A new solver for the ARZ traffic flow model on a junction

Guillaume Costeseque
collaboration with Paola Goatin, Simone Göttlich and Oliver Kolb

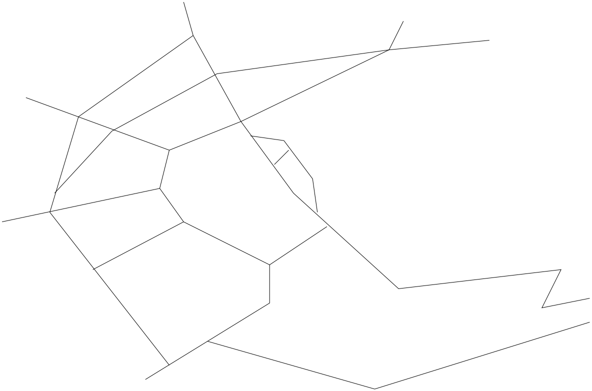
Inria Sophia-Antipolis Méditerranée

The finite volumes schemes and traffic modeling
Besançon, France – November 22, 2017

Traffic flows on a network

Besançon network [Google maps, Nov. 16, 2017]

Traffic flows on a network



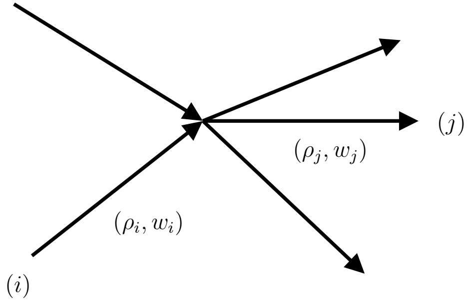
Road network ≡ graph made of edges and vertices

Outline

Outline

ARZ model on a junction

n incoming and m outgoing roads ($n, m \geq 1$)



ARZ model on a junction
(continued)

- ARZ model [1, 10] on each branch (i)

$$\begin{cases} \partial_t \rho_i + \partial_x (\rho_i v_i) = 0, \\ \partial_t (\rho_i w_i) + \partial_x (\rho_i v_i w_i) = 0, \\ w_i := v_i + p_i(\rho_i) \end{cases} \tag{1}$$

+ initial conditions (Riemann problem)

ARZ model on a junction
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+ initial conditions (Riemann problem)

- Coupling conditions needed to ensure conservation of
  - Mass flow $q = \rho v$
  - Momentum flow $qw = \rho v w$

  through the junction

Problem statement

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- What are we looking for?

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- What are we looking for?
  Well-posedness of Riemann solvers at the junction

Outline

Common assumptions

(A1) Conservation of the fluxes: $\displaystyle\sum_{i=1}^{n} \underbrace{\rho_i v_i}_{=:q_i} = \sum_{j=n+1}^{n+m} \underbrace{\rho_j v_j}_{=:q_j}$

(A2) Fixed assignment coefficients:

$$\exists\ (\alpha_{ji})_{i,j} \in [0,1], \text{ s.t. } \sum_{j=n+1}^{n+m} \alpha_{ji} = 1 \quad \text{and} \quad q_j = \sum_{i=1}^{n} \alpha_{ji} q_i$$

(A3) Bounds on the fluxes

$$\begin{cases} 0 \leq q_i \leq \Delta_i, & i = 1, \dots, n, \\ 0 \leq q_j \leq \Sigma_j, & j = n+1, \dots, n+m, \end{cases}$$

Δ_i demand and Σ_j supply

Common assumptions

(A4) Maximization of the total incoming fluxes:

$$\max \sum_{i=1}^{n} q_i$$

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Literature:

- ARZ model
  - Garavello-Piccoli [4]
  - Herty-Rascle [7]
  - Herty-Moutari-Rascle [6]
  - Haut-Bastin [5]
- Phase Transition model
  - Colombo, Goatin, Piccoli [2]
  - Garavello, Marcellini [3]
- Engineering community: Lebacque's works [9, 8]

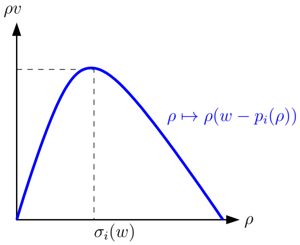
Common assumptions

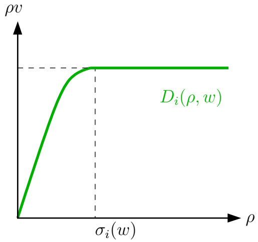
(A4) Multi-objective optimization of the incoming fluxes:

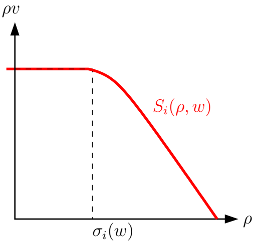
$$\max\,(q_1, \dots, q_n)$$

and for any fixed $\mathbf{P} = (P_1, \dots, P_n)$ such that $P_i \in]0, 1[$ and $\sum_{i=1}^n P_i = 1$, the ratio $\dfrac{q_l}{\sum_{i=1}^n q_i}$ is the closest to P_l

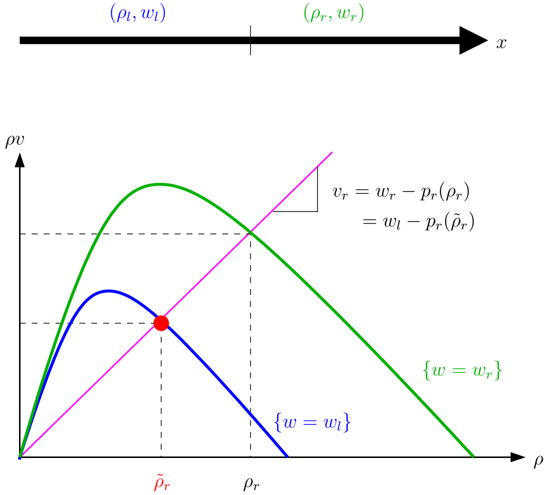
Demand and supply







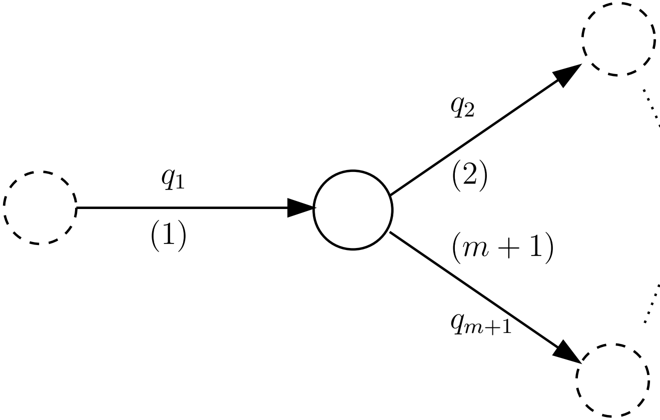
Downstream density perceived by upstream traffic



The velocity is conserved through a contact discontinuity!

Outline

RS for a 1-to-m diverge ($m \geq 1$)



RS for a 1-to-m diverge ($m \geq 1$)

- Initial states $((\rho_{1,0}, v_{1,0}), (\rho_{2,0}, v_{2,0}) \dots, (\rho_{m+1,0}, v_{m+1,0}))$
- Multi-optimization $\equiv$ optimization of the total through-flow

$$\max_{\Omega_{1\times m}} q_1$$

- Set of admissible states

$$\Omega_{1\times m} := \left\{ q_1 \in \mathbb{R} \;\middle|\; \begin{array}{c} 0 \leq q_1 \leq \Delta_1 \\ 0 \leq q_j = \alpha_{j1} q_1 \leq \Sigma_j, \ \forall j \end{array} \right\}$$

RS for a 1-to-m diverge ($m \geq 1$)

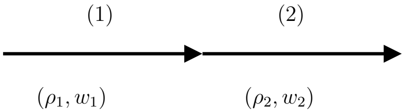
Solution for a 1-to-m diverge

$$q_1 = \min \left\{ \Delta_1, \min_{j=2,\dots,m+1} \frac{1}{\alpha_{j1}} \Sigma_j \right\},$$
$$q_j = \alpha_{j1} q_1, \qquad \forall j = 2, \dots, m+1$$

with

$$\Delta_1 = D_1 \left(\rho_{1,0}, w_1 \right)$$
$$\Sigma_j = S_j \left(p_j^{-1}(\max\{0, w_1 - v_{2,0}\}), w_1 \right), \quad \forall j = 2, \dots, m+1$$

Example of a 1×1 junction

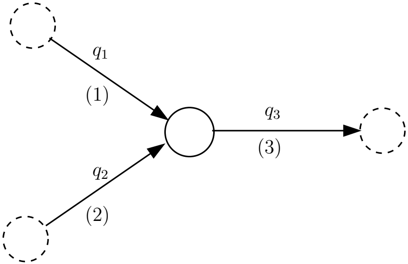


$$q_1 = q_2 = \min\left\{D_1(\rho_{1,0}, w_1)\ ,\ S_2(\tilde{\rho}_2, w_1)\right\}$$

where

$$\tilde{\rho}_2 = p_2^{-1}\left(\max\{0, w_1 - v_{2,0}\}\right)$$

Case of a 2×1 merge



$$q_3 = q_1 + q_2$$

+ initial conditions

$$((\rho_{1,0}, v_{1,0}), (\rho_{2,0}, v_{2,0}), (\rho_{3,0}, v_{3,0}))$$

Example of a 2×1 merge
(continued)

Multi-objective optimization problem

$$\max_{\Omega_{2\times 1}} (q_1, q_2)$$

with

$$\Omega_{2\times 1} = \left\{ (q_1, q_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} 0 \le q_1 \le \Delta_1, \\ 0 \le q_2 \le \Delta_2, \\ 0 \le q_3 = q_1 + q_2 \le \Sigma_3(q_1, q_2) \end{array} \right\}$$

$$\Delta_i = D_i(\rho_{i,0}, w_i), \quad i = 1, 2$$
$$\Sigma_3(q_1, q_2) = S_3\left(\tilde{\rho}_3, \tilde{w}\right)$$

Example of a 2×1 merge
(continued)

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$$\Delta_i = D_i(\rho_{i,0}, w_i), \quad i = 1, 2$$
$$\Sigma_3(q_1, q_2) = S_3\left(\tilde{\rho}_3, \tilde{w}\right)$$

$$\tilde{w} = \frac{q_1}{q_1 + q_2} w_1 + \frac{q_2}{q_1 + q_2} w_2$$
$$\tilde{\rho}_3 = p_3^{-1}\left(\max\{0, \tilde{w} - v_{3,0}\}\right)$$

First property

Proposition (Convexity of the feasible set)

The set of admissible states $\Omega_{2\times 1}$ *is non-empty and convex.*

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The set of admissible states $\Omega_{2\times 1}$ is non-empty and convex.

Sketch of the proof:

- $(0,0) \in \Omega_{2\times 1}$
- Classical convexity proof: take two points on the boundary of $\Omega_{2\times 1}$ and show that a convex combination of these two points still belongs to the set

Analysis of the supply

Assume $\exists z \in [0,1]$ such that

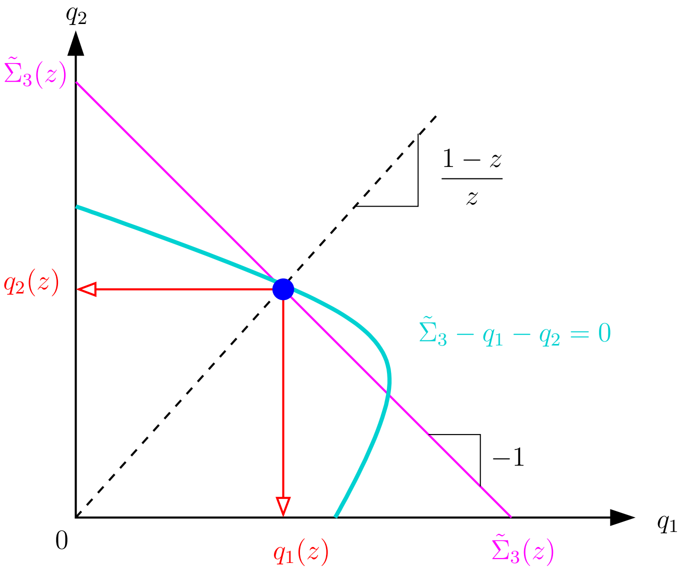
$$\begin{cases} q_1 = z(q_1 + q_2) \\ q_2 = (1-z)(q_1 + q_2) \end{cases}$$

Define

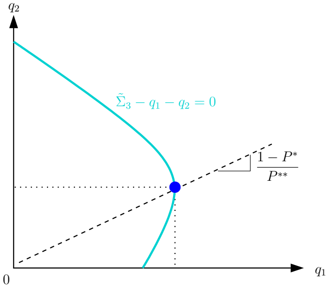
$$\tilde{\Sigma}_3(z) = \Sigma_3\left(q_1, q_2\right)$$

and set

$$\Delta w = w_1 - w_2$$


q_2
$\tilde{\Sigma}_3(z)$
$\frac{1-z}{z}$
$q_2(z)$
$\tilde{\Sigma}_3 - q_1 - q_2 = 0$
-1
0
$q_1(z)$
$\tilde{\Sigma}_3(z)$
q_1

Local optima



- <u>If $\Delta w > 0$</u>
  - P^* local minimum for $z \mapsto q_1(z) = z\tilde\Sigma_3(z)$
  - P^{**} local maximum for $z \mapsto q_2(z) = (1-z)\tilde\Sigma_3(z)$
- <u>If $\Delta w < 0$</u>
  - P^* is a local maximum for $z \mapsto q_1(z) = z\tilde\Sigma_3(z)$
  - P^{**} is a local minimum for $p \mapsto q_2(z) = (1-z)\tilde\Sigma_3(z)$

Riemann solver for the 2-to-1 merge

Algorithm

Consider given pressure functions $p_i(\rho)$ and initial conditions.

1. Fix a priority ratio $P \in]0,1[$
2. Compute
$$F(P) = \min\left\{\frac{\Delta_1}{P}, \frac{\Delta_2}{1-P}, \tilde{\Sigma}_3(P)\right\}$$
with
$$\tilde{\Sigma}_3(P) = S_3(\rho_P, \tilde{w}_P)$$
and $\tilde{w}_P = Pw_1 + (1-P)w_2$ and $\rho_P = p_3^{-1}\left(\max\{0, \tilde{w}_P - v_{3,0}\}\right)$
3. Distinguish the different cases

Set

$$\tilde{q}_1 = P\tilde{\Sigma}_3(P) \quad \text{and} \quad \tilde{q}_2 = (1-P)\tilde{\Sigma}_3(P)$$
$$q_1^* = P^*\tilde{\Sigma}_3(P^*) \quad \text{and} \quad q_2^* = (1-P^*)\tilde{\Sigma}_3(P^*)$$

1. If

$$\begin{aligned} &\Delta w = 0, && \text{or} \\ &\Delta w < 0 \quad \text{and} \quad P \leq P^*, && \text{or} \\ &\Delta w > 0 \quad \text{and} \quad P \geq P^{**}, \end{aligned} \tag{2}$$

   we choose

$$\begin{aligned} q_1 &= \min\left\{\Delta_1, \max\left\{\tilde{q}_1, \Sigma_3(q_1, q_2) - q_2\right\}\right\}, \\ q_2 &= \min\left\{\Delta_2, \max\left\{\tilde{q}_2, \Sigma_3(q_1, q_2) - q_1\right\}\right\}. \end{aligned} \tag{3}$$

2. If

$$\Delta w < 0 \quad \text{and} \quad P \geq P^*,$$

then

$$q_2 = \min\left\{\Delta_2, \max\left\{q_2^*, \Sigma_3(q_1, q_2) - q_1\right\}\right\}. \tag{4}$$

Computation of q_1:

1. If

$$F(P) = \tilde{\Sigma}_3(P) \qquad \text{and} \qquad q_2^* \leq \Delta_2,$$

we apply

$$q_1 = \min\{q_1^*, \Delta_1\}. \tag{5}$$

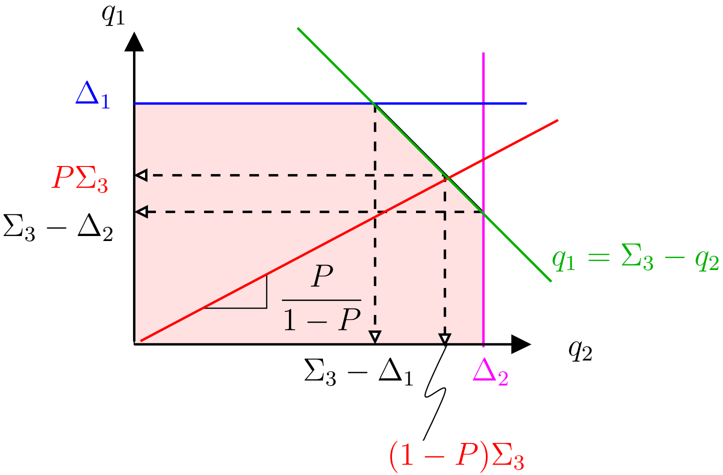
2. Otherwise

$$q_1 = \min\left\{\Delta_1, \max\left\{\tilde{q}_1, \Sigma_3(q_1, q_2) - q_2\right\}\right\}. \tag{6}$$

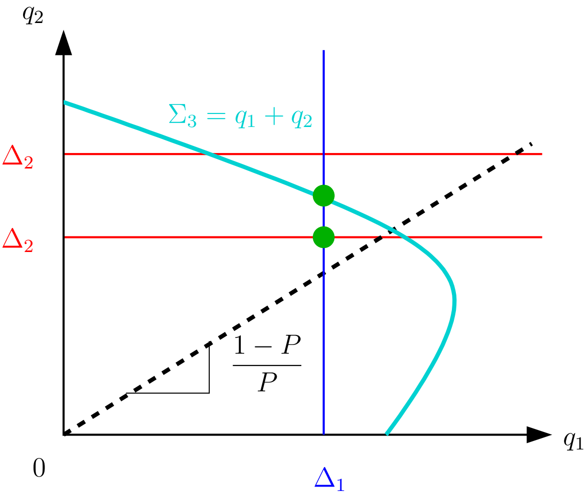
3. The case $\Delta w > 0 \quad \text{and} \quad P \leq P^{**}$ treated analogously to case 2.

Easy case $\Delta w = 0$

Σ_3 is a constant

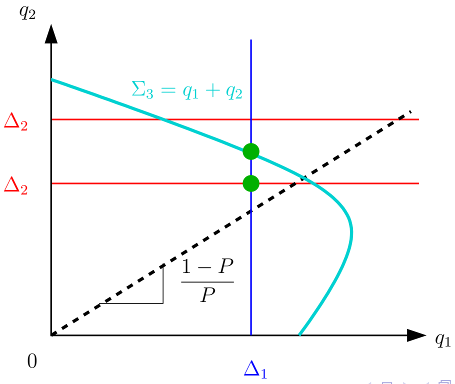


Subcase 1: $F(P) = \frac{\Delta_1}{P}$

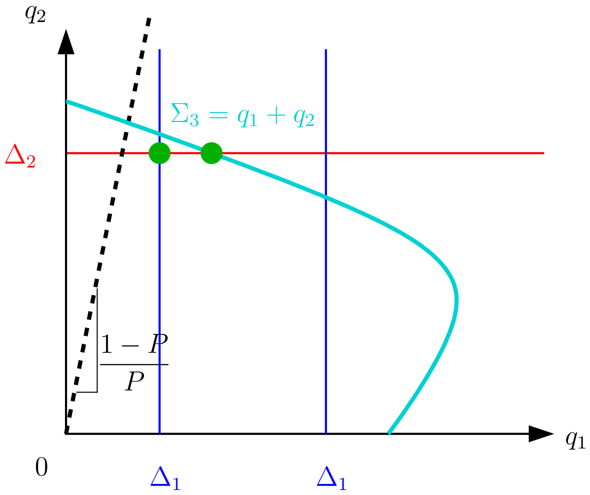


Subcase 1: $F(P) = \dfrac{\Delta_1}{P}$

$$\begin{cases} q_1 = \Delta_1 \\ q_2 = \min\left\{\Delta_2, \max\left[(1-P)\tilde{\Sigma}_3(P), \Sigma_3(\Delta_1, q_2) - \Delta_1\right]\right\} \end{cases}$$

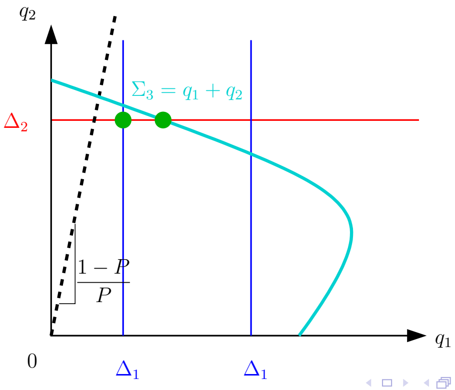


Subcase 2: $F(P) = \dfrac{\Delta_2}{1-P}$

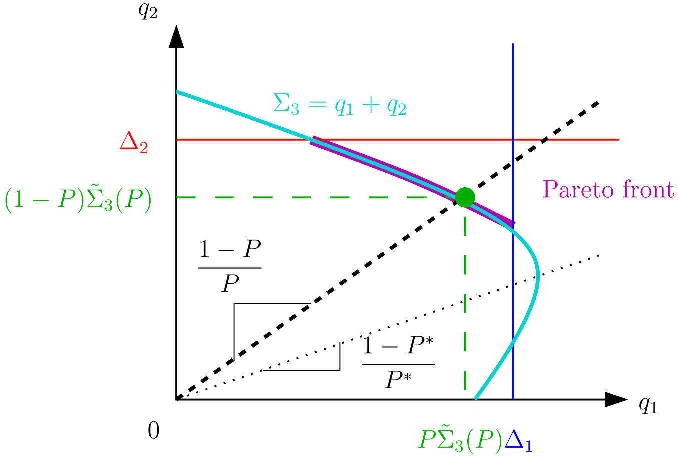


Subcase 2: $F(P) = \dfrac{\Delta_2}{1-P}$

$$\begin{cases} q_1 = \min\left\{\Delta_1, \max\left[P\tilde{\Sigma}_3(P), \Sigma_3(q_1,\Delta_2) - \Delta_2\right]\right\} \\ q_2 = \Delta_2 \end{cases}$$



Subcase 3: $F(P) = \tilde{\Sigma}_3(P)$ and $P \leq P^*$



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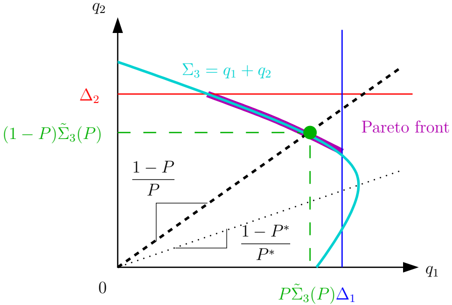
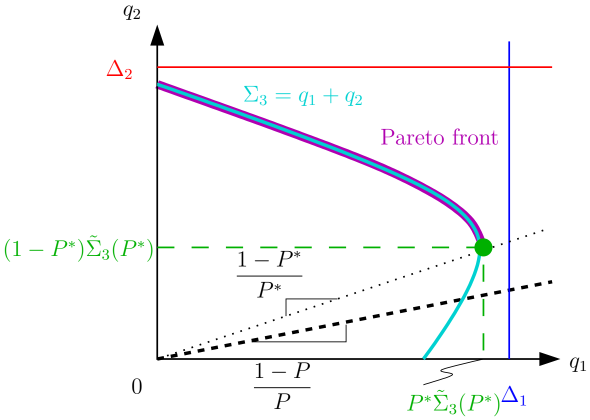


Subcase 4: $F(P) = \tilde{\Sigma}_3(P)$ and $P \geq P^*$

(a) If $q_1^* \leq \Delta_1$ and $q_2^* \leq \Delta_2$



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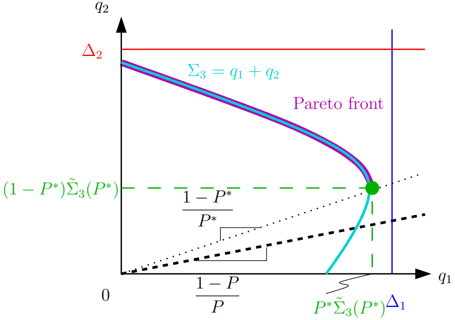
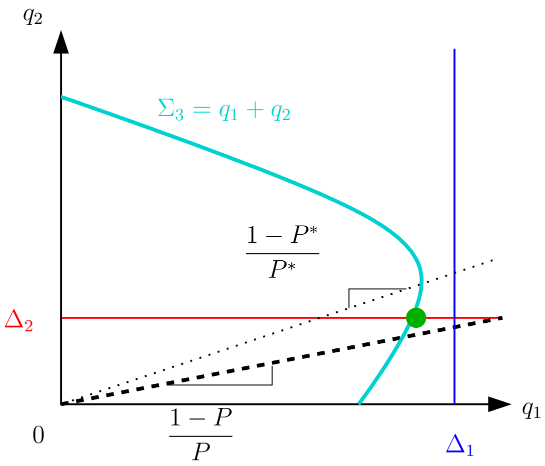


Subcase 4: $F(P) = \tilde{\Sigma}_3(P)$ and $P \geq P^*$

(b) If $q_1^* \leq \Delta_1$ and $q_2^* > \Delta_2$



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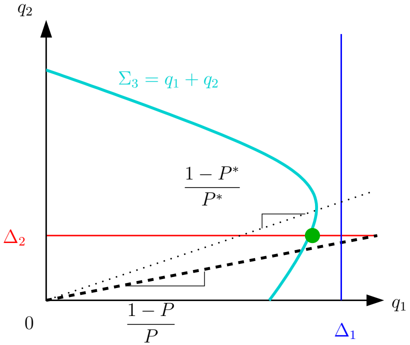
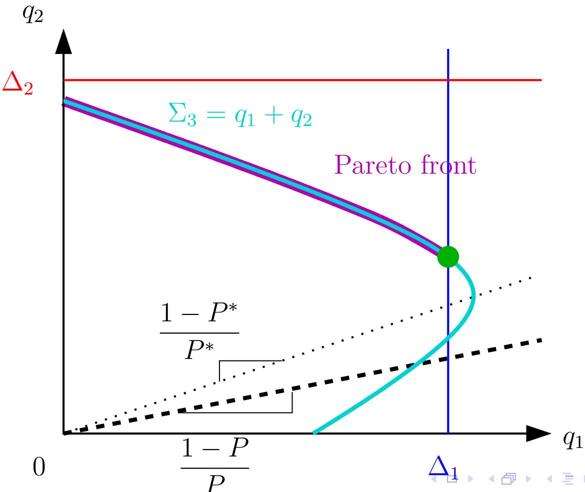
(b) If $q_1^* \leq \Delta_1$ and $q_2^* > \Delta_2$

$$\begin{cases} q_1 = \Sigma_3(q_1, \Delta_2) - \Delta_2 \\ q_2 = \Delta_2 \end{cases}$$



Subcase 4: $F(P) = \tilde{\Sigma}_3(P)$ and $P \geq P^*$

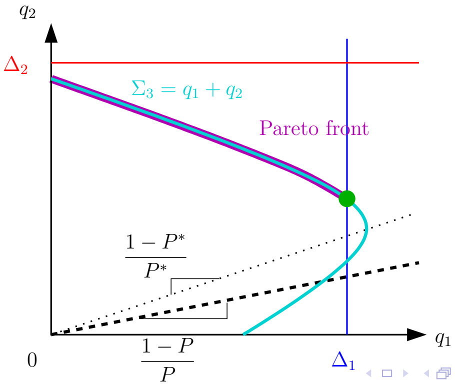
(c) If $q_1^* > \Delta_1$ and $\bar{q}_2 \leq \Delta_2$



Subcase 4: $F(P) = \tilde{\Sigma}_3(P)$ and $P \geq P^*$

(c) If $q_1^* > \Delta_1$ and $\bar{q}_2 \leq \Delta_2$

$$\begin{cases} q_1 = \Delta_1 \\ q_2 = \bar{q}_2 \quad \text{solution of} \quad q_2 = \Sigma_3(\Delta_1, q_2) - \Delta_1 \quad \text{and} \quad q_2 \geq q_2^* \end{cases}$$



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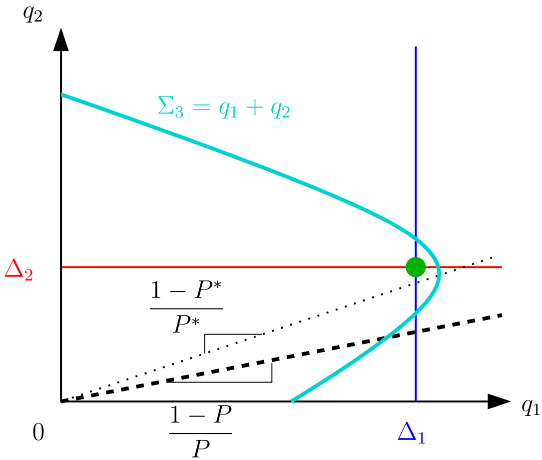
(d) If $q_1^* > \Delta_1$ and $\bar{q}_2 \geq \Delta_2 \geq \underline{q}_2$



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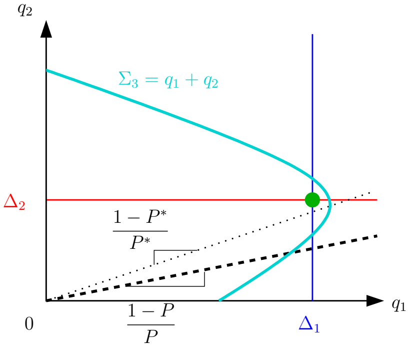

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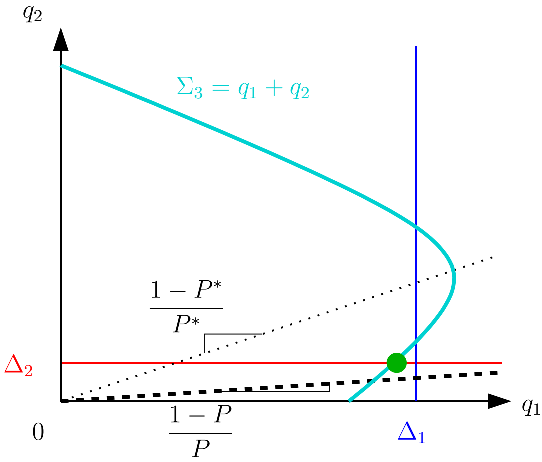

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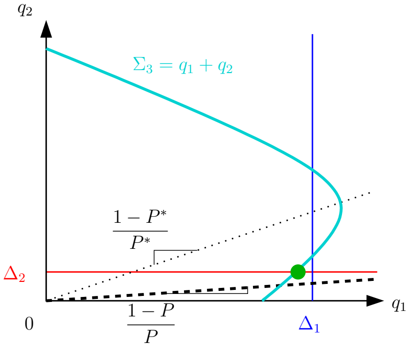

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Thanks for your attention

guillaume.costeseque@inria.fr