Traffic Flow Modeling on Road Networks Using Hamilton-Jacobi Equations

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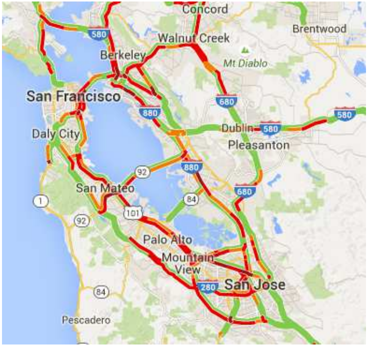
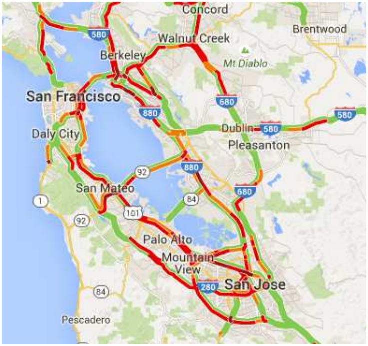
Traffic flows on a network



[Caltrans, Oct. 7, 2015]

Traffic flows on a network



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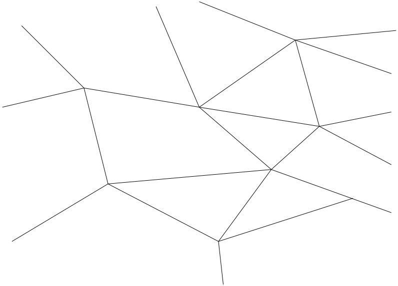


Road network $\equiv$ graph made of edges and vertices

Breakthrough in traffic monitoring

Traffic monitoring

- “old”: loop detectors at fixed locations (Eulerian)
- “new”: GPS devices moving within the traffic (Lagrangian)

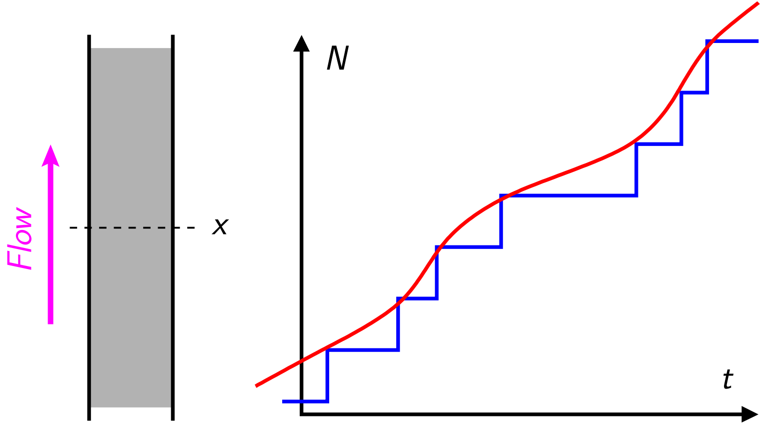
Data assimilation of Floating Car Data

[Mobile Millenium, 2008]

Outline

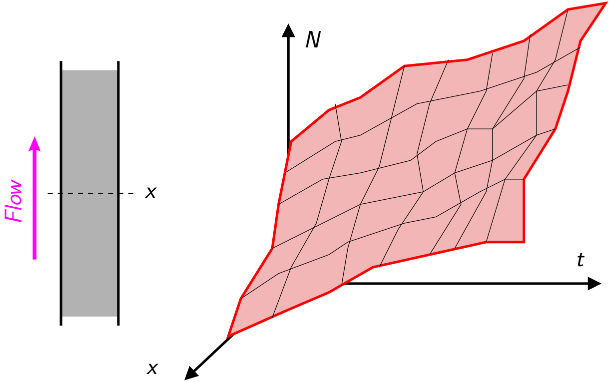
Outline

Convention for vehicle labeling



Three representations of traffic flow

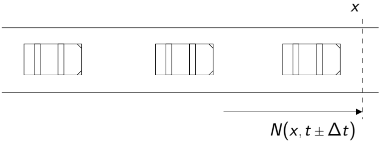
Moskowitz' surface



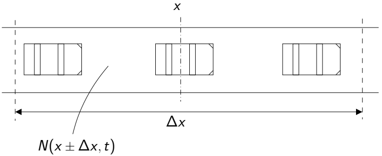
See also [Makigami et al, 1971], [Laval and Leclercq, 2013]

Notations: macroscopic

- $N(t,x)$ vehicle label at (t,x)
- the flow $Q(t,x) = \lim_{\Delta t \to 0} \dfrac{N(t+\Delta t, x) - N(t,x)}{\Delta t}$,



- the density $\rho(t,x) = \lim_{\Delta x \to 0} \dfrac{N(t,x) - N(t, x+\Delta x)}{\Delta x}$,

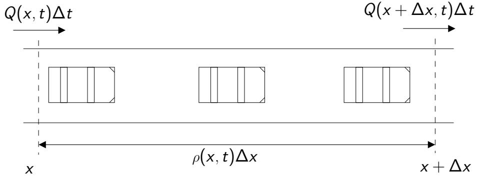


- the stream speed (mean spatial speed) $V(t,x)$.

Macroscopic models

- Hydrodynamics analogy
- Two main categories: first and second order models
- Two common equations:

$$\begin{cases} \partial_t \rho(t,x) + \partial_x Q(t,x) = 0 & \text{conservation equation} \\ \\ Q(t,x) = \rho(t,x) V(t,x) & \text{definition of flow speed} \end{cases} \quad (1)$$



First order: the LWR model

LWR model [Lighthill and Whitham, 1955], [Richards, 1956]

Scalar one dimensional conservation law

$$\partial_t \rho(t,x) + \partial_x \mathfrak{F}\left(\rho(t,x)\right) = 0 \tag{2}$$

with

$$\mathfrak{F} : \rho(t,x) \mapsto Q(t,x) =: \mathfrak{F}_x\left(\rho(t,x)\right)$$

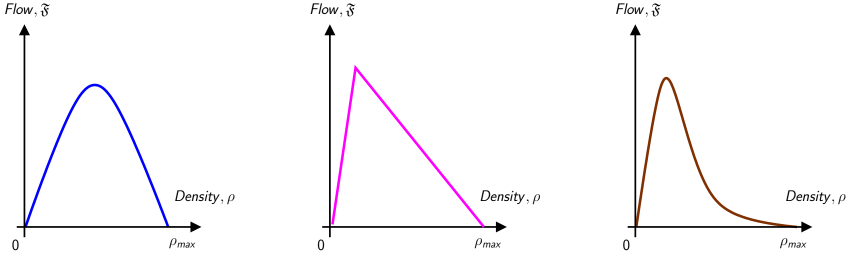
Overview: conservation laws (CL) / Hamilton-Jacobi (HJ)

		Eulerian $t - x$	Lagrangian $t - n$
CL	Variable	Density ρ	Spacing r
	Equation	$\boxed{\partial_t \rho + \partial_x \mathfrak{F}(\rho) = 0}$	$\boxed{\partial_t r + \partial_n V(r) = 0}$
HJ	Variable	Label N $N(t,x) = \int_x^{+\infty} \rho(t,\xi) d\xi$	Position $\mathcal{X}$ $\mathcal{X}(t,n) = \int_n^{+\infty} r(t,\eta) d\eta$
	Equation	$\boxed{\partial_t N + H(\partial_x N) = 0}$	$\boxed{\partial_t \mathcal{X} + \mathcal{V}(\partial_n \mathcal{X}) = 0}$
	Hamiltonian	$H(p) = -\mathfrak{F}(-p)$	$\mathcal{V}(p) = -V(-p)$

Fundamental diagram (FD)

Flow-density fundamental diagram $\mathfrak{F}$

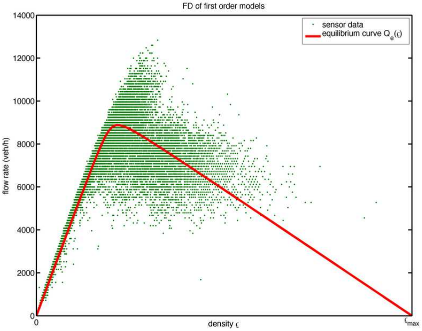
- Empirical function with
  - ρ_{max} the maximal or jam density,
  - ρ_c the critical density
- Flux is increasing for $\rho \leq \rho_c$: free-flow phase
- Flux is decreasing for $\rho \geq \rho_c$: congestion phase

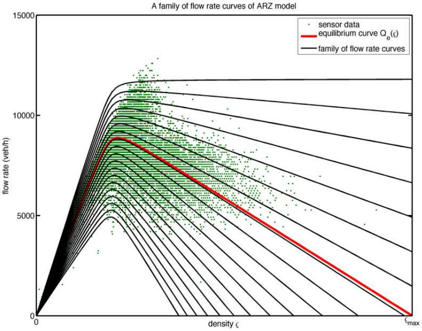


[Garavello and Piccoli, 2006]

Motivation for higher order models

- Experimental evidences
  - fundamental diagram: multi-valued in congested case





[S. Fan, U. Illinois], NGSIM dataset

Motivation for higher order models

- Experimental evidences
  - fundamental diagram: multi-valued in congested case
  - phenomena not accounted for: bounded acceleration, capacity drop...
- Need for models able to integrate measurements of different traffic quantities (acceleration, fuel consumption, noise)

GSOM family [Lebacque, Mammar, Haj-Salem 2007]

- Generic Second Order Models (GSOM) family

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 & \text{Conservation of vehicles,} \\ \partial_t(\rho I) + \partial_x(\rho v I) = \rho\varphi(I) & \text{Dynamics of the driver attribute } I, \\ v = \mathfrak{I}(\rho, I) & \text{Fundamental diagram,} \end{cases} \tag{3}$$

- Specific driver attribute I
  - the driver aggressiveness,
  - the driver origin/destination or path,
  - the vehicle class,
  - ...
- Flow-density fundamental diagram

$$\mathfrak{F} : (\rho, I) \mapsto \rho\mathfrak{I}(\rho, I).$$

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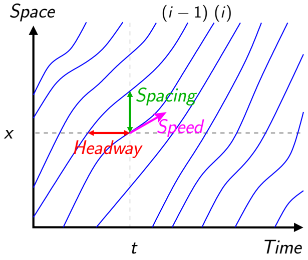
Outline

Setting

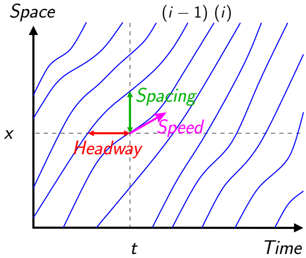


- $t \mapsto x_i(t)$ trajectory of vehicle i
- $i =$ discrete position index ($i \in \mathbb{Z}$)
- $n =$ continuous (Lagrangian) variable

$$n = i\varepsilon \quad \text{and} \quad t = \varepsilon s$$

- $\varepsilon > 0$ a scale factor

Setting



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- $\varepsilon > 0$ a scale factor

Proposition (Rescaled positions)

Define

$$x_i(s) = \frac{1}{\varepsilon} X^\varepsilon(\varepsilon s, i\varepsilon) \iff X^\varepsilon(t, n) = \varepsilon x_{\lfloor \frac{n}{\varepsilon} \rfloor}\left(\frac{t}{\varepsilon}\right)$$

General result

Let consider

- the simplest microscopic model;

$$\dot{x}_i(t) = F\left(x_{i-1}(t) - x_i(t)\right) \tag{4}$$

- the LWR macroscopic model (HJ equation in Lagrangian):

$$\partial_t X^0 = F(-\partial_n X^0) \tag{5}$$

General result

Let consider

- the simplest microscopic model;

$$\dot{x}_i(t) = F\left(x_{i-1}(t) - x_i(t)\right) \tag{4}$$

- the LWR macroscopic model (HJ equation in Lagrangian):

$$\partial_t X^0 = F(-\partial_n X^0) \tag{5}$$

Theorem ((Monneau) Convergence to the viscosity solution)

If $X^\varepsilon(t,n) := \varepsilon x_{\lfloor \frac{n}{\varepsilon} \rfloor}\left(\frac{t}{\varepsilon}\right)$ with $(x_i)_{i\in\mathbb{Z}}$ solution of (4) and X^0 the unique solution of HJ (5), then under suitable assumptions,

$$|X^\varepsilon - X^0|_{L^\infty(\mathcal{K})} \underset{\varepsilon\to 0}{\longrightarrow} 0, \qquad \forall \mathcal{K} \text{ compact set.}$$

Toy model

- Vehicles consider $m \geq 1$ leaders
- First order multi-anticipative model

$$\dot{x}_i(t+\tau) = \max \left[0, V_{max} - \sum_{j=1}^{m} f\left(x_{i-j}(t) - x_i(t)\right) \right] \qquad (6)$$

- f speed-spacing function
  - non-negative
  - non-increasing

Homogenization

Proposition ((Monneau) Convergence)

Assume $m \geq 1$ *fixed.*

If τ *is small enough, if* $X^\varepsilon(t,n) := \varepsilon x_{\lfloor \frac{n}{\varepsilon} \rfloor}\left(\frac{t}{\varepsilon}\right)$ *with* $(x_i)_{i \in \mathbb{Z}}$ *solution of (6) and if* $X^0(n,t)$ *solves*

$$\partial_t X^0 = F\left(-\partial_n X^0, m\right) \tag{7}$$

with

$$F(r,m) = \max\left[0, V_{max} - \sum_{j=1}^{m} f(jr)\right],$$

then under suitable assumptions,

$$|X^\varepsilon - X^0|_{L^\infty(\mathcal{K})} \underset{\varepsilon \to 0}{\longrightarrow} 0, \qquad \forall \mathcal{K} \text{ compact set.}$$

Multi-anticipatory macroscopic model

- χ_j the fraction of j-anticipatory vehicles
- traffic flow $\equiv$ mixture of traffic of j-anticipatory vehicles

$$\chi = (\chi_j)_{j=1,\ldots,m}, \quad \text{with} \quad 0 \leq \chi_j \leq 1 \quad \text{and} \quad \sum_{j=1}^{m} \chi_j = 1$$

- GSOM model with driver attribute $I = \chi$

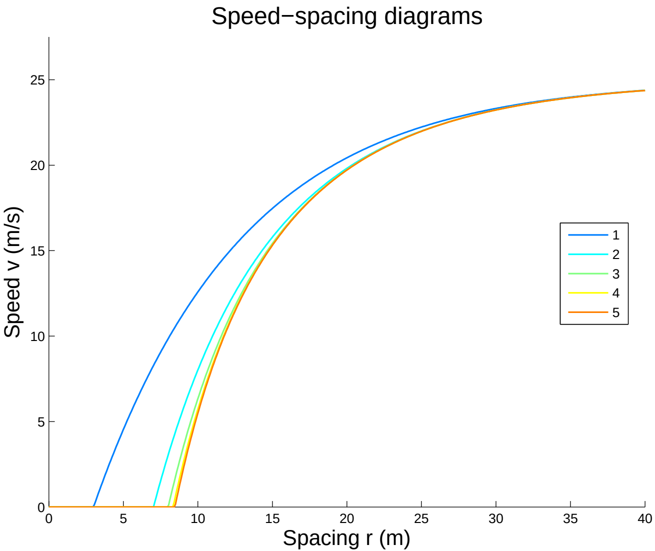
$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = 0, \\ \partial_t (\rho \chi) + \partial_x (\rho \chi v) = 0, \\ v := \displaystyle\sum_{j=1}^{m} \chi_j F(1/\rho, j) = W(1/\rho, \chi). \end{cases} \tag{8}$$

Numerical resolution

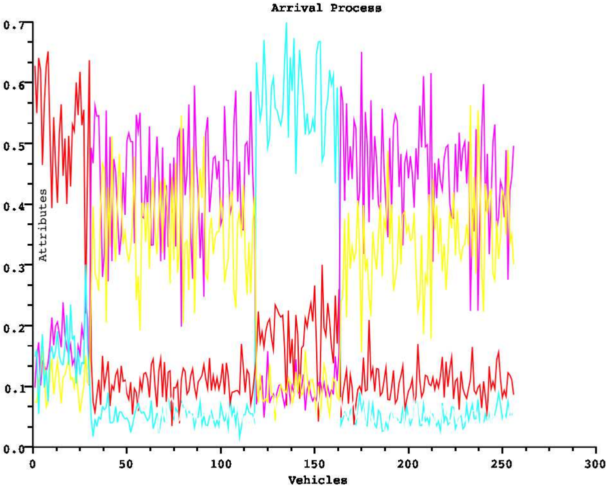
- Godunov scheme in Eulerian $t - x$
  with $(\Delta t, \Delta x_k)$ steps $\Rightarrow$ CFL condition
- Variational formulation and dynamic programming techniques [2]
- Particle methods in the Lagrangian framework $t - n$

$$\begin{cases} x_n^{t+1} = x_n^t + \Delta t W\left(\dfrac{x_{n-1}^t - x_n^t}{\Delta n}, \chi_n^t\right) \\ \chi_n^{t+1} = \chi_n^t \end{cases} \quad (9)$$

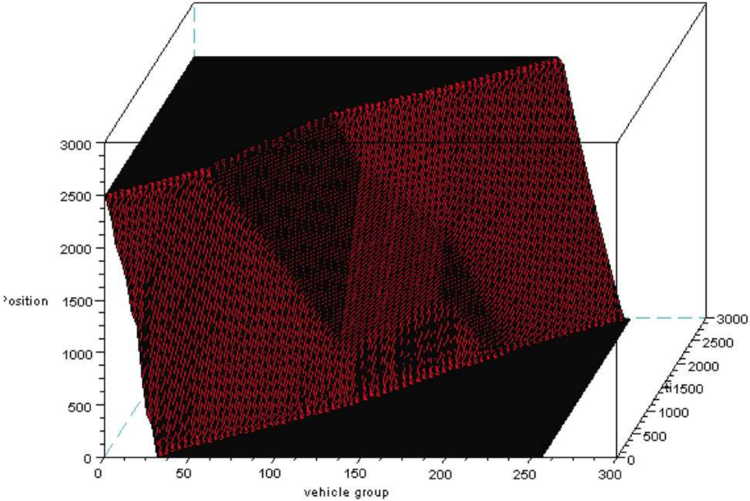
Numerical example



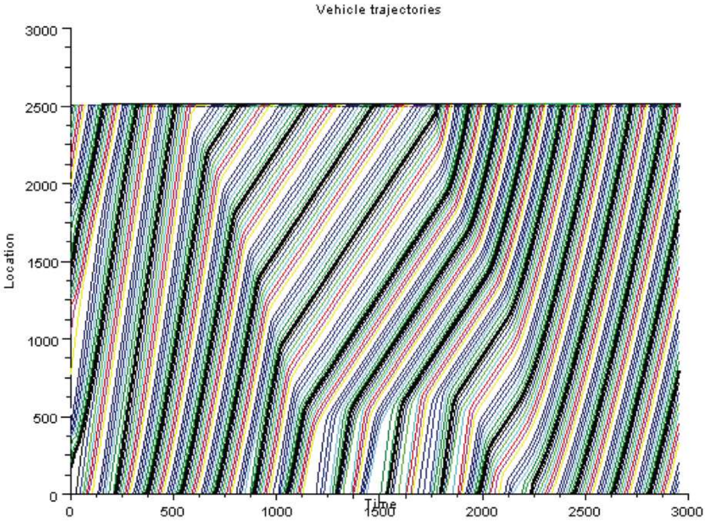
Numerical example: $(\chi_j)_j$



Numerical example: Lagrangian trajectories



Numerical example: Eulerian trajectories



Outline

LWR in Eulerian (t, x)

- Cumulative vehicles count (CVC) or Moskowitz surface $N(t, x)$

$$Q = \partial_t N \quad \text{and} \quad \rho = -\partial_x N$$

- If density ρ satisfies the scalar (LWR) conservation law

$$\partial_t \rho + \partial_x \mathfrak{F}(\rho) = 0$$

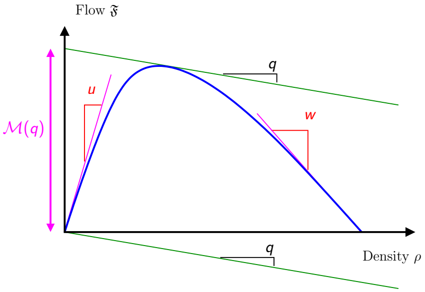
- Then N satisfies the first order Hamilton-Jacobi equation

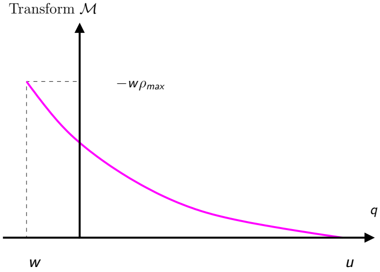
$$\partial_t N - \mathfrak{F}(-\partial_x N) = 0 \tag{10}$$

LWR in Eulerian (t, x)

- Legendre-Fenchel transform with $\mathfrak{F}$ concave (*relative capacity*)

$$\mathcal{M}(q) = \sup_{\rho} \; [\mathfrak{F}(\rho) - \rho q]$$

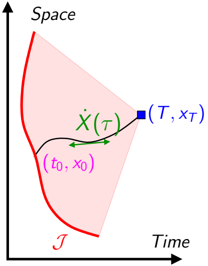




LWR in Eulerian (t, x)
(continued)

- Lax-Hopf formula (representation formula) [Daganzo, 2006]

$$N(T, x_T) = \min_{u(.),(t_0,x_0)} \int_{t_0}^{T} \mathcal{M}(u(\tau))d\tau + N(t_0, x_0),$$
$$\left| \begin{array}{l} \dot{X} = u \\ u \in \mathcal{U} \\ X(t_0) = x_0, \quad X(T) = x_T \\ (t_0, x_0) \in \mathcal{J} \end{array} \right. \tag{11}$$

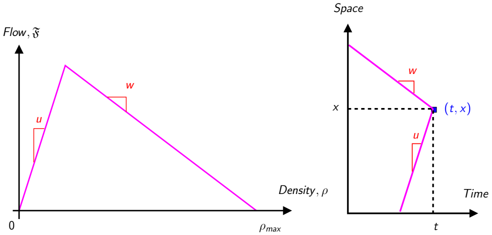


- Viability theory [Claudel and Bayen, 2010]

LWR in Eulerian (t, x)

(Historical note)

- Dynamic programming [Daganzo, 2006] for triangular FD (u and w free and congested speeds)



- Minimum principle [Newell, 1993]

$$N(t,x) = \min\left[N\left(t - \frac{x - x_u}{u}, x_u\right), \right. \\ \left. N\left(t - \frac{x - x_w}{w}, x_w\right) + \rho_{max}(x_w - x)\right], \quad (12)$$

LWR in Lagrangian (n, t)

- Consider $X(t, n)$ the location of vehicle n at time $t \geq 0$

$$v = \partial_t X \quad \text{and} \quad r = -\partial_n X$$

- If the spacing $r := 1/\rho$ satisfies the LWR model (Lagrangian coord.)

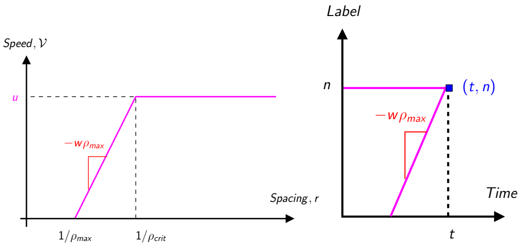
$$\partial_t r + \partial_n \mathcal{V}(r) = 0$$

- Then X satisfies the first order Hamilton-Jacobi equation

$$\partial_t X - \mathcal{V}(-\partial_n X) = 0. \tag{13}$$

LWR in Lagrangian (n, t)
(continued)

- Dynamic programming for triangular FD



- Minimum principle $\Rightarrow$ car following model [Newell, 2002]

$$X(t,n) = \min \Big[X(t_0, n) + u(t - t_0), \quad X(t_0, n + w\rho_{max}(t - t_0)) + w(t - t_0) \Big]. \tag{14}$$

GSOM in Lagrangian (n, t)

- From [Lebacque and Khoshyaran, 2013], GSOM in Lagrangian

$$\begin{cases} \partial_t r + \partial_N v = 0 & \text{Conservation of vehicles,} \\ \partial_t I = 0 & \text{Dynamics of } I, \\ v = \mathcal{W}(N, r, t) := \mathcal{V}(r, I(N, t)) & \text{Fundamental diagram.} \end{cases} \tag{15}$$

- Position $\mathcal{X}(N, t) := \int_{-\infty}^{t} v(N, \tau) d\tau$ satisfies the HJ equation

$$\partial_t \mathcal{X} - \mathcal{W}(N, -\partial_N \mathcal{X}, t) = 0, \tag{16}$$

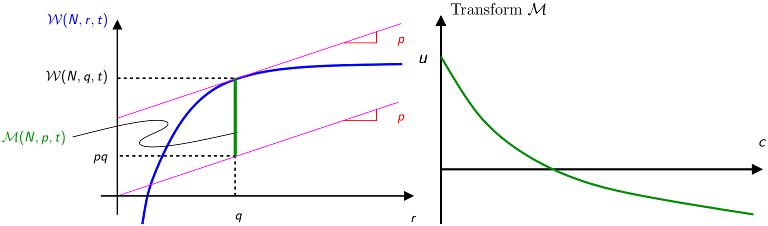
- And $I(N, t)$ solves the ODE

$$\left| \begin{array}{l} \partial_t I(N, t) = 0, \\ I(N, 0) = i_0(N), \qquad \text{for any} \quad N. \end{array} \right.$$

GSOM in Lagrangian (n, t)
(continued)

- Legendre-Fenchel transform of $\mathcal{W}$ according to r

$$\mathcal{M}(N, c, t) = \sup_{r \in \mathbb{R}} \{\mathcal{W}(N, r, t) - cr\}$$



GSOM in Lagrangian (n, t)
(continued)

- Lax-Hopf formula

$$\mathcal{X}(N_T, T) = \min_{u(.),(N_0,t_0)} \int_{t_0}^{T} \mathcal{M}(N, u, t)dt + \mathbf{c}(N_0, t_0), \tag{17}$$
$$\left| \begin{array}{l} \dot{N} = u \\ u \in \mathcal{U} \\ N(t_0) = N_0, \quad N(T) = N_T \\ (N_0, t_0) \in \mathcal{K} \end{array} \right.$$

GSOM in Lagrangian (n, t)
(continued)

- Optimal trajectories = characteristics

$$\begin{cases} \dot{N} = \partial_r \mathcal{W}(N, r, t), \\ \dot{r} = -\partial_N \mathcal{W}(N, r, t), \end{cases} \tag{18}$$

- System of ODEs to solve
- Difficulty: not straight lines in the general case

General ideas

First key element: Lax-Hopf formula

- Computations only for the characteristics

$$\mathcal{X}(N_T, T) = \min_{(N_0, r_0, t_0)} \int_{t_0}^{T} \mathcal{M}(N, \partial_r \mathcal{W}(N, r, t), t)dt + \mathbf{c}(N_0, r_0, t_0),$$
$$\left| \begin{array}{l} \dot{N}(t) = \partial_r \mathcal{W}(N, r, t) \\ \dot{r}(t) = -\partial_N \mathcal{W}(N, r, t) \\ N(t_0) = N_0, \quad r(t_0) = r_0, \quad N(T) = N_T \\ (N_0, r_0, t_0) \in \mathcal{K} \end{array} \right. \tag{19}$$

- $\mathcal{K} := \mathrm{Dom}(\mathbf{c})$ is the set of initial/boundary values

General ideas
(continued)

Second key element: inf-morphism prop. [Aubin et al, 2011]

- Consider a union of sets (initial and boundary conditions)

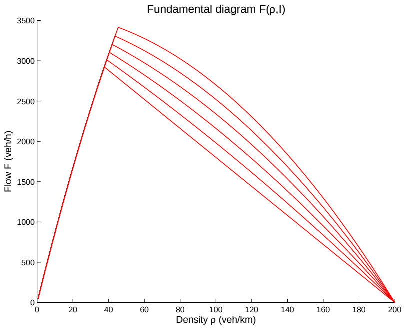
$$\mathcal{K} = \bigcup_l \mathcal{K}_l,$$

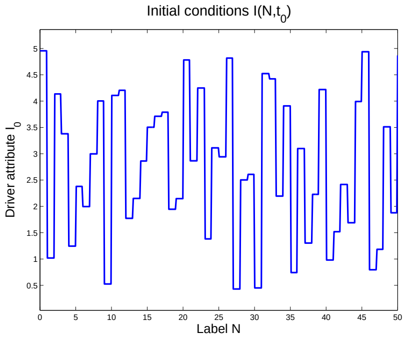
- then the global minimum is

$$\mathcal{X}(N_T, T) = \min_l \mathcal{X}_l(N_T, T), \tag{20}$$

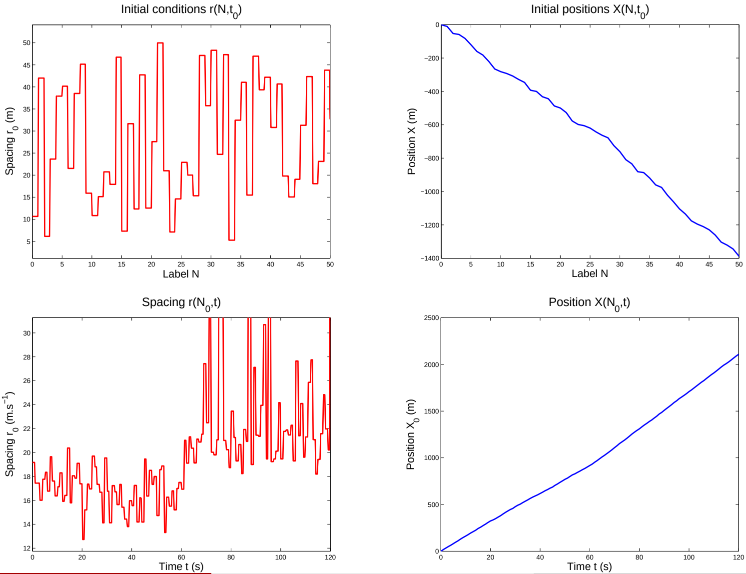
- with $\mathcal{X}_l$ partial solution to sub-problem $\mathcal{K}_l$.

Fundamental Diagram and Driver Attribute

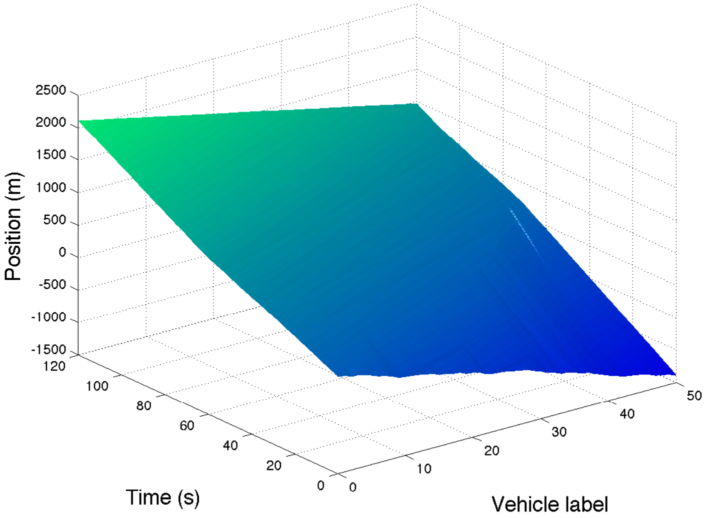




Initial and Boundaries Conditions



Numerical result (Initial cond. + first traj.)



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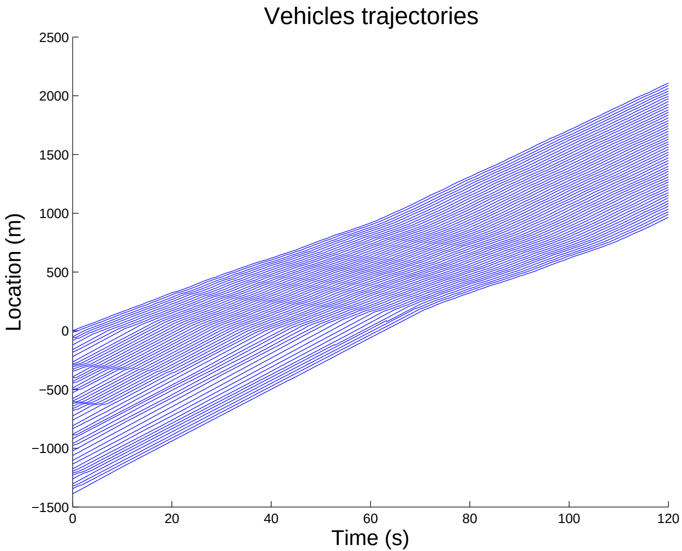
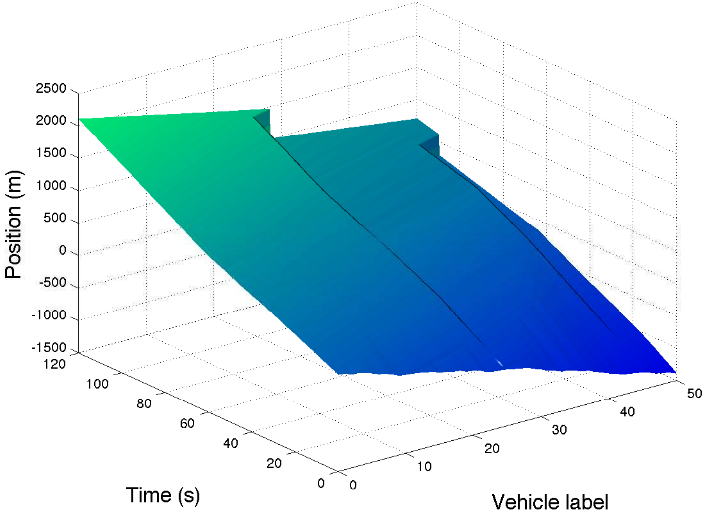
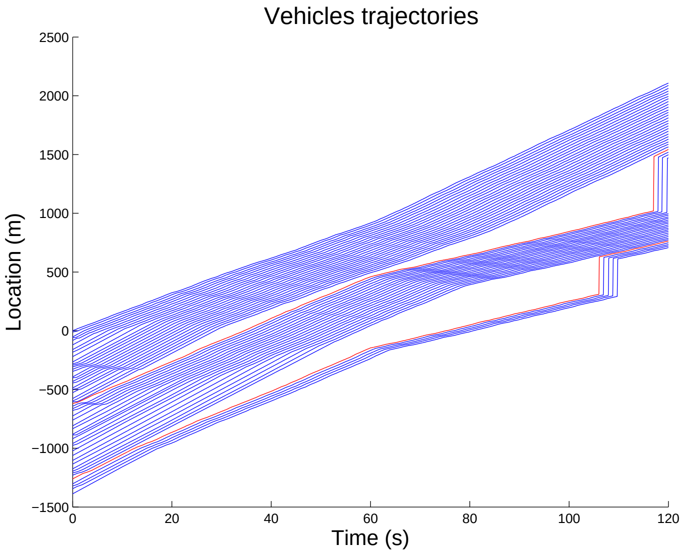


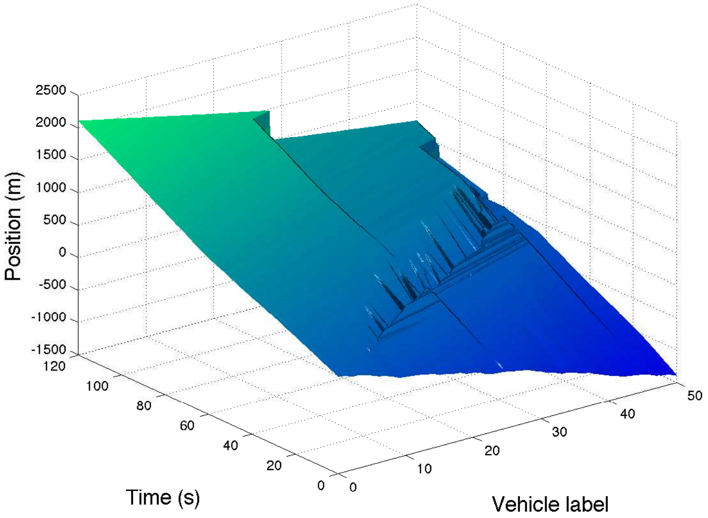
Numerical result (Initial cond.+ 3 traj.)



Numerical result (Initial cond. + 3 traj.)



Numerical result (Initial cond. + 3 traj. + Eulerian data)



Outline

Motivation

Classical approaches:

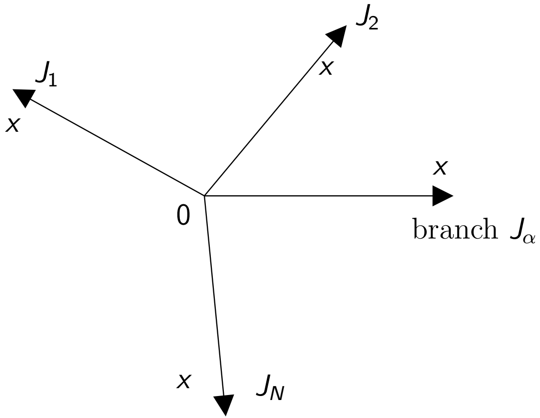
- Macroscopic modeling on (homogeneous) sections
- Coupling conditions at (pointwise) junction

For instance, consider

$$\begin{cases} \partial_t \rho + \partial_x Q(\rho) = 0, & \text{scalar conservation law,} \\ \rho(., t = 0) = \rho_0(.), & \text{initial conditions,} \\ \psi(\underbrace{\rho(x = 0^-, t)}_{\text{upstream}}, \underbrace{\rho(x = 0^+, t)}_{\text{downstream}}) = 0, & \text{coupling condition.} \end{cases} \tag{21}$$

See Garavello, Piccoli [3], Lebacque, Khoshyaran [6] and Bressan et al. [1]

Star-shaped junction



Junction model

Proposition (Junction model [IMZ, '13])

That leads to the following junction model (see [5])

$$\begin{cases} \partial_t u^\alpha + H_\alpha\left(\partial_x u^\alpha\right) = 0, & x > 0, \ \alpha = 1, \dots, N \\ u^\alpha = u^\beta =: u, & x = 0, \\ \partial_t u + \mathcal{H}\left(\partial_x u^1, \dots, \partial_x u^N\right) = 0, & x = 0 \end{cases} \tag{22}$$

with initial condition $u^\alpha(0,x) = u_0^\alpha(x)$ *and*

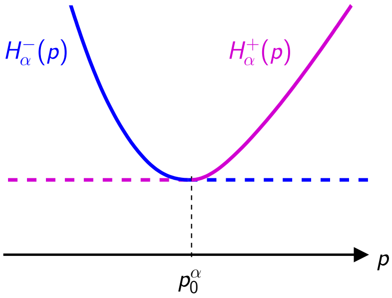
$$\mathcal{H}\left(\partial_x u^1, \dots, \partial_x u^N\right) = \underbrace{\max_{\alpha=1,\dots,N} \left\{H_\alpha^-\left(\partial_x u^\alpha\right)\right\}}_{\text{from optimal control}}.$$

Basic assumptions

For all $\alpha = 1, \ldots, N$,

(A0) The initial condition u_0^α is Lipschitz continuous.

(A1) The Hamiltonians H_α are $C^1(\mathbb{R})$ and convex such that:



Presentation of the scheme

Proposition (Numerical Scheme)

Let us consider the discrete space and time derivatives:

$$p_i^{\alpha,n} := \frac{U_{i+1}^{\alpha,n} - U_i^{\alpha,n}}{\Delta x} \quad \text{and} \quad (D_t U)_i^{\alpha,n} := \frac{U_i^{\alpha,n+1} - U_i^{\alpha,n}}{\Delta t}$$

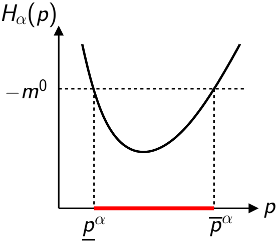
Then we have the following numerical scheme:

$$\begin{cases} (D_t U)_i^{\alpha,n} + \max\{H_\alpha^+(p_{i-1}^{\alpha,n}), H_\alpha^-(p_i^{\alpha,n})\} = 0, & i \geq 1 \\ U_0^n := U_0^{\alpha,n}, & i = 0, \quad \alpha = 1, \ldots, N \\ (D_t U)_0^n + \max_{\alpha=1,\ldots,N} H_\alpha^-(p_0^{\alpha,n}) = 0, & i = 0 \end{cases} \tag{23}$$

With the initial condition $U_i^{\alpha,0} := u_0^\alpha(i\Delta x)$.

Δx and Δt = space and time steps satisfying a CFL condition

Stronger CFL condition



As for any $\alpha = 1, \dots, N$, we have (gradient estimates)

$$\underline{p}_\alpha \leq p_i^{\alpha,n} \leq \overline{p}_\alpha \qquad \text{for all} \quad i, n \geq 0$$

Then the CFL condition becomes:

$$\frac{\Delta x}{\Delta t} \geq \sup_{\substack{\alpha=1,\dots,N \\ p_\alpha \in [\underline{p}_\alpha, \overline{p}_\alpha]}} |H'_\alpha(p_\alpha)| \qquad (24)$$

Existence and uniqueness

Theorem (Existence and uniqueness [IMZ, '13])

Under (A0)-(A1), there exists a unique viscosity solution u of (22) on the junction, satisfying for some constant $C_T > 0$

$$|u(t,y) - u_0(y)| \leq C_T \quad \text{for all} \quad (t,y) \in J_T.$$

Moreover the function u is Lipschitz continuous with respect to (t,y).

Convergence

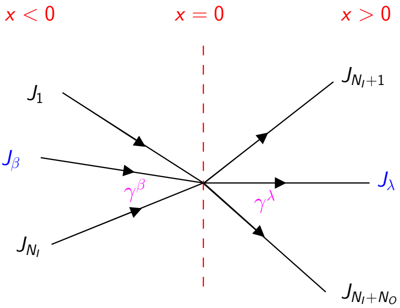
Theorem (Convergence from discrete to continuous [CML, '13])

Assume that (A0)-(A1) and the CFL condition (24) are satisfied. Then the numerical solution converges uniformly to u the unique viscosity solution of the junction model (22) when $\varepsilon := (\Delta t, \Delta x) \to 0$

$$\limsup_{\varepsilon \to 0} \sup_{(n\Delta t, i\Delta x) \in \mathcal{K}} \left| u^\alpha(n\Delta t, i\Delta x) - U_i^{\alpha,n} \right| = 0$$

▸ Proof

Setting



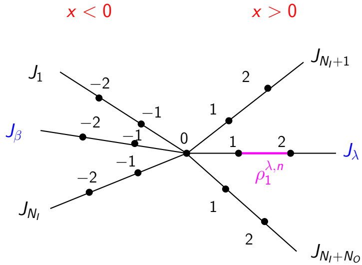
N_I incoming and N_O outgoing roads

Links with "classical" approach

Definition (Discrete car density)

The discrete vehicle density $\rho_i^{\alpha,n}$ *with* $n \geq 0$ *and* $i \in \mathbb{Z}$ *is given by:*

$$\rho_i^{\alpha,n} := \begin{cases} \gamma^\alpha p_{|i|-1}^{\alpha,n} & \text{for} \quad \alpha = 1, \dots, N_I, \quad i \leq -1 \\ -\gamma^\alpha p_i^{\alpha,n} & \text{for} \quad \alpha = N_I + 1, \dots, N_I + N_O, \quad i \geq 0 \end{cases} \tag{25}$$



Traffic interpretation

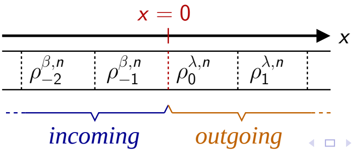
Proposition (Scheme for vehicles densities)

The scheme deduced from (23) for the discrete densities is given by:

$$\frac{\Delta x}{\Delta t}\{\rho_i^{\alpha,n+1} - \rho_i^{\alpha,n}\} = \begin{cases} F^\alpha(\rho_{i-1}^{\alpha,n}, \rho_i^{\alpha,n}) - F^\alpha(\rho_i^{\alpha,n}, \rho_{i+1}^{\alpha,n}) & \text{for } i \neq 0, -1 \\ F_0^\alpha(\rho_0^{\cdot,n}) - F^\alpha(\rho_i^{\alpha,n}, \rho_{i+1}^{\alpha,n}) & \text{for } i = 0 \\ F^\alpha(\rho_{i-1}^{\alpha,n}, \rho_i^{\alpha,n}) - F_0^\alpha(\rho_0^{\cdot,n}) & \text{for } i = -1 \end{cases}$$

With

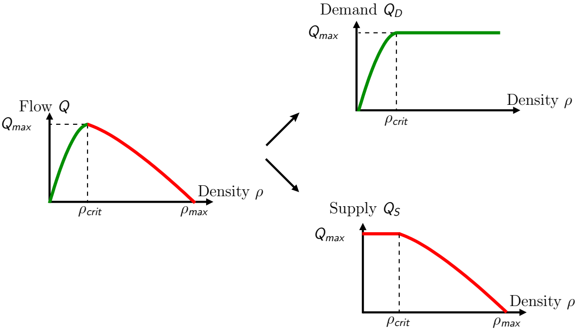
$$\begin{cases} F^\alpha(\rho_{i-1}^{\alpha,n}, \rho_i^{\alpha,n}) := \min\left\{ Q_D^\alpha(\rho_{i-1}^{\alpha,n}), \; Q_S^\alpha(\rho_i^{\alpha,n}) \right\} \\ F_0^\alpha(\rho_0^{\cdot,n}) := \gamma^\alpha \min\left\{ \min\limits_{\beta \le N_I} \frac{1}{\gamma^\beta} Q_D^\beta(\rho_0^{\beta,n}), \; \min\limits_{\lambda > N_I} \frac{1}{\gamma^\lambda} Q_S^\lambda(\rho_0^{\lambda,n}) \right\} \end{cases}$$



Supply and demand functions

Remark

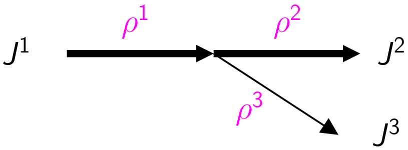
It recovers the seminal Godunov scheme with passing flow = minimum between upstream demand Q_D and downstream supply Q_S.



From [Lebacque '93, '96] and [Daganzo '95]

Example of a Diverge

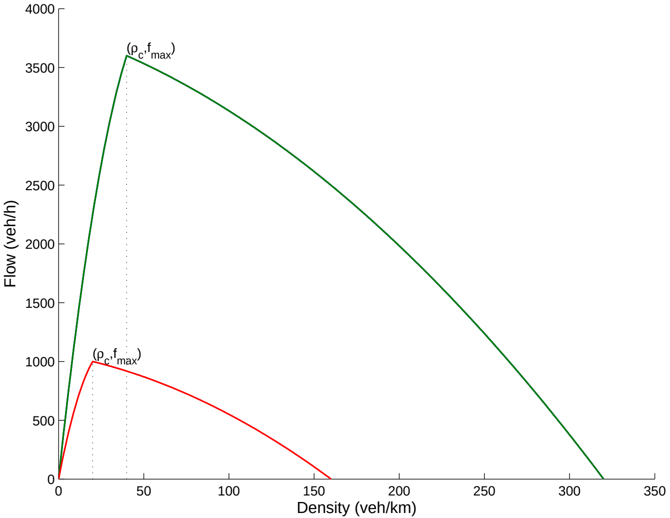
An off-ramp:



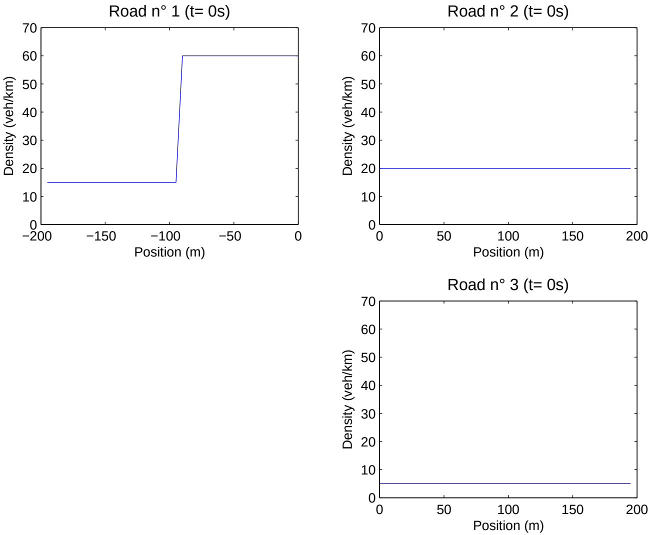
with

$$\begin{cases} \gamma^e = 1, \\ \gamma^l = 0.75, \\ \gamma^r = 0.25 \end{cases}$$

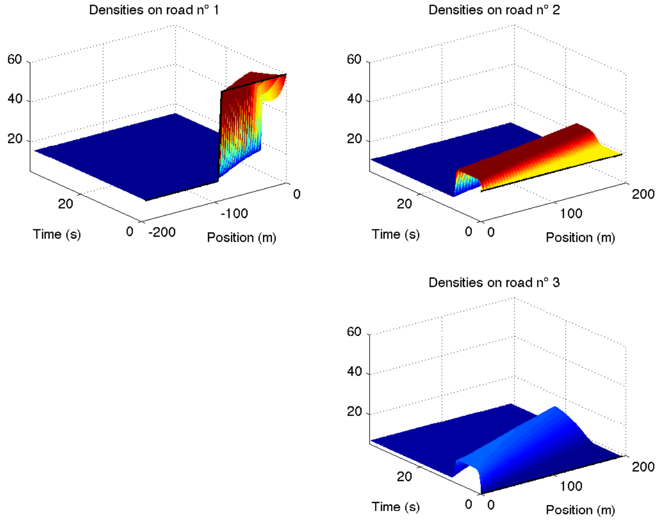
Fundamental Diagrams



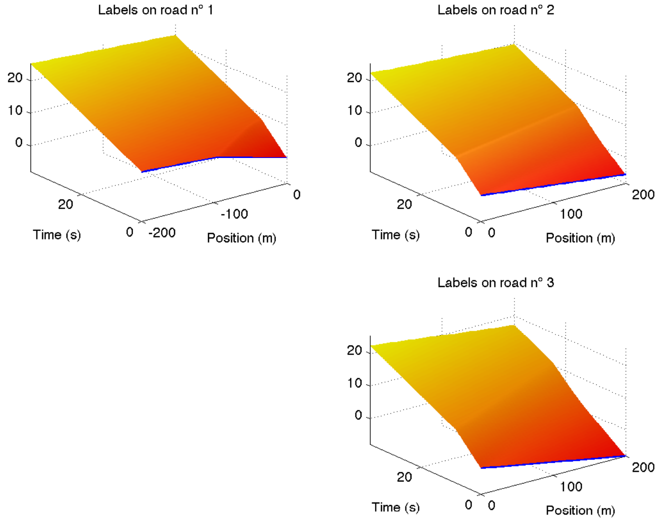
Initial conditions (t=0s)



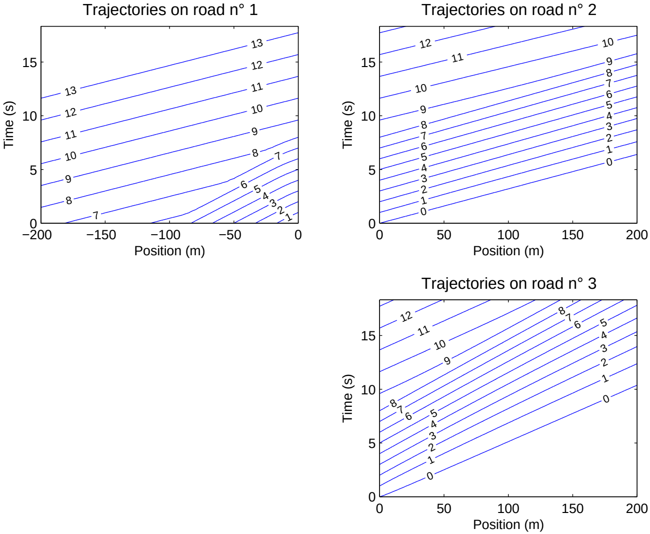
Numerical solution: densities



Numerical solution: Hamilton-Jacobi



Trajectories



New junction model

Proposition (Junction model [IM, '14])

From [4], we have

$$\begin{cases} \partial_t u^\alpha + H_\alpha\left(\partial_x u^\alpha\right) = 0, & x > 0,\ \alpha = 1, \dots, N \\ u^\alpha = u^\beta =: u, & x = 0, \\ \partial_t u + \mathcal{H}\left(\partial_x u^1, \dots, \partial_x u^N\right) = 0, & x = 0 \end{cases} \tag{26}$$

with initial condition $u^\alpha(0, x) = u_0^\alpha(x)$ and

$$\mathcal{H}\left(\partial_x u^1, \dots, \partial_x u^N\right) = \max\Bigg[\overbrace{\mathcal{L}}^{\text{flux limiter}}, \underbrace{\max_{\alpha=1,\dots,N}\left\{H_\alpha^-\left(\partial_x u^\alpha\right)\right\}}_{\substack{\text{minimum between}\\ \text{demand and supply}}}\Bigg].$$

Weaker assumptions on the Hamiltonians

For all $\alpha = 1, \dots, N$,
(A0) The initial condition u_0^α is Lipschitz continuous.
(A1) The Hamiltonians H_α are continuous and quasi-convex i.e. there exists points p_0^α such that

$$\begin{cases} H_\alpha \quad \text{is non-increasing on} \quad (-\infty, p_0^\alpha], \\ \\ H_\alpha \quad \text{is non-decreasing on} \quad [p_0^\alpha, +\infty). \end{cases}$$

Homogenization on a network

Proposition (Homogenization on a periodic network [IM'14])

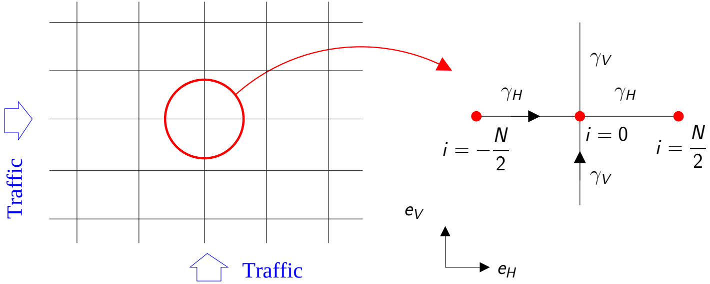
Assume (A0)-(A1). Consider a periodic network.
If $u^\varepsilon \in \mathbb{R}^d$ satisfies HJ equation on network,
then u^ε converges uniformly towards u^0 when $\varepsilon \to 0$,
with $u^0 \in \mathbb{R}^d$ solution of

$$\partial_t u^0 + \overline{H}\left(Du^0\right) = 0, \quad t > 0, \ x \in \mathbb{R}^d \tag{27}$$

See [Imbert, Monneau '14] [4]

Numerical homogenization on a network

Numerical scheme adapted to the cell problem $(d = 2)$



First example

Proposition (Effective Hamiltonian for fixed coefficients [IM'14])

If (γ^H, γ^V) are fixed, then the

- *(Hamiltonian) effective Hamiltonian $\overline{H}$ is given by*

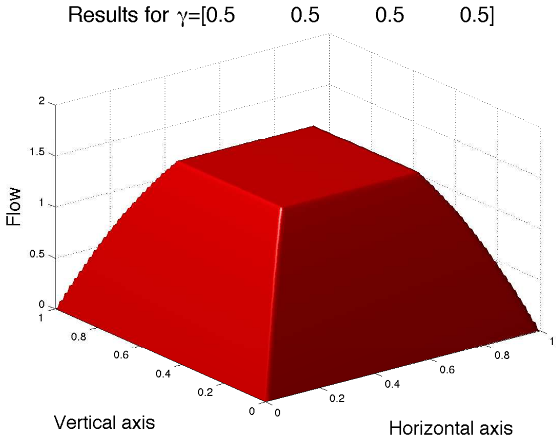
$$\overline{H}(\partial_x u_H, \partial_x u_V) = \max\left\{\mathcal{L}, \max_{i=\{H,V\}} H(\partial_x u_i)\right\},$$

- *(traffic flow) effective flow $\overline{Q}$ is given by*

$$\overline{\mathfrak{F}}(\rho_H, \rho_V) = \min\left\{-\mathcal{L}, \frac{\mathfrak{F}(\rho_H)}{\gamma^H}, \frac{\mathfrak{F}(\rho_V)}{\gamma^V}\right\}.$$

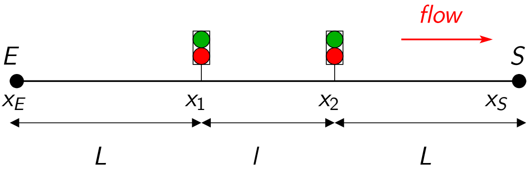
First example

Numerics: assume $\mathfrak{F}(\rho) = 4\rho(1-\rho)$ and $\mathcal{L} = -1.5$,



Second example

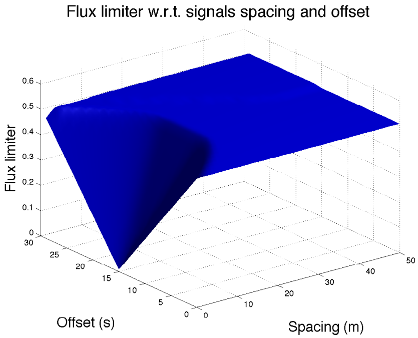
Two consecutive traffic signals on a 1D road

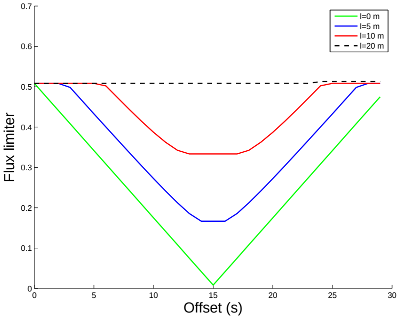


Homogenization theory by [Galise, Imbert, Monneau, '14]

Second example

Effective flux limiter $-\overline{\mathcal{L}}$ (numerics only)





Outline

Personal conclusions

	Advantages	**Drawbacks**
Homogeneous link	Micro-macro link Explicit solutions Data assimilation	Limited time delay Concavity of the FD Exactness (2nd order) Multilane
Junction	Uniqueness of the solution Homogenization result	Fixed proportions Multilane

Perspectives

Some open questions:

- Micro-macro: higher time delay?
- Confront the results with real data (micro datasets)
- Explicit Lax-Hopf formula for time/space dependent Hamiltonians?

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THANKS FOR YOUR ATTENTION

Any question?

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