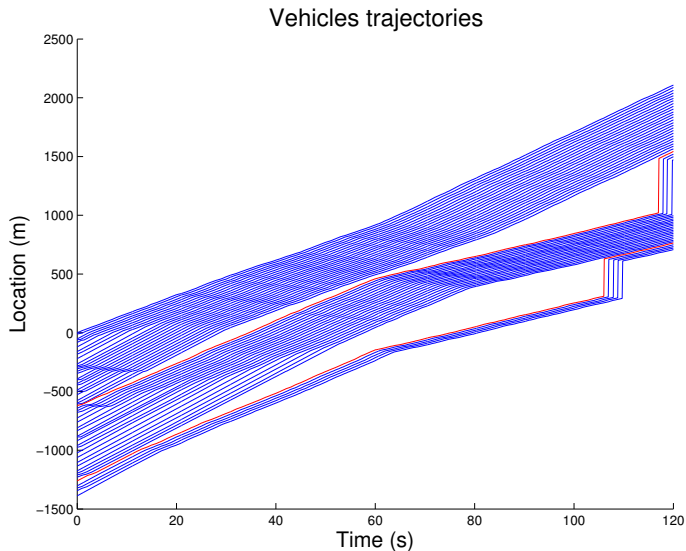
Numerical result (Initial cond. + first traj.)



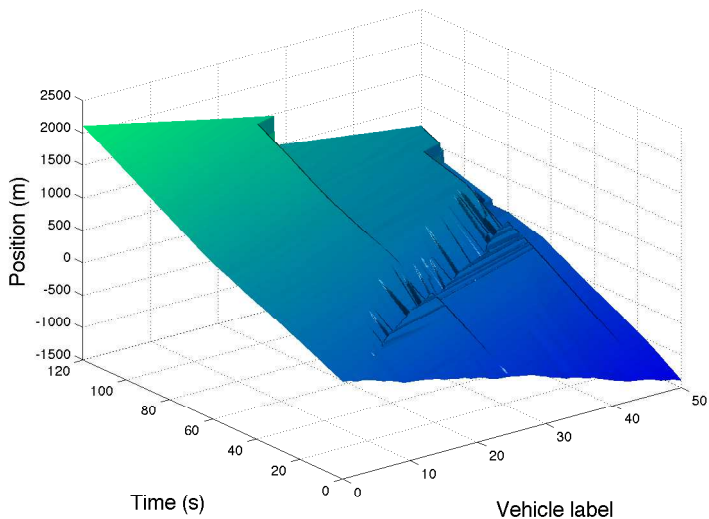
Numerical result (Initial cond.+ 3 traj.)



Numerical result (Initial cond. + 3 traj.)



Numerical result (Initial cond. + 3 traj. + Eulerian data)



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The whole picture

We need

- (i) a link model
- (ii) a junction model
- (iii) the upstream (resp. downstream) boundary conditions for an incoming (resp. outgoing) link
- (iv) link-node and node-link interfaces

GSOM lagrangian

General expressions of GSOM family

In **Eulerian**,

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 & \text{Conservation of vehicles,} \\ \partial_t(\rho l) + \partial_x(\rho v l) = \rho \varphi(l) & \text{Dynamics of the driver attribute } l, \quad (15) \\ v = \mathfrak{I}(\rho, l) & \text{Fundamental diagram,} \end{cases}$$

Transformed in **Lagrangian**,

$$\begin{cases} \partial_T r + \partial_n v = 0 & \text{Conservation of vehicles,} \\ \partial_T l = \varphi(l) & \text{Dynamics of the driver attribute } l, \quad (16) \\ v = \mathcal{V}(r, l) & \text{Fundamental diagram.} \end{cases}$$

GSOM lagrangian

Following classical approach [3, 4] we set

- Δt , ΔN time and particle steps;
- $r_n^t := r(t\Delta t, n\Delta N)$, for any $t \in \mathbb{N}$ and any $n \in \mathbb{Z}$
- and $I_n^t := I(t\Delta t, n\Delta N)$.

Numerical scheme

$$\begin{cases} r_n^{t+1} := r_n^t + \frac{\Delta t}{\Delta N} [V_{n-1}^t - V_n^t], \\ V_n^t := \mathcal{V}(r_n^t, I_n^t), \\ I_n^{t+1} = I_n^t + \Delta t \varphi(I_n^t) \end{cases} \quad (17)$$

CFL condition:

$$\frac{\Delta N}{\Delta t} \geq \sup_{N,r,t} |\partial_r \mathcal{V}(r, I(t, N))|. \quad (18)$$

GSOM lagrangian (HJ)

Introduce $\mathcal{X}(T, N)$ the position of particle N at time T and satisfying

$$r = -\partial_N \mathcal{X} \quad \text{and} \quad v = \partial_T \mathcal{X}$$

such that

$$\begin{cases} \partial_T \mathcal{X} = \mathcal{V}(-\partial_N \mathcal{X}, I), \\ \partial_T I = \varphi(I). \end{cases} \quad (19)$$

Numerical scheme for HJ equation

$$\begin{cases} \mathcal{X}_n^{t+1} = \mathcal{X}_n^t + \Delta t V_n^t, \\ V_n^t := \mathcal{V}\left(\frac{\mathcal{X}_{n-1}^t - \mathcal{X}_n^t}{\Delta N}, I_n^t\right), \\ I_n^{t+1} = I_n^t + \Delta t \varphi(I_n^t) \end{cases} \quad (20)$$

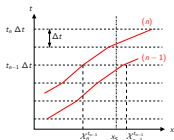
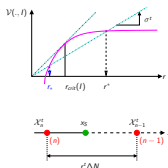
Boundary conditions

We have two different solutions:

- “Classical” supply-demand methodology [3, 2, 1]
but it implies to work with flows;
- Using tools developed in [Lebacque, Khoshyaran, (2013)] [4] that allow to compute directly spacing.

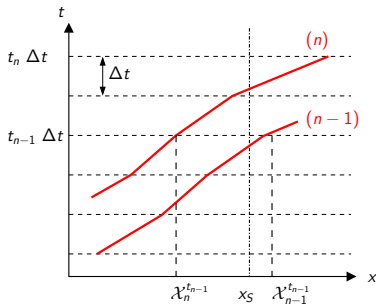
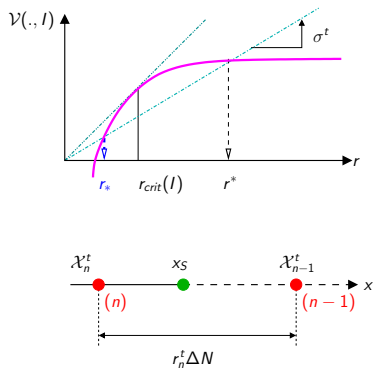
Downstream boundary conditions

(Continued)



- Exit point S located at x_S
- Boundary data = downstream supply $\sigma^t = \sigma(t\Delta t)$.
- (n) the last particle located on the link (or at least a fraction $\eta\Delta N$ of it is still on the link, with $0 \leq \eta < 1$).

Downstream boundary conditions



Downstream boundary conditions

(Continued)

Computational steps:

- 1 Define the spacing associated to particle (n) as $r_n^t := \frac{\mathcal{X}_{n-1}^t - \mathcal{X}_n^t}{\Delta N}$

Downstream boundary conditions

(Continued)

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- 1 Define the spacing associated to particle (n) as $r_n^t := \frac{\mathcal{X}_{n-1}^t - \mathcal{X}_n^t}{\Delta N}$
- 2 Define the proportion of (n) already out $\eta := \frac{x_S - \mathcal{X}_n^t}{r_n^t \Delta N}$

Downstream boundary conditions

(Continued)

Computational steps:

- 1 Define the spacing associated to particle (n) as $r_n^t := \frac{\mathcal{X}_{n-1}^t - \mathcal{X}_n^t}{\Delta N}$
- 2 Define the proportion of (n) already out $\eta := \frac{x_S - \mathcal{X}_n^t}{r_n^t \Delta N}$
- 3 Distinguish two cases:
 - either $\mathcal{V}(r_n^t, l_n^t) \leq \sigma^t r_n^t$: spacing is conserved.
 - or $\mathcal{V}(r_n^t, l_n^t) > \sigma^t r_n^t$: then, we solve $\mathcal{V}(r_n^t, l_n^t) = \sigma^t r_n^t$ and choice of the **smallest** value $r_n^t \leftarrow r_*$ (congested)

Downstream boundary conditions

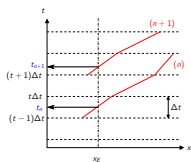
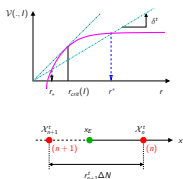
(Continued)

Computational steps:

- 1 Define the spacing associated to particle (n) as $r_n^t := \frac{\mathcal{X}_{n-1}^t - \mathcal{X}_n^t}{\Delta N}$
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- 4 Update \mathcal{X}_n^{t+1} (Euler scheme)
 - If $\mathcal{X}_n^{t+1} > x_S$, go to next particle $n \leftarrow n + 1$
 - Else, update $\eta \leftarrow \eta - \frac{\Delta t}{r_n^t \Delta N} \mathcal{V}(r_n^t, l_n^t)$.

Upstream boundary conditions

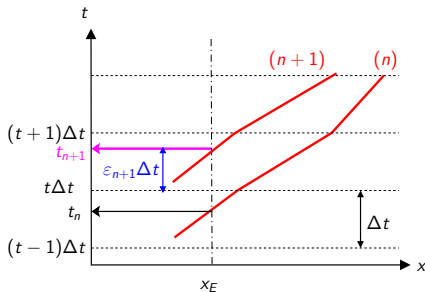
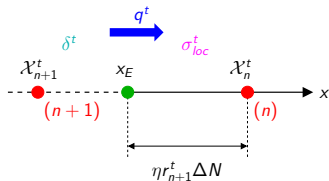
(Continued)



- Entry point E located at x_E
- Boundary data = (discrete) upstream demand $\delta^t = \delta(t\Delta t)$
- n the last vehicle entered in the link
- next particle $(n + 1)$ is still part of the **demand** and will enter in the link at time $(t + \epsilon)\Delta t$

Upstream boundary conditions

We don't know the position of *next* particle!



Upstream boundary conditions

(Continued)

Computational steps:

1 Instantiation:

- We initialize the fraction η

$$\eta = q^{t-1} \frac{(t\Delta t - t_n)}{\Delta N}$$

and

$$r_{n+1}^t = \frac{\mathcal{X}_n^t - x_E}{\eta \Delta N}.$$

- We introduce the local supply

$$\sigma_{loc}^t = \Xi \left(\frac{1}{r_{n+1}^t}, I_{n+1}^t, I_n^t; x_E \right) \quad \text{for any } t \in \mathbb{N}, \quad n \in \mathbb{Z},$$

- Let F^t be the number of particles stored inside the upstream “queue”.

Upstream boundary conditions

(Continued)

- 2 **Stock model:** The evolution of the stock F^t is given by

$$F^{t+1} = F^t + (\delta^t - q^t)\Delta t, \quad (21)$$

where δ^t is the (cumulative) demand and q^t is the effective inflow.

- if $F^t > 0$, then there is a (vertical) queue upstream and

$$q^t = \min \left\{ \sigma_{loc}^t, Q_{max}(I_{n+1}^t), \frac{F^t}{\Delta t} + \delta^t \right\},$$

- if $F^t = 0$, then there is no queue and

$$q^t = \min \{ \sigma_{loc}^t, \delta^t \}.$$

Upstream boundary conditions

(Continued)

- 3 **Update:** Particle $(n + 1)$ is generated if and only if $\eta\Delta N + q^t\Delta t \geq \Delta N$.
- if $q^t\Delta t < (1 - \eta)\Delta N$, then

$$\eta \leftarrow \eta + \frac{q^t\Delta t}{(1 - \eta)\Delta N}.$$

- if $q^t\Delta t \geq (1 - \eta)\Delta N$, then the particle $(n + 1)$ has entered the link at time $t_{n+1} = (t + \varepsilon_{n+1})\Delta t$ where

$$\varepsilon_{n+1} = \frac{(1 - \eta)\Delta N}{q^t\Delta t}.$$

The position of particle $(n + 1)$ is updated

$$\mathcal{X}_{n+1}^{t+1} = x_E + (1 - \varepsilon_{n+1})\Delta t \mathcal{V} (r_{n+1}^t, l_{n+1}^t).$$

Go to next particle $n \leftarrow n + 1$.

Upstream boundary conditions

(Continued)

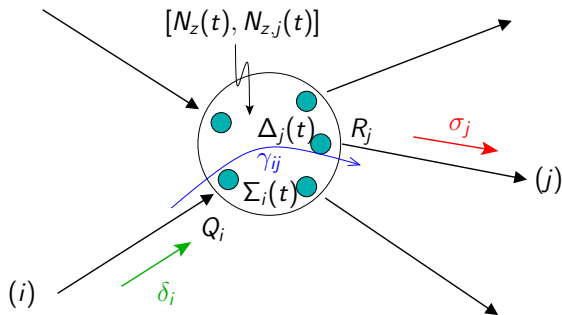
- **Final update:** We compute the attribute

$$l_{n+1}^{t+1} = l_{n+1}^t + \Delta t \varphi(l_n^t)$$

and update the time step $t \leftarrow t + 1$.

Junction model

Internal state model (acts like a buffer)







Assignment of particles through the junction




3 methods:

- The assignment of particles is known: $\exists (\gamma_{ij})_{i,j}$ that describe the proportion of particles coming from any road $i \in \mathcal{I}$ that want to exit the junction on road $j \in \mathcal{J}$
- The path through the junction of each particle $n \in \mathbb{Z}$ is known: included in the particle attribute $l(t, n)$ and does not evolve in time [straightforward]
- The origin-destination (OD) information for each particle is known (may depend on time): consider a reactive assignment model that give us the path followed by particles.

Some references I

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-  J.-P. LEBACQUE, S. MAMMAR, AND H. H. SALEM, *Generic second order traffic flow modelling*, in Transportation and Traffic Theory 2007. Papers Selected for Presentation at ISTTT17, 2007.
-  M. J. LIGHTHILL AND G. B. WHITHAM, *On kinematic waves II. A theory of traffic flow on long crowded roads*, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 229 (1955), pp. 317–345.
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THANKS FOR YOUR ATTENTION

Any question?

guillaume.costeseque@inria.fr