Second Order Traffic Flow Models on Networks

Guillaume Costeseque
in collaboration with J-P. Lebacque (IFSTTAR)

Inria Sophia-Antipolis Méditerranée

Mannheim University
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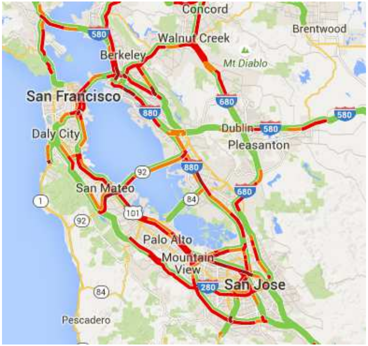
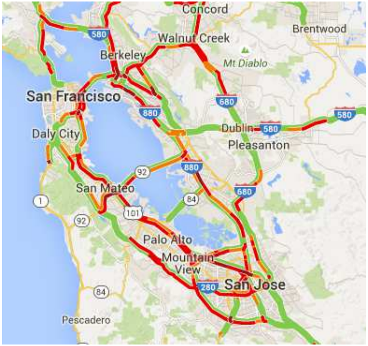
Traffic flows on a network



[Caltrans, Oct. 7, 2015]

Traffic flows on a network



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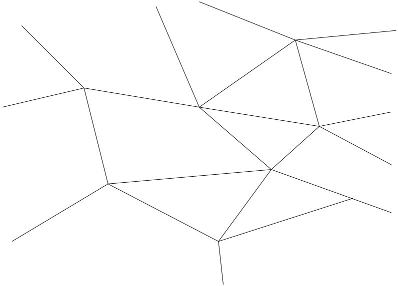


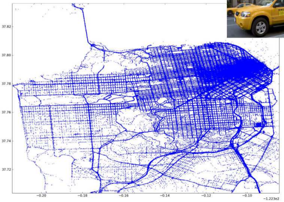
Road network ≡ graph made of edges and vertices

Breakthrough in traffic monitoring

Traffic monitoring

- “old”: loop detectors at fixed locations (Eulerian)
- “new”: GPS devices moving within the traffic (Lagrangian)

Data assimilation of Floating Car Data

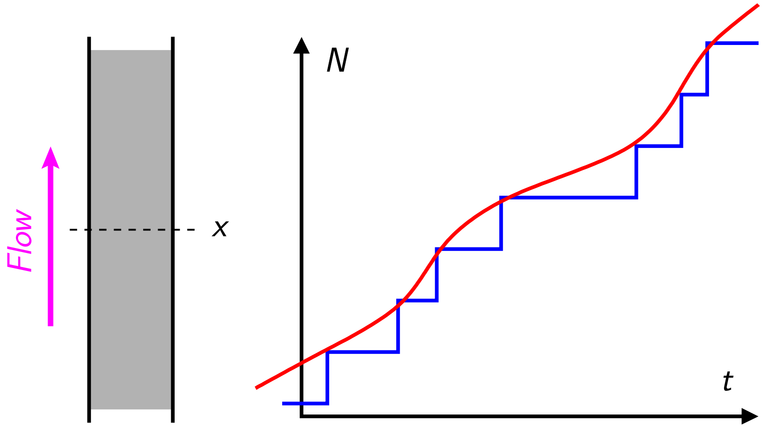

[Mobile Millenium, 2008]

Outline

1. Introduction to traffic
2. Variational principle applied to GSOM models
3. GSOM models on a junction

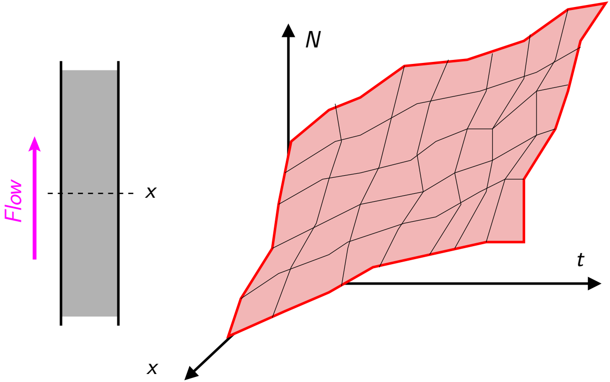
Outline

Convention for vehicle labeling



Three representations of traffic flow

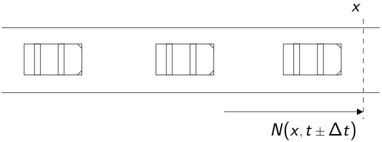
Moskowitz' surface



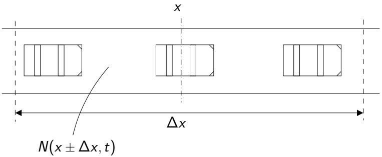
See also [Makigami et al, 1971], [Laval and Leclercq, 2013]

Notations: macroscopic

- $N(t,x)$ vehicle label at (t,x)
- the flow $Q(t,x) = \lim_{\Delta t \to 0} \dfrac{N(t+\Delta t, x) - N(t,x)}{\Delta t} = \partial_t N(t,x)$



- the density $\rho(t,x) = \lim_{\Delta x \to 0} \dfrac{N(t,x) - N(t, x+\Delta x)}{\Delta x} = -\partial_x N(t,x)$

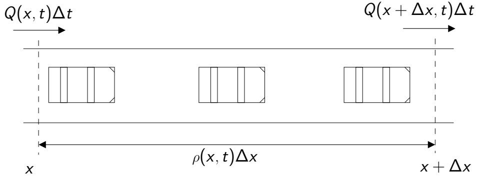


- the stream speed (mean spatial speed) $V(t,x)$.

Macroscopic models

- Hydrodynamics analogy
- Two main categories: first and second order models
- Two common equations:

$$\begin{cases} \partial_t \rho(t,x) + \partial_x Q(t,x) = 0 & \text{conservation equation} \\ Q(t,x) = \rho(t,x)V(t,x) & \text{definition of flow speed} \end{cases} \quad (1)$$



First order: the LWR model

LWR model [Lighthill and Whitham, 1955], [Richards, 1956] [6, 7]

Scalar one dimensional conservation law

$$\partial_t \rho(t,x) + \partial_x \mathfrak{F}(\rho(t,x)) = 0 \tag{2}$$

with

$$\mathfrak{F} : \rho(t,x) \mapsto \mathfrak{F}(\rho(t,x)) := Q(t,x)$$

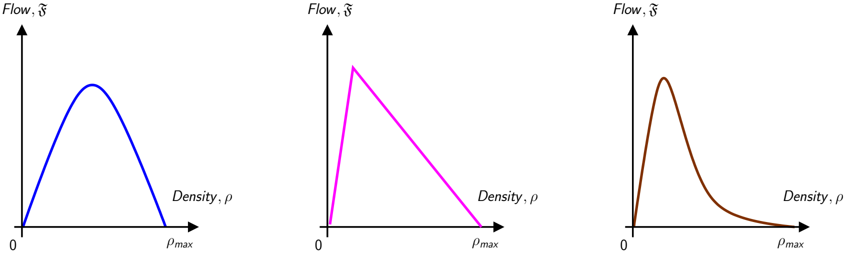
Overview: conservation laws (CL) / Hamilton-Jacobi (HJ)

		Eulerian $t - x$	Lagrangian $t - n$
CL	Variable	Density ρ	Spacing r
	Equation	$\partial_t \rho + \partial_x \mathfrak{F}(\rho) = 0$	$\partial_t r + \partial_n V(r) = 0$
HJ	Variable	Label N	Position $\mathcal{X}$
		$N(t,x) = \int_x^{+\infty} \rho(t,\xi) d\xi$	$\mathcal{X}(t,n) = \int_n^{+\infty} r(t,\eta) d\eta$
	Equation	$\partial_t N + H(\partial_x N) = 0$	$\partial_t \mathcal{X} + \mathcal{V}(\partial_n \mathcal{X}) = 0$
	Hamiltonian	$H(p) = -\mathfrak{F}(-p)$	$\mathcal{V}(p) = -V(-p)$

Fundamental diagram (FD)

Flow-density fundamental diagram $\mathfrak{F}$

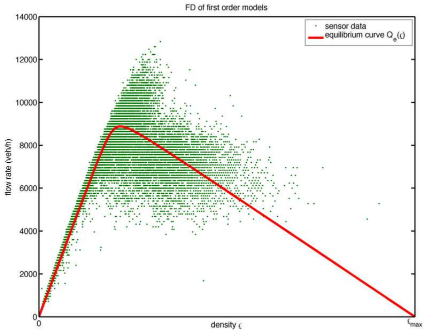
- Empirical function with
  - ρ_{max} the maximal or jam density,
  - ρ_c the critical density
- Flux is increasing for $\rho \leq \rho_c$: free-flow phase
- Flux is decreasing for $\rho \geq \rho_c$: congestion phase

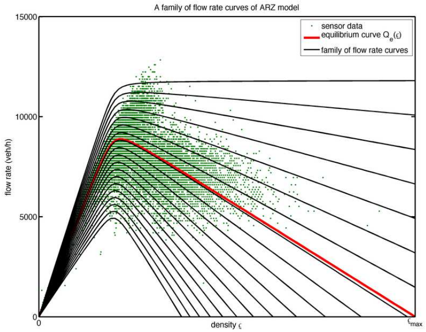


[Garavello and Piccoli, 2006]

Motivation for higher order models

- Experimental evidences
  - fundamental diagram: multi-valued in congested case





[S. Fan, U. Illinois], NGSIM dataset

Motivation for higher order models

- Experimental evidences
  - fundamental diagram: multi-valued in congested case
  - phenomena not accounted for: bounded acceleration, capacity drop...
- Need for models able to integrate measurements of different traffic quantities (acceleration, fuel consumption, noise)

GSOM family [Lebacque, Mammar, Haj-Salem 2007] [5]

- Generic Second Order Models (GSOM) family

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 & \text{Conservation of vehicles,} \\ \partial_t(\rho I) + \partial_x(\rho v I) = \rho\varphi(I) & \text{Dynamics of the driver attribute } I, \\ v = \mathfrak{I}(\rho, I) & \text{Speed-density fundamental diagram,} \end{cases} \tag{3}$$

- Specific driver attribute I
  - the driver aggressiveness,
  - the driver origin/destination or path,
  - the vehicle class,
  - ...
- Flow-density fundamental diagram

$$\mathfrak{F} : (\rho, I) \mapsto \rho\mathfrak{I}(\rho, I).$$

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GSOM family [Lebacque, Mammar, Haj-Salem 2007] [5] (continued)

- Kinematic waves or 1-waves:
  - similar to the seminal LWR model
  - density variations at speed $\nu = \partial_\rho \mathfrak{I}(\rho, I)$
  - driver attribute I is continuous
- Contact discontinuities or 2-waves:
  - variations of driver attribute I at speed $\nu = \mathfrak{I}(\rho, I)$
  - the flow speed v is constant.

Examples of GSOM models

- LWR model = GSOM model with no specific driver attribute
- LWR model with bounded acceleration = GSOM model with $I := v$
- ARZ model = GSOM with $I := v + p(\rho)$

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho w) + \partial_x(\rho v w) = 0, \\ w = v + p(\rho) \end{cases}$$

- Generalized ARZ model [Fan, Herty, Seibold]
- Multi-commodity models (multi-class, multi-lanes) of [Jin and Zhang], [Bagnerini and Rascle] or [Herty, Kirchner, Moutari and Rascle], [Klar, Greenberg and Rascle]
- Colombo 1-phase model
- Stochastic GSOM model [Khoshyaran and Lebacque]

Outline

LWR in Eulerian (t, x)

- Cumulative vehicles count (CVC) or Moskowitz surface $N(t, x)$

$$Q = \partial_t N \quad \text{and} \quad \rho = -\partial_x N$$

- If density ρ satisfies the scalar (LWR) conservation law

$$\partial_t \rho + \partial_x \mathfrak{F}(\rho) = 0$$

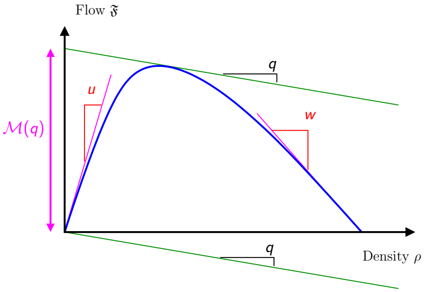
- Then N satisfies the first order Hamilton-Jacobi equation

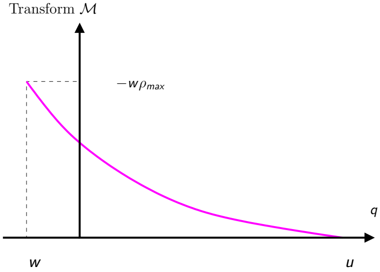
$$\partial_t N - \mathfrak{F}(-\partial_x N) = 0 \tag{4}$$

LWR in Eulerian (t, x)

- Legendre-Fenchel transform with $\mathfrak{F}$ concave (*relative capacity*)

$$\mathcal{M}(q) = \sup_{\rho} \; [\mathfrak{F}(\rho) - \rho q]$$





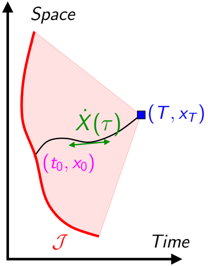
LWR in Eulerian (t, x)

(continued)

- Lax-Hopf formula (representation formula) [Daganzo, 2006]

$$N(T, x_T) = \min_{u(.),(t_0,x_0)} \int_{t_0}^{T} \mathcal{M}(u(\tau))d\tau + N(t_0, x_0), \quad (5)$$

$$\left| \begin{array}{l} \dot{X} = u \\ u \in \mathcal{U} \\ X(t_0) = x_0, \quad X(T) = x_T \\ (t_0, x_0) \in \mathcal{J} \end{array} \right.$$

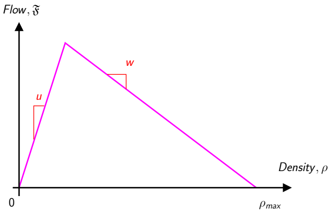


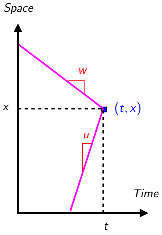
- Viability theory [Claudel and Bayen, 2010]

LWR in Eulerian (t, x)

(Historical note)

- Dynamic programming [Daganzo, 2006] for triangular FD (u and w free and congested speeds)





- Minimum principle [Newell, 1993]

$$N(t,x) = \min \Bigg[N\left(t - \frac{x - x_u}{u}, x_u\right), \quad N\left(t - \frac{x - x_w}{w}, x_w\right) + \rho_{max}(x_w - x) \Bigg], \tag{6}$$

LWR in Lagrangian (n, t)

- Consider $X(t, n)$ the location of vehicle n at time $t \geq 0$

$$v = \partial_t X \quad \text{and} \quad r = -\partial_n X$$

- If the spacing $r := 1/\rho$ satisfies the LWR model (Lagrangian coord.)

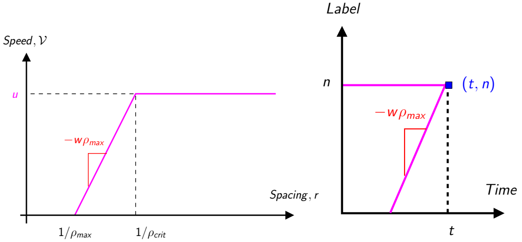
$$\partial_t r + \partial_n \mathcal{V}(r) = 0$$

- Then X satisfies the first order Hamilton-Jacobi equation

$$\partial_t X - \mathcal{V}(-\partial_n X) = 0. \tag{7}$$

LWR in Lagrangian (n, t)
(continued)

- Dynamic programming for triangular FD



- Minimum principle ⇒ car following model [Newell, 2002]

$$X(t,n) = \min \Big[X(t_0, n) + u(t - t_0), \\ X(t_0, n + w\rho_{max}(t - t_0)) + w(t - t_0) \Big]. \quad (8)$$

GSOM in Lagrangian (n, t)

- From [Lebacque and Khoshyaran, 2013], GSOM in Lagrangian

$$\begin{cases} \partial_t r + \partial_N v = 0 & \text{Conservation of vehicles,} \\ \partial_t I = 0 & \text{Dynamics of } I, \\ v = \mathcal{W}(N, r, t) := \mathcal{V}(r, I(N, t)) & \text{Fundamental diagram.} \end{cases} \tag{9}$$

- Position $\mathcal{X}(N, t) := \int_{-\infty}^{t} v(N, \tau) d\tau$ satisfies the HJ equation

$$\partial_t \mathcal{X} - \mathcal{W}(N, -\partial_N \mathcal{X}, t) = 0, \tag{10}$$

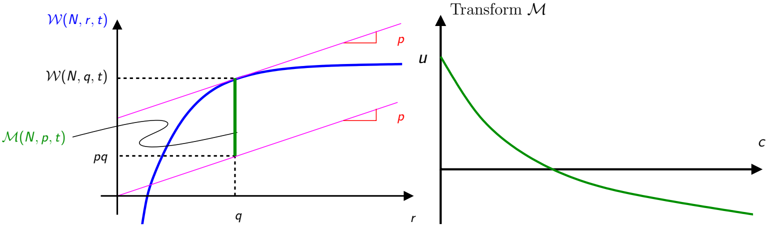
- And $I(N, t)$ solves the ODE

$$\left| \begin{array}{ll} \partial_t I(N, t) = 0, & \\ I(N, 0) = i_0(N), & \text{for any} \quad N. \end{array} \right.$$

GSOM in Lagrangian (n, t)
(continued)

- Legendre-Fenchel transform of $\mathcal{W}$ according to r

$$\mathcal{M}(N, c, t) = \sup_{r \in \mathbb{R}} \{\mathcal{W}(N, r, t) - cr\}$$



GSOM in Lagrangian (n, t)
(continued)

- Lax-Hopf formula

$$\mathcal{X}(N_T, T) = \min_{u(.),(N_0,t_0)} \int_{t_0}^{T} \mathcal{M}(N, u, t)dt + \mathbf{c}(N_0, t_0), \qquad (11)$$
$$\left| \begin{array}{l} \dot{N} = u \\ u \in \mathcal{U} \\ N(t_0) = N_0, \quad N(T) = N_T \\ (N_0, t_0) \in \mathcal{K} \end{array} \right.$$

GSOM in Lagrangian (n, t)

(continued)

- Optimal trajectories = characteristics

$$\begin{cases} \dot{N} = \partial_r \mathcal{W}(N, r, t), \\ \dot{r} = -\partial_N \mathcal{W}(N, r, t), \end{cases} \tag{12}$$

- System of ODEs to solve
- Difficulty: not straight lines in the general case

General ideas

First key element: Lax-Hopf formula

- Computations only for the characteristics

$$\mathcal{X}(N_T, T) = \min_{(N_0, r_0, t_0)} \int_{t_0}^{T} \mathcal{M}(N, \partial_r \mathcal{W}(N, r, t), t)dt + \mathbf{c}(N_0, r_0, t_0),$$
$$\left| \begin{array}{l} \dot{N}(t) = \partial_r \mathcal{W}(N, r, t) \\ \dot{r}(t) = -\partial_N \mathcal{W}(N, r, t) \\ N(t_0) = N_0, \quad r(t_0) = r_0, \quad N(T) = N_T \\ (N_0, r_0, t_0) \in \mathcal{K} \end{array} \right. \tag{13}$$

- $\mathcal{K} := \mathrm{Dom}(\mathbf{c})$ is the set of initial/boundary values

General ideas
(continued)

Second key element: inf-morphism prop. [Aubin et al, 2011]

- Consider a union of sets (initial and boundary conditions)

$$\mathcal{K} = \bigcup_{l} \mathcal{K}_l,$$

- then the global minimum is

$$\mathcal{X}(N_T, T) = \min_{l} \mathcal{X}_l(N_T, T), \tag{14}$$

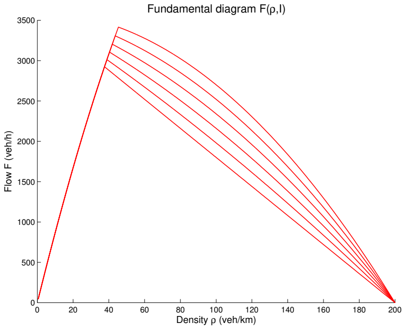
- with $\mathcal{X}_l$ partial solution to sub-problem $\mathcal{K}_l$.

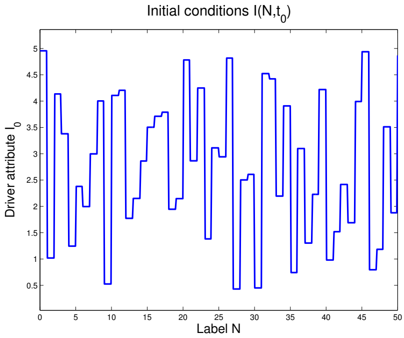
IBVP

Consider piecewise affine initial and boundary conditions:

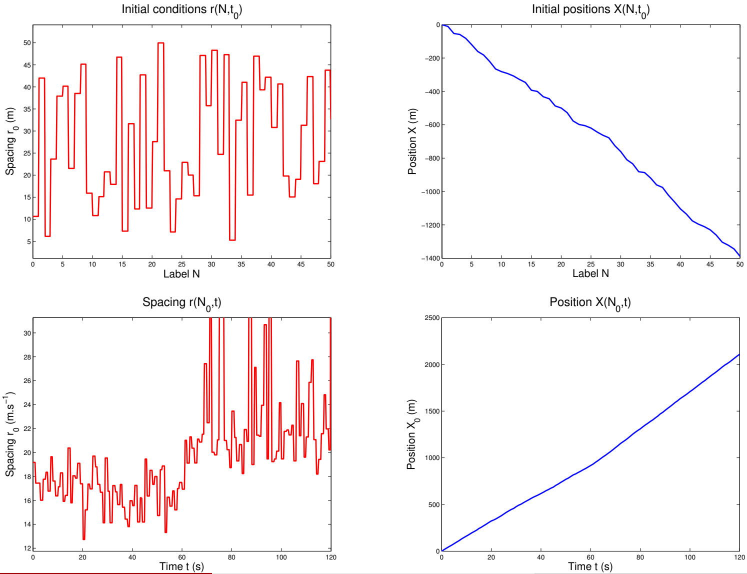
- initial condition at time $t = t_0$ = initial position of vehicles $\xi(\cdot, t_0)$
- "upstream" boundary condition = trajectory $\xi(N_0, \cdot)$ of the first vehicle,
- and internal boundary conditions given for instance by cumulative vehicle counts at fixed location $\mathcal{X} = x_0$.

Fundamental Diagram and Driver Attribute

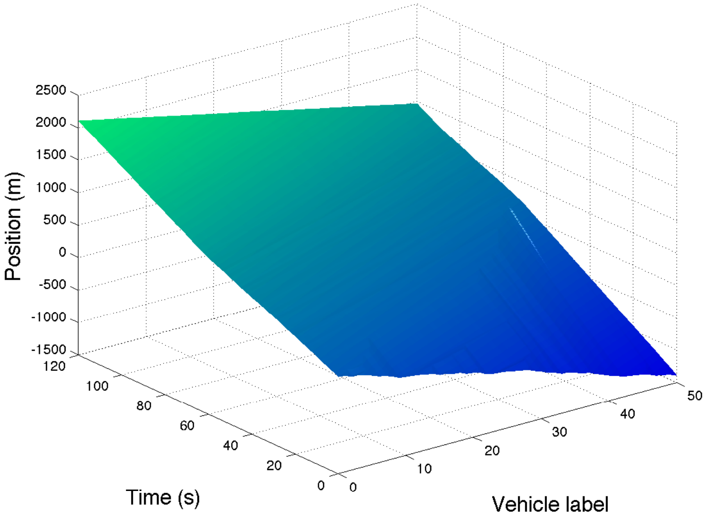




Initial and Boundaries Conditions



Numerical result (Initial cond. + first traj.)



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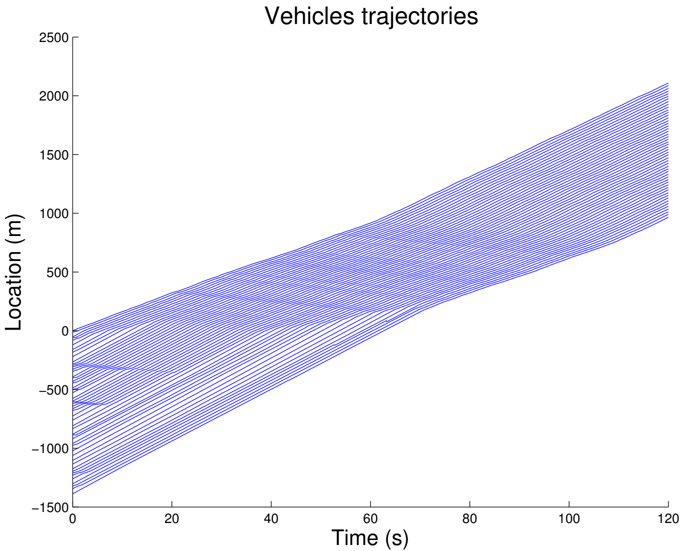
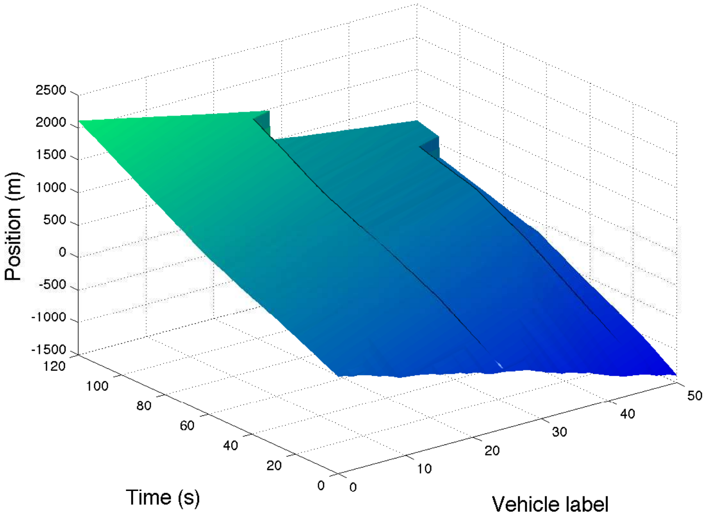
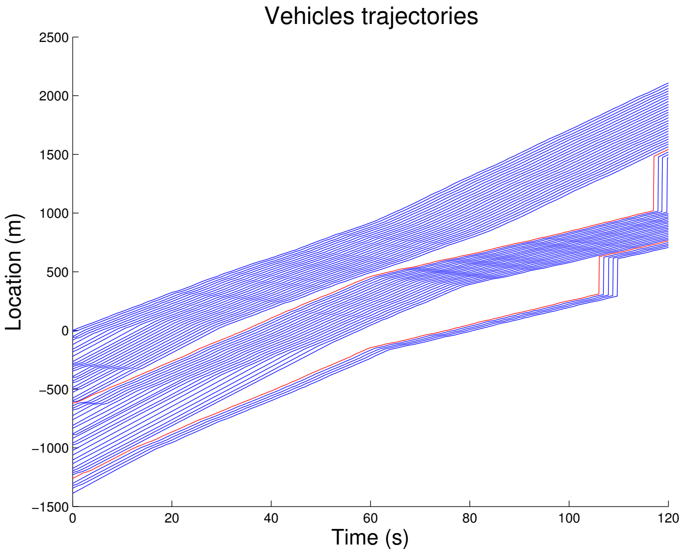


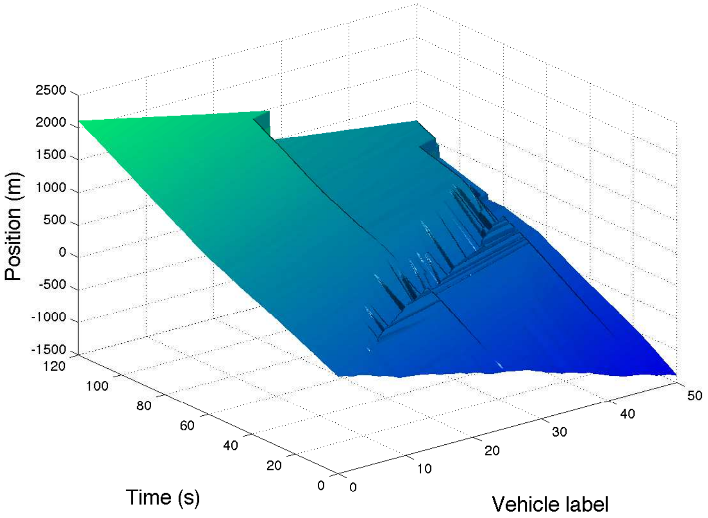
Numerical result (Initial cond.+ 3 traj.)



Numerical result (Initial cond. + 3 traj.)



Numerical result (Initial cond. + 3 traj. + Eulerian data)



Outline

The whole picture

We need

(i) a link model

(ii) a junction model

(iii) the upstream (resp. downstream) boundary conditions for an incoming (resp. outgoing) link

(iv) link-node and node-link interfaces

GSOM lagrangian

General expressions of GSOM family

In Eulerian,

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 & \text{Conservation of vehicles,} \\ \partial_t(\rho I) + \partial_x(\rho v I) = \rho\varphi(I) & \text{Dynamics of the driver attribute } I, \\ v = \mathfrak{I}(\rho, I) & \text{Fundamental diagram,} \end{cases} \tag{15}$$

Transformed in Lagrangian,

$$\begin{cases} \partial_T r + \partial_n v = 0 & \text{Conservation of vehicles,} \\ \partial_T I = \varphi(I) & \text{Dynamics of the driver attribute } I, \\ v = \mathcal{V}(r, I) & \text{Fundamental diagram.} \end{cases} \tag{16}$$

GSOM lagrangian

Following classical approach [3, 4] we set

- Δt, ΔN time and particle steps;
- $r_n^t := r(t\Delta t, n\Delta N)$, for any $t \in \mathbb{N}$ and any $n \in \mathbb{Z}$
- and $I_n^t := I(t\Delta t, n\Delta N)$.

Numerical scheme

$$\begin{cases} r_n^{t+1} := r_n^t + \dfrac{\Delta t}{\Delta N}\left[V_{n-1}^t - V_n^t\right], \\ V_n^t := \mathcal{V}\left(r_n^t, I_n^t\right), \\ I_n^{t+1} = I_n^t + \Delta t \varphi\left(I_n^t\right) \end{cases} \tag{17}$$

CFL condition:

$$\frac{\Delta N}{\Delta t} \geq \sup_{N,r,t} \left|\partial_r \mathcal{V}(r, I(t,N))\right|. \tag{18}$$

GSOM lagrangian (HJ)

Introduce $\mathcal{X}(T, N)$ the position of particle N at time T and satisfying

$$r = -\partial_N \mathcal{X} \quad \text{and} \quad v = \partial_T \mathcal{X}$$

such that

$$\begin{cases} \partial_T \mathcal{X} = \mathcal{V}\left(-\partial_N \mathcal{X}, I\right), \\ \partial_T I = \varphi(I). \end{cases} \tag{19}$$

Numerical scheme for HJ equation

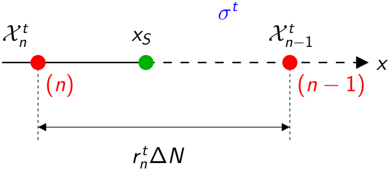
$$\begin{cases} \mathcal{X}_n^{t+1} = \mathcal{X}_n^t + \Delta t \ V_n^t, \\ V_n^t := \mathcal{V}\left(\dfrac{\mathcal{X}_{n-1}^t - \mathcal{X}_n^t}{\Delta N}, I_n^t\right), \\ I_n^{t+1} = I_n^t + \Delta t \ \varphi\left(I_n^t\right) \end{cases} \tag{20}$$

Boundary conditions

We have two different solutions:

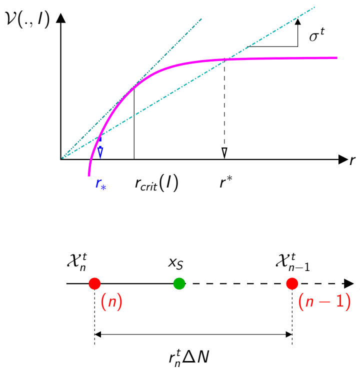
- "Classical" supply-demand methodology [3, 2, 1]
  but it implies to work with flows;
- Using tools developed in [Lebacque, Khoshyaran, (2013)] [4] that allow to compute directly spacing.

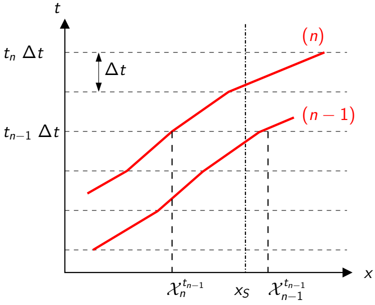
Downstream boundary conditions
(Continued)



- Exit point S located at x_S
- Boundary data = downstream supply $\sigma^t = \sigma(t\Delta t)$.
- (n) the last particle located on the link (or at least a fraction $\eta \Delta N$ of it is still on the link, with $0 \leq \eta < 1$).

Downstream boundary conditions





Downstream boundary conditions
(Continued)

Computational steps:

1. Define the spacing associated to particle (n) as $r_n^t := \dfrac{\mathcal{X}_{n-1}^t - \mathcal{X}_n^t}{\Delta N}$

Downstream boundary conditions

(Continued)

Computational steps:

1. Define the spacing associated to particle (n) as $r_n^t := \dfrac{\mathcal{X}_{n-1}^t - \mathcal{X}_n^t}{\Delta N}$
2. Define the proportion of (n) already out $\eta := \dfrac{x_S - \mathcal{X}_n^t}{r_n^t \Delta N}$

Downstream boundary conditions
(Continued)

Computational steps:

1. Define the spacing associated to particle (n) as $r_n^t := \dfrac{\mathcal{X}_{n-1}^t - \mathcal{X}_n^t}{\Delta N}$
2. Define the proportion of (n) already out $\eta := \dfrac{x_S - \mathcal{X}_n^t}{r_n^t \Delta N}$
3. Distinguish two cases:
   - either $\mathcal{V}(r_n^t, I_n^t) \leq \sigma^t r_n^t$: spacing is conserved.
   - or $\mathcal{V}(r_n^t, I_n^t) > \sigma^t r_n^t$: then, we solve $\mathcal{V}(r_n^t, I_n^t) = \sigma^t r_n^t$ and choice of the smallest value $r_n^t \leftarrow r_*$ (congested)

Downstream boundary conditions
(Continued)

Computational steps:

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4. Update $\mathcal{X}_n^{t+1}$ (Euler scheme)
   - If $\mathcal{X}_n^{t+1} > x_S$, go to next particle $n \leftarrow n+1$
   - Else, update $\eta \leftarrow \eta - \dfrac{\Delta t}{r_n^t \Delta N} \mathcal{V}\left(r_n^t, I_n^t\right)$

Downstream boundary conditions
(Continued)

Computational steps:

1. Define the spacing associated to particle (n) as $r_n^t := \dfrac{\mathcal{X}_{n-1}^t - \mathcal{X}_n^t}{\Delta N}$
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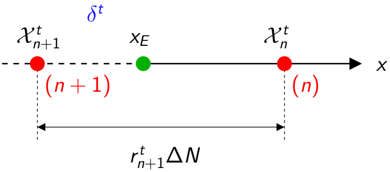
   - Else, update $\eta \leftarrow \eta - \dfrac{\Delta t}{r_n^t \Delta N} \mathcal{V}\left(r_n^t, I_n^t\right)$ and update I_n^{t+1} (Euler scheme)
5. Go to next time step $t \leftarrow t+1$

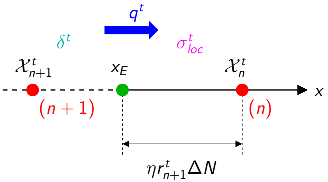
Upstream boundary conditions
(Continued)

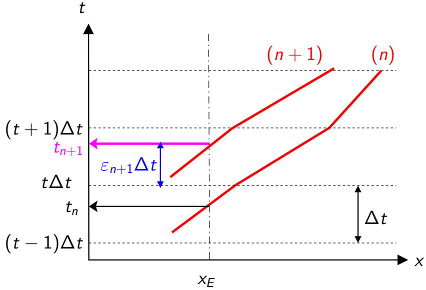


- Entry point E located at x_E
- Boundary data = (discrete) upstream demand $\delta^t = \delta(t\Delta t)$
- n the last vehicle entered in the link
- next particle $(n+1)$ is still part of the demand and will enter in the link at time $(t+\varepsilon)\Delta t$

Upstream boundary conditions

We don't know the position of *next* particle!





Upstream boundary conditions

(Continued)

Computational steps:

1. **Instantiation:**
   - We initialize the fraction η

$$\eta = q^{t-1} \frac{(t\Delta t - t_n)}{\Delta N}$$

   and

$$r^t_{n+1} = \frac{\mathcal{X}^t_n - x_E}{\eta \Delta N}.$$

   - We introduce the local supply

$$\sigma^t_{loc} = \Xi \left(\frac{1}{r^t_{n+1}}, I^t_{n+1}, I^t_n; x_E \right) \quad \text{for any} \quad t \in \mathbb{N}, \quad n \in \mathbb{Z},$$

   - Let F^t be the number of particles stored inside the upstream "queue".

Upstream boundary conditions
(Continued)

2. **Stock model:** The evolution of the stock F^t is given by

$$F^{t+1} = F^t + (\delta^t - q^t)\Delta t, \tag{21}$$

where δ^t is the (cumulative) demand and q^t is the effective inflow.

- if $F^t > 0$, then there is a (vertical) queue upstream and

$$q^t = \min\left\{\sigma^t_{loc} \ , \ Q_{max}(I^t_{n+1}) \ , \ \frac{F^t}{\Delta t} + \delta^t\right\},$$

- if $F^t = 0$, then there is no queue and

$$q^t = \min\left\{\sigma^t_{loc} \ , \ \delta^t\right\}.$$

Upstream boundary conditions
(Continued)

3. **Update:** Particle $(n+1)$ is generated if and only if $\eta \Delta N + q^t \Delta t \geq \Delta N$.
   - if $q^t \Delta t < (1-\eta)\Delta N$, then
   $$\eta \leftarrow \eta + \frac{q^t \Delta t}{(1-\eta)\Delta N}.$$
   - if $q^t \Delta t \geq (1-\eta)\Delta N$, then the particle $(n+1)$ has entered the link at time $t_{n+1} = (t + \varepsilon_{n+1})\Delta t$ where
   $$\varepsilon_{n+1} = \frac{(1-\eta)\Delta N}{q^t \Delta t}.$$
   The position of particle $(n+1)$ is updated
   $$\mathcal{X}_{n+1}^{t+1} = x_E + (1-\varepsilon_{n+1})\Delta t \; \mathcal{V}\left(r_{n+1}^t, I_{n+1}^t\right).$$
   Go to next particle $n \leftarrow n+1$.

Upstream boundary conditions
(Continued)

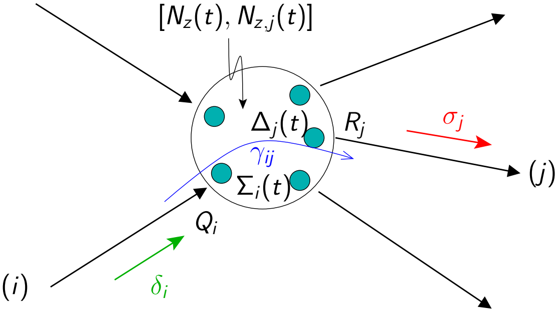
4. **Final update:** We compute the attribute

$$I_{n+1}^{t+1} = I_{n+1}^{t} + \Delta t \, \varphi \left(I_n^t \right)$$

and update the time step $t \leftarrow t + 1$.

Junction model

Internal state model (acts like a buffer)



Assignment of particles through the junction

3 methods:

- The assignment of particles is known: $\exists\ (\gamma_{ij})_{i,j}$ that describe the proportion of particles coming from any road $i \in \mathcal{I}$ that want to exit the junction on road $j \in \mathcal{J}$
- The path through the junction of each particle $n \in \mathbb{Z}$ is known: included in the particle attribute $I(t, n)$ and does not evolve in time [straightforward]
- The origin-destination (OD) information for each particle is known (may depend on time): consider a reactive assignment model that give us the path followed by particles.

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THANKS FOR YOUR ATTENTION

Any question?

guillaume.costeseque@inria.fr