

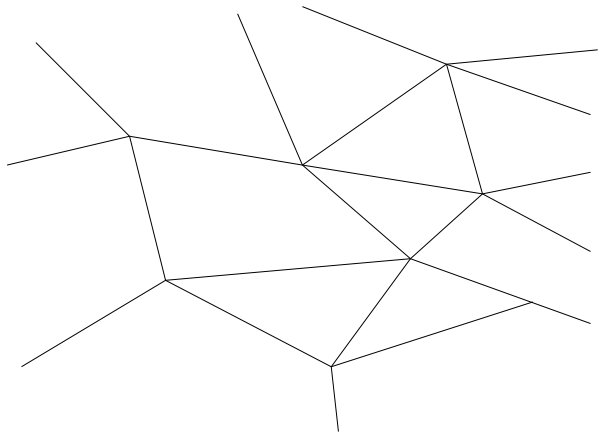
Numerical approach for Hamilton-Jacobi equations on a network: application to traffic

Guillaume Costeseque
(PhD with supervisors R. Monneau & J-P. Lebacque)

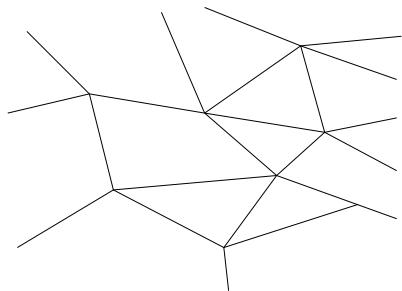
Université Paris Est, Ecole des Ponts ParisTech & IFSTTAR

NETCO Conference - Tours,
June 24, 2014

Flows on a network



Flows on a network



A network is like a (oriented) **graph** made of **edges** and **vertices**

Examples:

- traffic flow,
- gas pipelines,
- blood vessels,
- shallow water,
- internet communications...

Outline

- 1 Introduction
- 2 Numerical scheme
- 3 Traffic interpretation
- 4 Numerical simulation
- 5 Recent developments

Motivation

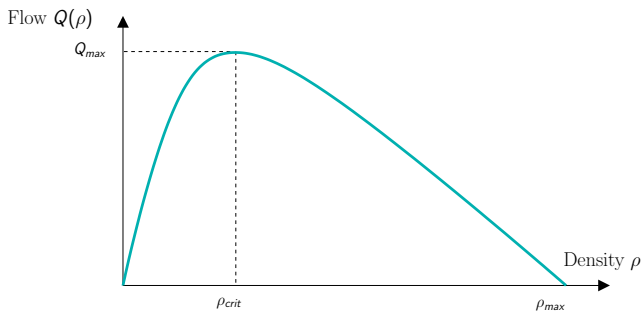
Classical approaches (see [A. Bressan's lectures](#)):

- Macroscopic modeling on (homogeneous) **sections**
- **Coupling conditions** at (pointwise) **junction**

For instance, consider

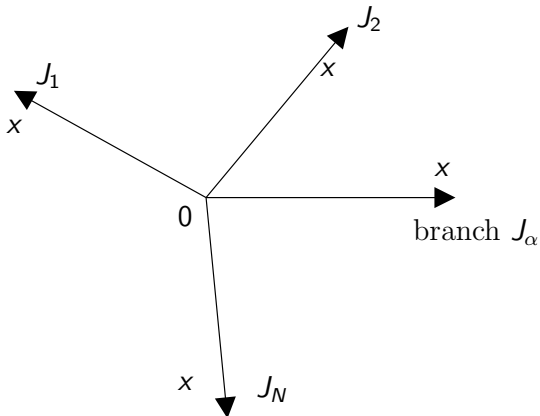
$$\begin{cases} \rho_t + (Q(\rho))_x = 0, & \text{scalar conservation law,} \\ \rho(\cdot, t = 0) = \rho_0(\cdot), & \text{initial conditions,} \\ \psi(\rho(x = 0^-, t), \rho(x = 0^+, t)) = 0, & \text{coupling condition.} \end{cases} \quad (1)$$

See Garavello, Piccoli [3], Lebacque, Khoshyaran [6] and Bressan et al. [1]



$$Q(\rho) = \rho V(\rho) \quad \text{with} \quad V(\rho) = \text{velocity function}$$

Star-shaped junction



Junction model

Proposition (Junction model [IMZ, '13])

That leads to the following junction model (see [5])

$$\begin{cases} u_t^\alpha + H_\alpha(u_x^\alpha) = 0, & x > 0, \alpha = 1, \dots, N \\ u^\alpha = u^\beta =: u, & x = 0, \\ u_t + \mathcal{H}(u_x^1, \dots, u_x^N) = 0, & x = 0 \end{cases} \quad (2)$$

with initial condition $u^\alpha(0, x) = u_0^\alpha(x)$ and

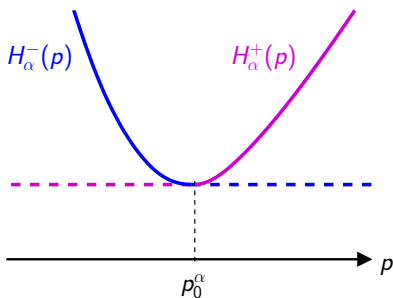
$$\mathcal{H}(u_x^1, \dots, u_x^N) = \underbrace{\max_{\alpha=1, \dots, N} \{H_\alpha^-(u_x^\alpha)\}}_{\text{from optimal control}}.$$

Basic assumptions

For all $\alpha = 1, \dots, N$,

(A0) The initial condition u_0^α is Lipschitz continuous.

(A1) The Hamiltonians H_α are $C^1(\mathbb{R})$ and convex such that:



Numerics on networks

Godunov scheme mainly used for conservation laws:

- [Bretti, Natalini, Piccoli '06, '07]: Godunov scheme compared to kinetic schemes / fast algorithms
- [Blandin, Bretti, Cutolo, Piccoli '09]: Godunov scheme adapted for Colombo model (only tested for 1×1 junctions)

Numerics on networks

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For Hamilton-Jacobi equations on networks:

- [Göttlich, Ziegler, Herty '13]: **Lax-Freidrichs scheme** outside the junction + coupling conditions (density) at the junction
- [Han, Piccoli, Friesz, Yao '12]: **Lax-Hopf formula** for HJ equation coupled with a Riemann solver at junction
- [Camilli, Festa, Schieborn '13]: semi-Lagrangian scheme only designed for **Eikonal** equations

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Presentation of the scheme

Proposition (Numerical Scheme)

Let us consider the discrete space and time derivatives:

$$p_i^{\alpha,n} := \frac{U_{i+1}^{\alpha,n} - U_i^{\alpha,n}}{\Delta x} \quad \text{and} \quad (D_t U)_i^{\alpha,n} := \frac{U_i^{\alpha,n+1} - U_i^{\alpha,n}}{\Delta t}$$

Then we have the following numerical scheme:

$$\begin{cases} (D_t U)_i^{\alpha,n} + \max\{H_\alpha^+(p_{i-1}^{\alpha,n}), H_\alpha^-(p_i^{\alpha,n})\} = 0, & i \geq 1 \\ U_0^n := U_0^{\alpha,n}, & i = 0, \quad \alpha = 1, \dots, N \\ (D_t U)_0^n + \max_{\alpha=1, \dots, N} H_\alpha^-(p_0^{\alpha,n}) = 0, & i = 0 \end{cases} \quad (3)$$

With the initial condition $U_i^{\alpha,0} := u_0^\alpha(i\Delta x)$.

Δx and $\Delta t =$ **space and time steps** satisfying a **CFL condition**

CFL condition

The natural CFL condition is given by:

$$\frac{\Delta x}{\Delta t} \geq \sup_{\substack{\alpha=1,\dots,N \\ i \geq 0, 0 \leq n \leq n_T}} |H'_\alpha(p_i^{\alpha,n})| \quad (4)$$

Gradient estimates

Theorem (Time and Space Gradient estimates)

Assume (A0)-(A1). If the CFL condition (4) is satisfied, then we have that:

- (i) Considering $M^n = \sup_{\alpha, i} (D_t U)_i^{\alpha, n}$ and $m^n = \inf_{\alpha, i} (D_t U)_i^{\alpha, n}$, we have the following time derivative estimate:

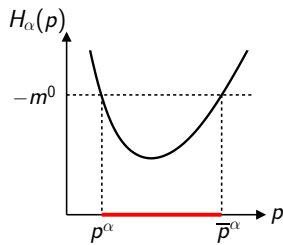
$$m^0 \leq m^n \leq m^{n+1} \leq M^{n+1} \leq M^n \leq M^0$$

- (ii) Considering $\underline{p}_\alpha = (H_\alpha^-)^{-1}(-m^0)$ and $\bar{p}_\alpha = (H_\alpha^+)^{-1}(-m^0)$, we have the following gradient estimate:

$$\underline{p}_\alpha \leq p_i^{\alpha, n} \leq \bar{p}_\alpha, \quad \text{for all } i \geq 0, \quad n \geq 0 \quad \text{and} \quad \alpha = 1, \dots, N$$

► Proof

Stronger CFL condition



As for any $\alpha = 1, \dots, N$, we have that:

$$\underline{p}_\alpha \leq p_i^{\alpha,n} \leq \bar{p}_\alpha \quad \text{for all } i, n \geq 0$$

Then the CFL condition becomes:

$$\frac{\Delta x}{\Delta t} \geq \sup_{\substack{\alpha=1,\dots,N \\ p_\alpha \in [\underline{p}_\alpha, \bar{p}_\alpha]}} |H'_\alpha(p_\alpha)| \quad (5)$$

Existence and uniqueness

(A2) Technical assumption (Legendre-Fenchel transform)

$$H_\alpha(p) = \sup_{q \in \mathbb{R}} (pq - L_\alpha(q)) \quad \text{with} \quad L''_\alpha \geq \delta > 0, \quad \text{for all index } \alpha$$

Existence and uniqueness

(A2) Technical assumption (Legendre-Fenchel transform)

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Theorem (Existence and uniqueness [IMZ, '13])

Under (A0)-(A1)-(A2), there exists a *unique viscosity solution* u of (2) on the junction, satisfying for some constant $C_T > 0$

$$|u(t, y) - u_0(y)| \leq C_T \quad \text{for all } (t, y) \in J_T.$$

Moreover the function u is Lipschitz continuous with respect to (t, y) .

Convergence

Theorem (Convergence from discrete to continuous [CML, '13])

Assume that (A0)-(A1)-(A2) and the CFL condition (5) are satisfied. Then the numerical solution converges uniformly to u the unique viscosity solution of (2) when $\varepsilon \rightarrow 0$, locally uniformly on any compact set \mathcal{K} :

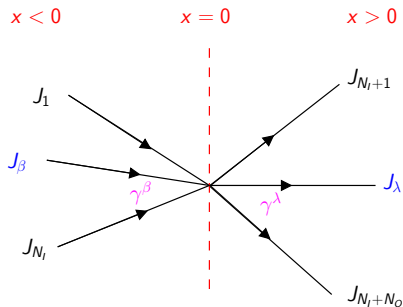
$$\limsup_{\varepsilon \rightarrow 0} \sup_{(n\Delta t, i\Delta x) \in \mathcal{K}} |u^\alpha(n\Delta t, i\Delta x) - U_i^{\alpha, n}| = 0$$

► Proof

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Setting



N_I incoming and N_O outgoing roads

Car densities

The **car density** ρ^α solves the **LWR equation** on branch α :

$$\rho_t^\alpha + (Q^\alpha(\rho^\alpha))_x = 0$$

By definition

$$\rho^\alpha = \gamma^\alpha \partial_x U^\alpha \quad \text{on branch } \alpha$$

And

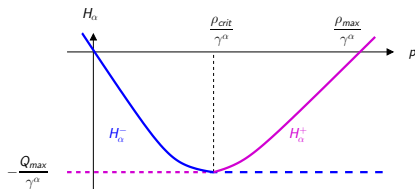
$$\begin{cases} u^\alpha(x, t) = -U^\alpha(-x, t), & x > 0, \text{ for incoming roads} \\ u^\alpha(x, t) = -U^\alpha(x, t), & x > 0, \text{ for outgoing roads} \end{cases}$$

where the **car index** u^α solves the **HJ equation** on branch α :

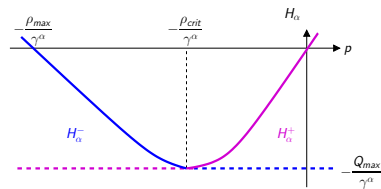
$$u_t^\alpha + H^\alpha(u_x^\alpha) = 0, \quad \text{for } x > 0$$

Flow

$$H_\alpha(p) := \begin{cases} -\frac{1}{\gamma^\alpha} Q^\alpha(\gamma^\alpha p) & \text{for } \alpha = 1, \dots, N_I \\ -\frac{1}{\gamma^\alpha} Q^\alpha(-\gamma^\alpha p) & \text{for } \alpha = N_I + 1, \dots, N_I + N_O \end{cases}$$



Incoming roads



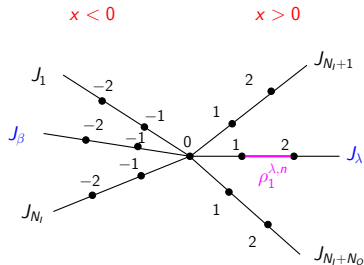
Outgoing roads

Links with “classical” approach

Definition (Discrete car density)

The discrete car density $\rho_i^{\alpha,n}$ with $n \geq 0$ and $i \in \mathbb{Z}$ is given by:

$$\rho_i^{\alpha,n} := \begin{cases} \gamma^\alpha p_{|i|-1}^{\alpha,n} & \text{for } \alpha = 1, \dots, N_I, \quad i \leq -1 \\ -\gamma^\alpha p_i^{\alpha,n} & \text{for } \alpha = N_I + 1, \dots, N_I + N_O, \quad i \geq 0 \end{cases} \quad (6)$$



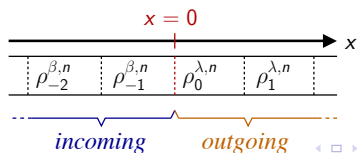
Traffic interpretation

Proposition (Scheme for vehicles densities)

The scheme deduced from (3) for the discrete densities is given by:

$$\frac{\Delta x}{\Delta t} \{\rho_i^{\alpha, n+1} - \rho_i^{\alpha, n}\} = \begin{cases} F^\alpha(\rho_{i-1}^{\alpha, n}, \rho_i^{\alpha, n}) - F^\alpha(\rho_i^{\alpha, n}, \rho_{i+1}^{\alpha, n}) & \text{for } i \neq 0, -1 \\ F_0^\alpha(\rho_0^{\alpha, n}) - F^\alpha(\rho_i^{\alpha, n}, \rho_{i+1}^{\alpha, n}) & \text{for } i = 0 \\ F^\alpha(\rho_{i-1}^{\alpha, n}, \rho_i^{\alpha, n}) - F_0^\alpha(\rho_0^{\alpha, n}) & \text{for } i = -1 \end{cases}$$

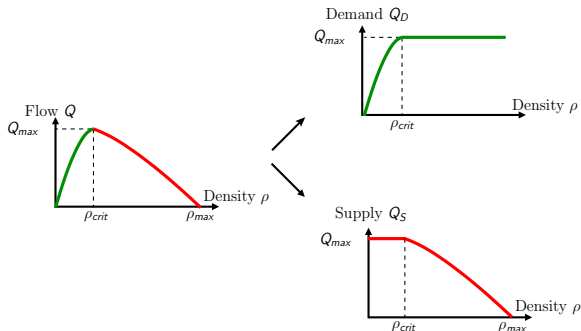
With
$$\begin{cases} F^\alpha(\rho_{i-1}^{\alpha, n}, \rho_i^{\alpha, n}) := \min \{ Q_D^\alpha(\rho_{i-1}^{\alpha, n}), Q_S^\alpha(\rho_i^{\alpha, n}) \} \\ F_0^\alpha(\rho_0^{\alpha, n}) := \gamma^\alpha \min \left\{ \min_{\beta \leq N_I} \frac{1}{\gamma^\beta} Q_D^\beta(\rho_0^{\beta, n}), \min_{\lambda > N_I} \frac{1}{\gamma^\lambda} Q_S^\lambda(\rho_0^{\lambda, n}) \right\} \end{cases}$$



Supply and demand functions

Remark

It recovers the seminal *Godunov scheme* with passing flow = minimum between *upstream demand* Q_D and *downstream supply* Q_S .



From [Lebacque '93, '96]

Supply and demand VS Hamiltonian

$$H_{\alpha}^{-}(p) = \begin{cases} -\frac{1}{\gamma^{\alpha}} Q_D^{\alpha}(\gamma^{\alpha} p) & \text{for } \alpha = 1, \dots, N_I \\ -\frac{1}{\gamma^{\alpha}} Q_S^{\alpha}(-\gamma^{\alpha} p) & \text{for } \alpha = N_I + 1, \dots, N_I + N_O \end{cases}$$

And

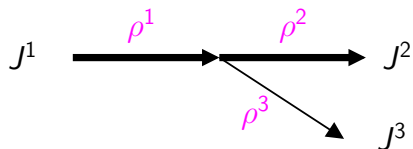
$$H_{\alpha}^{+}(p) = \begin{cases} -\frac{1}{\gamma^{\alpha}} Q_S^{\alpha}(\gamma^{\alpha} p) & \text{for } \alpha = 1, \dots, N_I \\ -\frac{1}{\gamma^{\alpha}} Q_D^{\alpha}(-\gamma^{\alpha} p) & \text{for } \alpha = N_I + 1, \dots, N_I + N_O \end{cases}$$

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Example of a Diverge

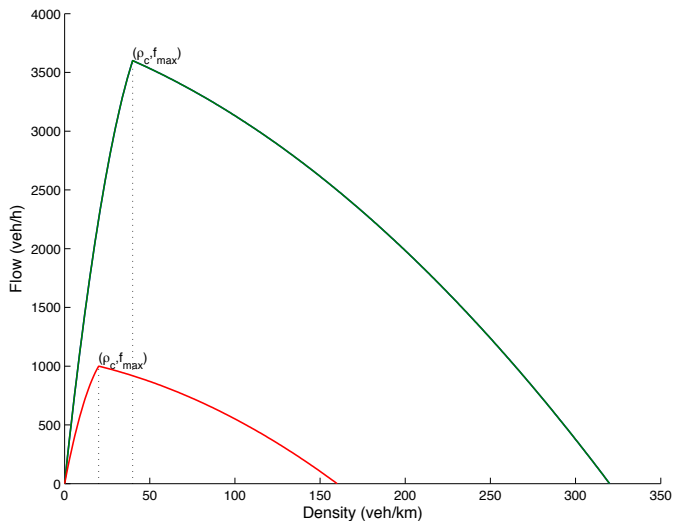
An **off-ramp**:



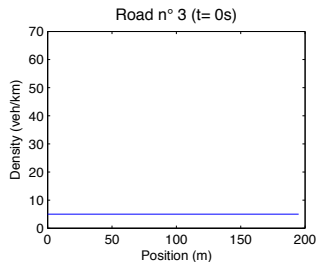
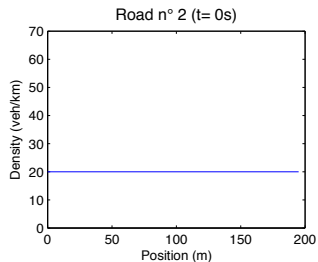
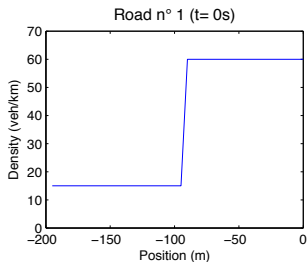
with

$$\begin{cases} \gamma^e = 1, \\ \gamma^l = 0.75, \\ \gamma^r = 0.25 \end{cases}$$

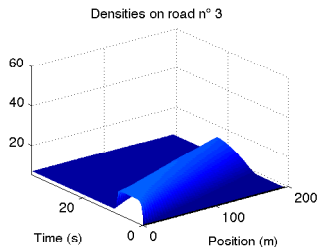
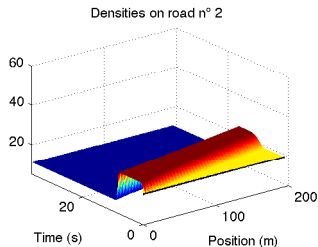
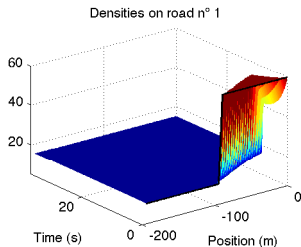
Fundamental Diagrams



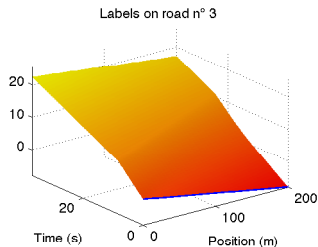
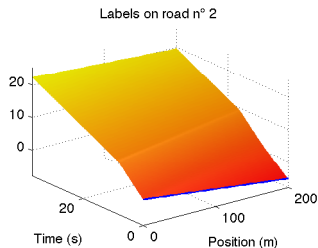
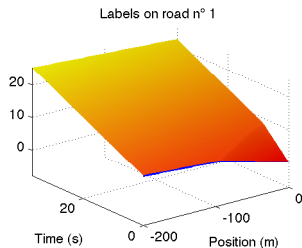
Initial conditions ($t=0s$)



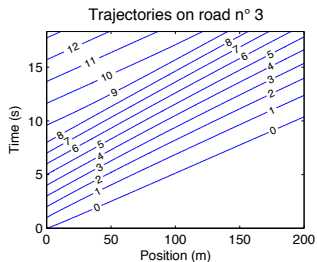
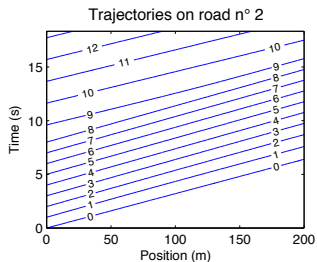
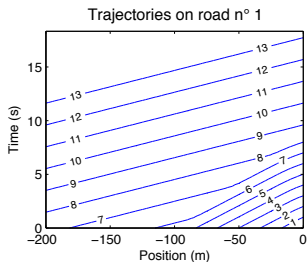
Numerical solution: densities



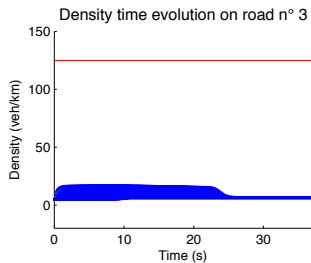
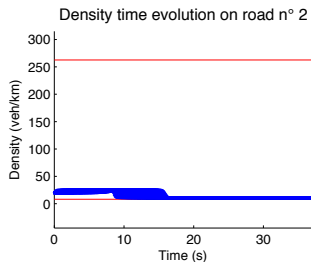
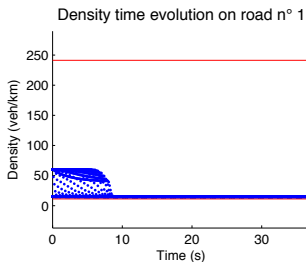
Numerical solution: Hamilton-Jacobi



Trajectories



Gradient estimates



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New junction model

Proposition (Junction model [IM, '14])

From [4], we have

$$\begin{cases} u_t^\alpha + H_\alpha(u_x^\alpha) = 0, & x > 0, \alpha = 1, \dots, N \\ u^\alpha = u^\beta =: u, & x = 0, \\ u_t + \mathcal{H}(u_x^1, \dots, u_x^N) = 0, & x = 0 \end{cases} \quad (7)$$

with initial condition $u^\alpha(0, x) = u_0^\alpha(x)$ and

$$\mathcal{H}(u_x^1, \dots, u_x^N) = \max \left[\underbrace{\mathcal{L}}_{\text{flux limiter}}, \underbrace{\max_{\alpha=1, \dots, N} \{H_\alpha^-(u_x^\alpha)\}}_{\text{minimum between demand and supply}} \right].$$

Weaker assumptions on the Hamiltonians

For all $\alpha = 1, \dots, N$,

(A0) The initial condition u_0^α is Lipschitz continuous.

(A1) The Hamiltonians H_α are continuous and **quasi-convex** i.e.

there exists points p_0^α such that

$$\begin{cases} H_\alpha & \text{is non-increasing on } (-\infty, p_0^\alpha], \\ H_\alpha & \text{is non-decreasing on } [p_0^\alpha, +\infty). \end{cases}$$

Homogenization on a network

Proposition (Homogenization on a periodic network [IM'14])

Assume (A0)-(A1). Consider a *periodic* network.

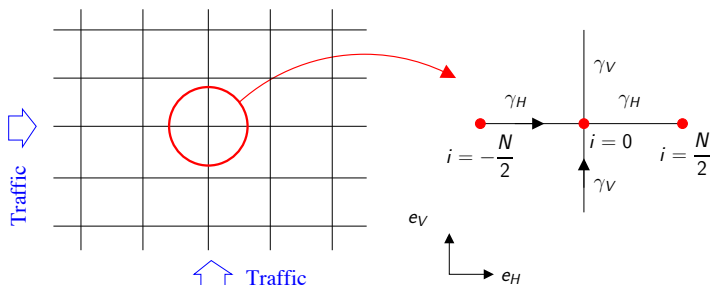
If u^ε satisfies (oscillating) HJ equation on network,
then u^ε converges uniformly towards u^0 when $\varepsilon \rightarrow 0$,
with u^0 solution of

$$u_t^0 + \overline{H}(\nabla_x u^0) = 0, \quad t > 0, \quad x \in \mathbb{R}^d \quad (8)$$

See Prof. [R. Monneau's lecture](#) and [4]

Numerical homogenization on a network

Numerical **scheme** adapted to the **cell problem**



First example

Proposition (Effective Hamiltonian for fixed coefficients [IM'14])

If (γ^H, γ^V) are fixed, then the

- (Hamiltonian) *effective Hamiltonian* \bar{H} is given by

$$\bar{H}(u_{H,x}, u_{V,x}) = \max \left\{ \mathcal{L}, \max_{i=\{H,V\}} H(u_{i,x}) \right\},$$

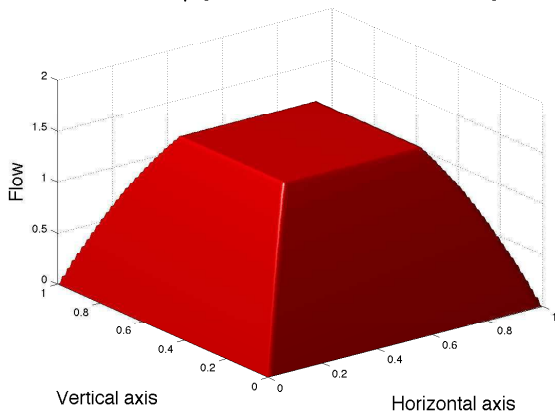
- (traffic flow) *effective flow* \bar{Q} is given by

$$\bar{Q}(\rho_H, \rho_V) = \min \left\{ -\mathcal{L}, \frac{Q(\rho_H)}{\gamma^H}, \frac{Q(\rho_V)}{\gamma^V} \right\}.$$

First example

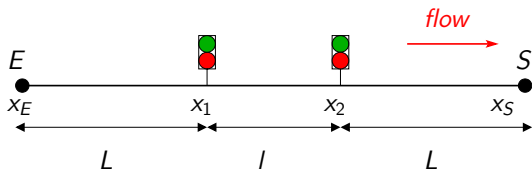
Numerics: assume $Q(\rho) = 4\rho(1 - \rho)$ and $\mathcal{L} = -1.5$,

Results for $\gamma=[0.5 \quad 0.5 \quad 0.5 \quad 0.5]$



Second example

Two consecutive traffic signals on a 1D road

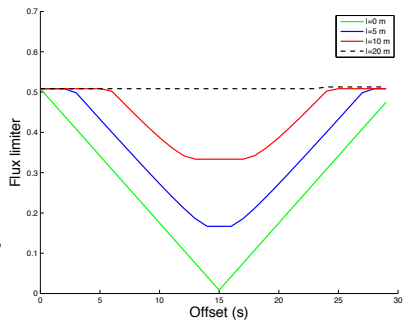
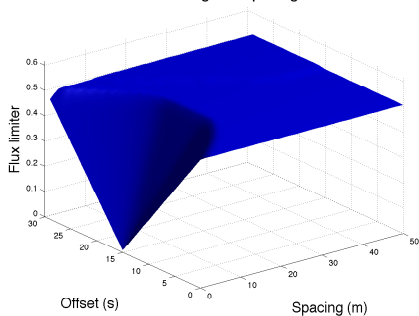


Homogenization theory by [G. Galise, C. Imbert, R. Monneau, '14]

Second example

Effective flux limiter $-\bar{\mathcal{L}}$ (numerics only)

Flux limiter w.r.t. signals spacing and offset



THANKS FOR YOUR ATTENTION

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Some references I



A. BRESSAN, S. CANIC, M. GARAVELLO, M. HERTY, AND B. PICCOLI, *Flows on networks: recent results and perspectives*, EMS Surveys in Mathematical Sciences, (2014).



G. COSTESEQUE, J.-P. LEBACQUE, AND R. MONNEAU, *A convergent scheme for hamilton-jacobi equations on a junction: application to traffic*, arXiv preprint arXiv:1306.0329, (2013).



M. GARAVELLO AND B. PICCOLI, *Traffic flow on networks*, American institute of mathematical sciences Springfield, MO, USA, 2006.



C. IMBERT AND R. MONNEAU, *Level-set convex hamilton-jacobi equations on networks*, (2014).



C. IMBERT, R. MONNEAU, AND H. ZIDANI, *A hamilton-jacobi approach to junction problems and application to traffic flows*, ESAIM: Control, Optimisation and Calculus of Variations, 19 (2013), pp. 129–166.

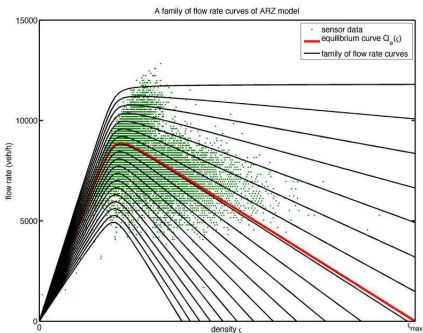
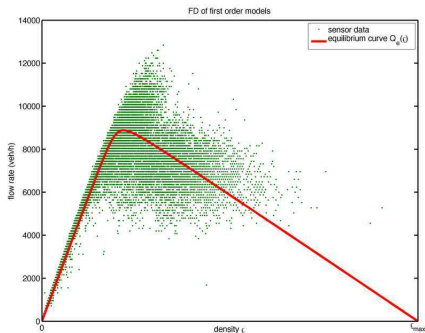
Some references II



J.-P. LEBACQUE AND M. M. KHOSHYARAN, *First-order macroscopic traffic flow models: Intersection modeling, network modeling*, in Transportation and Traffic Theory. Flow, Dynamics and Human Interaction. 16th International Symposium on Transportation and Traffic Theory, 2005.

Fundamental diagram

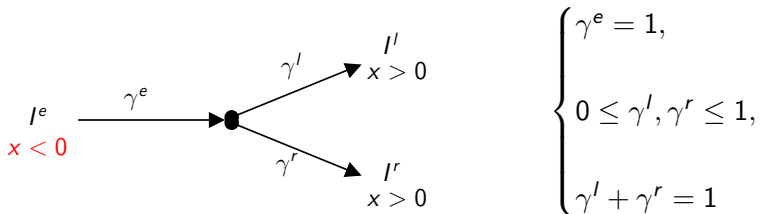
Fundamental diagram: **multi-valued** in congested case



[S. Fan, M. Herty, B. Seibold, 2013], NGSIM dataset

◀ Back

Motivation: the simple divergent road



LWR model [Lighthill, Whitham '55; Richards '56] on each **branch** α :

$$\rho_t^\alpha + (Q^\alpha(\rho^\alpha))_x = 0$$

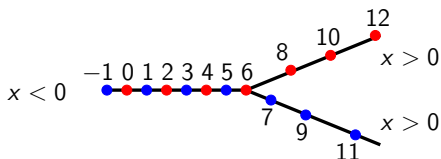
Getting the Hamilton-Jacobi equation

LWR model on each branch (outside the junction point)

$$\rho_t^\alpha + (Q^\alpha(\rho^\alpha))_x = 0 \quad \text{on branch } \alpha$$

Primitive:

$$\begin{cases} U^\alpha(x, t) = U^\alpha(0, t) + \frac{1}{\gamma^\alpha} \int_0^x \rho^\alpha(y, t) dy, \\ U^\alpha(0, t) = g(t) = \text{index of the single car at the junction point} \end{cases}$$



Sketch of the proof (gradient estimates):

Time derivative estimate:

1. Estimate on $m^{\alpha,n} = \inf_i (D_t U)_i^{\alpha,n}$ and partial result for $m^n = \inf_\alpha m^{\alpha,n}$
2. Similar estimate for M^n
3. Conclusion

Space derivative estimate:

1. New bounded Hamiltonian $\tilde{H}_\alpha(p)$ for $p \leq \underline{p}_\alpha$ and $p \geq \bar{p}_\alpha$
2. Time derivative estimate from above
3. Lemma: if for any (i, n, α) , $(D_t U)_i^{\alpha,n} \geq m^0$ then

$$\underline{p}_\alpha \leq p_i^{\alpha,n} \leq \bar{p}_\alpha$$

4. Conclusion as $\tilde{H}_\alpha = H_\alpha$ on $[\underline{p}_\alpha, \bar{p}_\alpha]$

See [2]

Convergence with uniqueness assumption

Sketch of the proof: (Comparison principle very helpful)

1. $\bar{u}^\alpha(t, x) := \limsup_{\varepsilon} U_i^{\alpha, n}$ is a subsolution of (2) (contradiction on Definition inequality with a test function φ)
2. Similarly, \underline{u}^α is a supersolution of (2)
3. Conclusion: $\bar{u}^\alpha = \underline{u}^\alpha$ viscosity solution of (2)

See [2]

Convergence without uniqueness assumption

Sketch of the proof: (No comparison principle)

1. Discrete Lipschitz bounds on $u_\varepsilon^\alpha(n\Delta t, i\Delta x) := U_i^{\alpha, n}$
2. Extension by continuity of u_ε^α
3. Ascoli theorem (convergent subsequence on every compact set)
4. The limit of one convergent subsequence $(u_\varepsilon^\alpha)_\varepsilon$ is super and sub-solution of (2)

See [2]

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