# Representation formula for traffic flow estimation on a network

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# HJ & Lax-Hopf formula

Hamilton-Jacobi equations: why and what for?

- Smoothness of the solution (no shocks)
- Physically meaningful quantity

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- Analytical expression of the solution
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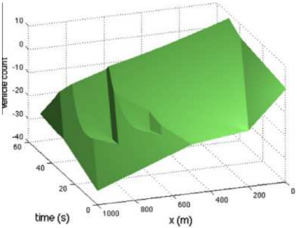


[MAZARÉ ET AL, 2012]

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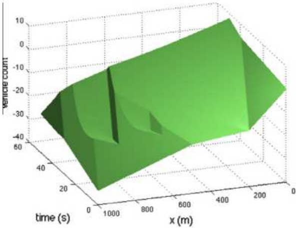
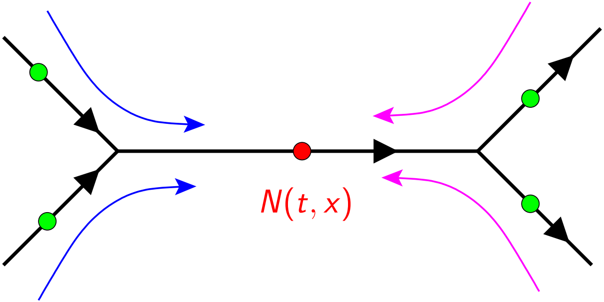



[Mazaré et al, 2012]

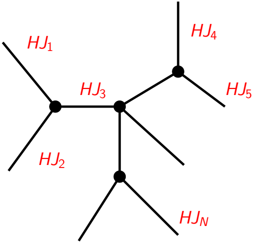
Everything broken for network applications?

# Network model

Simple case study: generalized three-detector problem (Newell (1993))



# A special network = junction




Network

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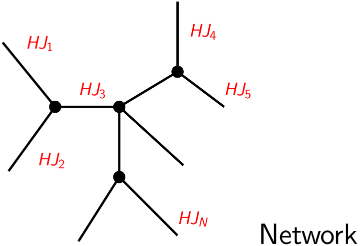


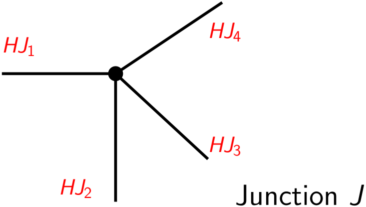

# Outline

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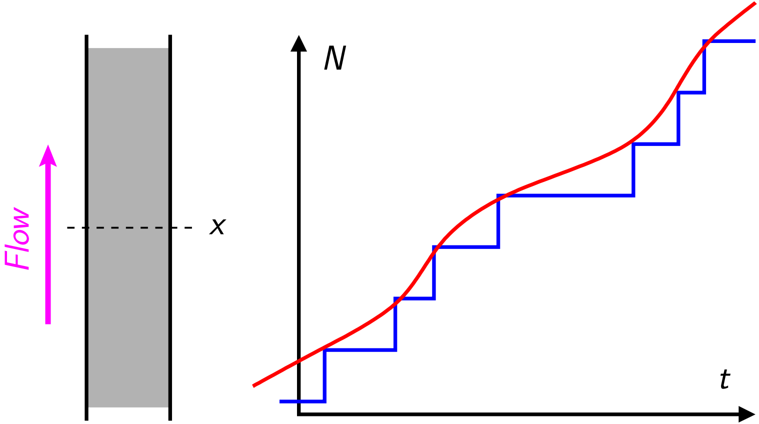
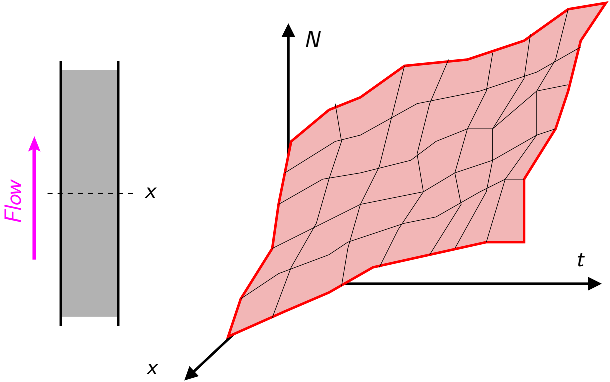
# Convention for vehicle labeling



# Three representations of traffic flow

## Moskowitz' surface




See also [Makigami et al, 1971], [Laval and Leclercq, 2013]

# Overview: conservation laws (CL) / Hamilton-Jacobi (HJ)

| | | Eulerian<br>$t-x$ | Lagrangian<br>$t-n$ |
|---|---|---|---|
| CL | Variable | Density $\rho$ | Spacing $r$ |
| | Equation | $\boxed{\partial_t \rho + \partial_x Q(\rho) = 0}$ | $\boxed{\partial_t r + \partial_x V(r) = 0}$ |
| HJ | Variable | Label $N$<br>$N(t,x) = \int_x^{+\infty} \rho(t,\xi)d\xi$ | Position $\mathcal{X}$<br>$\mathcal{X}(t,n) = \int_n^{+\infty} r(t,\eta)d\eta$ |
| | Equation | $\boxed{\partial_t N + H(\partial_x N) = 0}$ | $\boxed{\partial_t \mathcal{X} + \mathcal{V}(\partial_x \mathcal{X}) = 0}$ |
| | Hamiltonian | $H(p) = -Q(-p)$ | $\mathcal{V}(p) = -V(-p)$ |

# Outline

# Basic idea

First order Hamilton-Jacobi equation

$$u_t + H(Du) = 0, \quad \text{in} \quad \mathbb{R}^n \times (0, +\infty) \tag{1}$$

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Family of simple linear solutions

$$u^{\alpha,\beta}(t,x) = \alpha x - H(\alpha)t + \beta, \quad \text{for any} \quad \alpha \in \mathbb{R}^n, \ \beta \in \mathbb{R}$$

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Idea: envelope of elementary solutions (E. Hopf 1965 [5])

# Lax-Hopf formulæ

Consider Cauchy problem

$$\begin{cases} u_t + H(Du) = 0, & \text{in} \quad \mathbb{R}^n \times (0, +\infty), \\ u(., 0) = u_0(.), & \text{on} \quad \mathbb{R}^n. \end{cases} \tag{2}$$

Two formulas according to the smoothness of

- the Hamiltonian $H$
- the initial data $u_0$

# Lax-Hopf formulæ

Assumptions: **case 1**

(A1) $H : \mathbb{R}^n \to \mathbb{R}$ is convex

(A2) $u_0 : \mathbb{R}^n \to \mathbb{R}$ is uniformly Lipschitz

Theorem (First Lax-Hopf formula)

*If (A1)-(A2) hold true, then*

$$u(x,t) := \inf_{z \in \mathbb{R}^n} \sup_{y \in \mathbb{R}^n} \left[ u_0(z) + y.(x-z) - tH(y) \right] \tag{3}$$

*is the unique uniformly continuous viscosity solution of (2).*

# Lax-Hopf formulæ
(Continued)

Assumptions: **case 2**

(A3) $H : \mathbb{R}^n \to \mathbb{R}$ is continuous

(A4) $u_0 : \mathbb{R}^n \to \mathbb{R}$ is uniformly Lipschitz and convex

**Theorem (Second Lax-Hopf formula)**

*If (A3)-(A4) hold true, then*

$$u(x,t) := \sup_{y \in \mathbb{R}^n} \inf_{z \in \mathbb{R}^n} \left[ u_0(z) + y.(x-z) - tH(y) \right] \tag{4}$$

*is the unique uniformly continuous viscosity solution of (2).*

# Legendre-Fenchel transform

First Lax-Hopf formula (3) can be recast as

$$u(x,t) := \inf_{z \in \mathbb{R}^n} \left[ u_0(z) - tH^* \left( \frac{x-z}{t} \right) \right]$$

thanks to Legendre-Fenchel transform

$$L(z) = H^*(z) := \sup_{y \in \mathbb{R}^n} \left( y.z - H(y) \right).$$

**Proposition (Bi-conjugate)**

*If $H$ is strictly convex, 1-coercive i.e.* $\lim_{|p| \to \infty} \frac{H(p)}{|p|} = +\infty$,

*then $H^*$ is also convex and*

$$(H^*)^* = H.$$

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# LWR in Eulerian $(t, x)$

- Cumulative vehicles count (CVC) or Moskowitz surface $N(t, x)$

$$q = \partial_t N \quad \text{and} \quad \rho = -\partial_x N$$

- If density $\rho$ satisfies the scalar (LWR) conservation law

$$\partial_t \rho + \partial_x Q(\rho) = 0$$

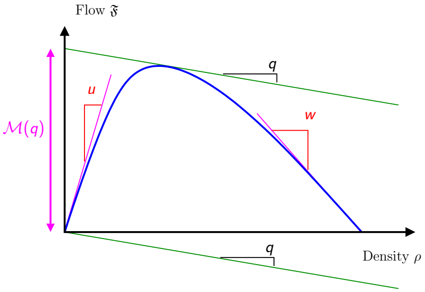
- Then $N$ satisfies the first order Hamilton-Jacobi equation

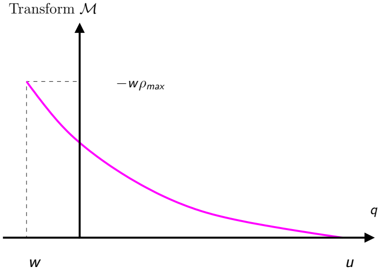
$$\partial_t N - Q(-\partial_x N) = 0 \tag{5}$$

# LWR in Eulerian $(t, x)$

- Legendre-Fenchel transform with $Q$ concave (*relative capacity*)

$$\mathcal{M}(q) = \sup_{\rho} \left[ Q(\rho) - \rho q \right]$$



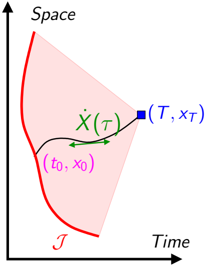



# LWR in Eulerian $(t, x)$
(continued)

- Lax-Hopf formula (representation formula) [Daganzo, 2006]

$$N(T, x_T) = \min_{u(.),(t_0,x_0)} \int_{t_0}^{T} \mathcal{M}(u(\tau))d\tau + N(t_0, x_0),$$

$$\left| \begin{array}{l} \dot{X} = u \\ u \in \mathcal{U} \\ X(t_0) = x_0, \quad X(T) = x_T \\ (t_0, x_0) \in \mathcal{J} \end{array} \right. \tag{6}$$

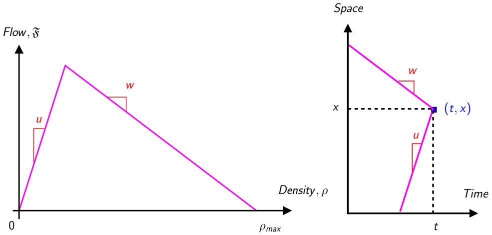



- Viability theory [Claudel and Bayen, 2010]

# LWR in Eulerian $(t, x)$

## (Historical note)

- Dynamic programming [Daganzo, 2006] for triangular FD ($u$ and $w$ free and congested speeds)




- Minimum principle [Newell, 1993]

$$N(t,x) = \min \left[ N\left(t - \frac{x - x_u}{u}, x_u\right), \right. \\ \left. N\left(t - \frac{x - x_w}{w}, x_w\right) + \rho_{max}(x_w - x) \right], \tag{7}$$

# LWR in Lagrangian $(n, t)$

- Consider $X(t, n)$ the location of vehicle $n$ at time $t \geq 0$

$$v = \partial_t X \quad \text{and} \quad r = -\partial_n X$$

- If the spacing $r := 1/\rho$ satisfies the LWR model (Lagrangian coord.)

$$\partial_t r + \partial_n \mathcal{V}(r) = 0$$

with the speed-spacing FD $\mathcal{V} : r \mapsto \mathfrak{I}(1/r)$,

- Then $X$ satisfies the first order Hamilton-Jacobi equation

$$\partial_t X - \mathcal{V}(-\partial_n X) = 0. \tag{8}$$

# LWR in Lagrangian $(n, t)$
(continued)

- Legendre-Fenchel transform with $\mathcal{V}$ concave

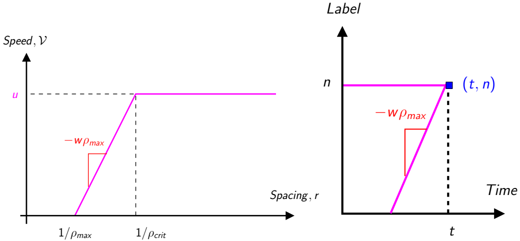
$$\mathcal{M}(u) = \sup_r \left[\mathcal{V}(r) - ru\right].$$

- Lax-Hopf formula

$$X(T, n_T) = \min_{u(.),(t_0,n_0)} \int_{t_0}^{T} \mathcal{M}(u(\tau))d\tau + X(t_0, n_0),$$
$$\left| \begin{array}{l} \dot{N} = u \\ u \in \mathcal{U} \\ N(t_0) = n_0, \quad N(T) = n_T \\ (t_0, n_0) \in \mathcal{J} \end{array} \right. \tag{9}$$

# LWR in Lagrangian $(n, t)$
(continued)

- Dynamic programming for triangular FD




- Minimum principle $\Rightarrow$ car following model [Newell, 2002]

$$X(t,n) = \min \Big[ X(t_0, n) + u(t - t_0), \quad X(t_0, n + w\rho_{max}(t - t_0)) + w(t - t_0) \Big]. \tag{10}$$

# Outline

# Space dependent Hamiltonian

Consider HJ equation posed on a junction $J$

$$\begin{cases} u_t + H(x, u_x) = 0, & \text{on} \quad J \times (0, +\infty), \\ u(t = 0, x) = g(x), & \text{on} \quad J \end{cases} \tag{11}$$

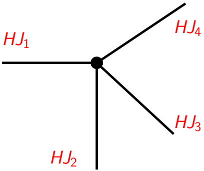
Extension of Lax-Hopf formula(s)?

- No simple linear solutions for (11)
- No definition of convexity for discontinuous functions

# Hamilton-Jacobi on networks

(Imbert-Monneau, 2014)

$$\begin{cases} u_t^i + H_i\left(u_x^i\right) = 0, & \text{for} \quad x \in J_i,\ x \neq 0, \\ u_t + F\left(x, u_{x_1}, \ldots, u_{x_N}\right) = 0, & \text{for} \quad x = 0 \end{cases}$$




Hamiltonians $H_i$: $\begin{cases} \text{continuous} \\ \text{quasi-convex} \\ \text{coercive} \end{cases}$

Junction functions $F$: continuous and non-increasing

**Comments:**

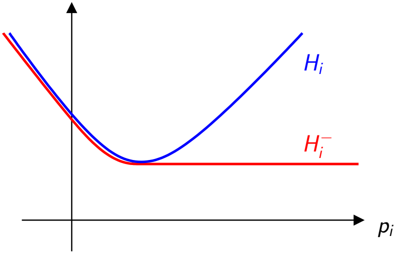
- Discontinuous HJ equations
- Can be seen as systems of HJ equations

# Junction condition of optimal control type

(Imbert-Monneau, 2014)

$$F_A(p) = \max\left\{A, \max_{i=1,\dots,N} H_i^-(p_i)\right\}$$

with $H_i^-$ the non-increasing envelope of $H_i$



# Any junction model reduces to a $F_A$ solution

(IMBERT-MONNEAU, 2014)

**Theorem (Equivalence between junction models (IMBERT-MONNEAU [6]))**

*For any function $G : \mathbb{R}^N \to \mathbb{R}$ that is continuous and non-increasing, solving the following equation*

$$\begin{cases} u_t^i + H_i(u_x^i) = 0, & \text{for} \quad x \in J_i, \ x \neq 0, \\ u_t + G(Du) = 0, & \text{for} \quad x = 0, \end{cases}$$

*is strictly equivalent to solve*

$$\begin{cases} u_t^i + H_i\left(u_x^i\right) = 0, & \text{for} \quad x \in J_i, \ x \neq 0, \\ u_t + F_A\left(u_{x_1}, \dots, u_{x_N}\right) = 0, & \text{for} \quad x = 0 \end{cases}$$

*for an appropriate $F_A$ where A depends on G.*

# Optimal control in $J$

(Imbert-Monneau-Zidani, 2013)

If $p \mapsto H(x,p)$ convex,
Then

$$u(t,x) = \inf_{\{X(0)=y,\ X(t)=x\}} \left\{ u_0(y) + \int_0^t L\left(X(\tau), \dot{X}(\tau)\right) d\tau \right\}$$

where $L(x,q) = \begin{cases} L_i(q) & \text{if} \quad x \in J_i, \\ \min\left(-A, \min_{i=1,\dots,N} L_i(q)\right), & \text{if} \quad x = 0. \end{cases}$

solves the Cauchy problem

$$\begin{cases} \partial_t u + H(x, \partial_x u) = 0, & \text{if} \quad x \in J,\ x \neq 0, \\ \partial_t u + F_A(\partial_x u) = 0, & \text{if} \quad x = 0, \\ u(0,x) = u_0(x) \end{cases}$$

# Dirichlet boundary conditions

(Imbert-Monneau, 2014)

$$\begin{cases} \partial_t u + \partial_x (H(u)) = 0, & \text{if} \quad x > 0, \\ u(t,0) = u_b, & \text{at} \quad x = 0. \end{cases}$$

Bardos, LeRoux, Nédélec boundary condition

$$H(u_\tau) = \max\left\{H^-(u_\tau), H^+(u_b)\right\}$$

where $u_\tau = u(t,0)$.

# Link with literature

(Imbert-Monneau, 2014)

- Optimal control
  - $N = 2$: Adimurthi-Gowda-Mishra (2005)
  - $N \geq 3$:
    - Achdou-Camilli-Cutrí-Tchou (2012)
    - Imbert-Monneau-Zidani (2013)
- Constrained scalar conservation laws
  - Colombo-Goatin (2007)
  - Andreianov-Goatin-Seguin (2010)
- BLN condition
  - Lebacque condition (1996) if $N = 1$ and $A = H^+(u_b)$

# Link with literature
(continued)

- HJ and state constraints
  - Soner, Capuzzo-Dolcetta – Lions, Blanc
  - Frankowska – Plaskacz
- HJ on networks
  - Schieborn (PhD thesis 2006)
  - Achdou – Camilli – Cutrí – Tchou (NoDEA 2012)
  - Camilli – Schieborn (2013)
  - Imbert – Monneau – Zidani (COCV 2013)
- Regional optimal control and ramified spaces
  - Bressan – Hong (2007)
  - Barles – Briani – Chasseigne (2013, 2014)
  - Rao – Zidani (2013), Rao – Siconolfi – Zidani (2014)
  - Barles – Chasseigne (2015)
- HJ equations with discontinuous source terms
  - Giga – Hamamuki (CPDE 2013)

# Junction models

Classical approaches for CL:

- Macroscopic modeling on (homogeneous) sections
- Coupling conditions at (pointwise) junction

For instance, consider

$$\begin{cases} \rho_t + (Q(\rho))_x = 0, & \text{scalar conservation law,} \\ \rho(., t=0) = \rho_0(.), & \text{initial conditions,} \\ \psi(\rho(x=0^-, t), \rho(x=0^+, t)) = 0, & \text{coupling condition.} \end{cases} \quad (12)$$

See Garavello, Piccoli [4], Lebacque, Khoshyaran [8] and Bressan et al. [1]

# Examples of junction models

- Model with internal state (= buffer(s))
  Bressan & Nguyen (NHM 2015) [2]
  - $\rho \mapsto Q(\rho)$ strictly concave
  - advection of $\gamma_{ij}(t,x)$ turning ratios from $(i)$ to $(j)$ (GSOM model with passive attribute)
  - internal dynamics of the buffers (ODEs): queue lengths
- Extended Link Transmission Model
  Jin (TR-B 2015) [7]
  - Link Transmission Model (LTM) Yperman (2005, 2007)
  - Triangular diagram

$$Q(\rho) = \min\{u\rho,\ w(\rho_{max} - \rho)\} \quad \text{for any} \quad \rho \in [0, \rho_{max}]$$

  - Commodity = turning ratios $\gamma_{ij}(t)$
  - Definition of boundary supply and demand functions

# Outline

# First remarks

If $N$ solves

$$N_t + H(N_x) = 0$$

then $\bar{N} = N + c$ for any $c \in \mathbb{R}$ is also a solution

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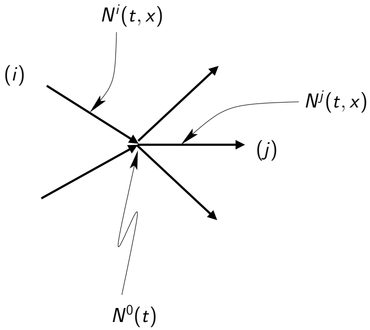



- No a priori relationship between initial conditions
- $N^k(t,x)$ consistent along the same branch $J_k$ and

$$\partial_t N^0(t) = \sum_i \partial_t N^i \left(t, x = 0^-\right) = \sum_j \partial_t N^j \left(t, x = 0^+\right)$$

# Key idea

Assume that $H$ is piecewise linear (triangular FD)

$$N_t + H(N_x) = 0$$

with

$$H(p) = \max\{\underbrace{H^+(p)}_{\text{supply}}, \underbrace{H^-(p)}_{\text{demand}}\}$$

# Key idea

Assume that $H$ is piecewise linear (triangular FD)

$$N_t + H\left(N_x\right) = 0$$

with

$$H(p) = \max\{\underbrace{H^+(p)}_{\text{supply}}, \underbrace{H^-(p)}_{\text{demand}}\}$$

Partial solutions $N^+$ and $N^-$ that solve resp.

$$\begin{cases} N_t^+ + H^+\left(N_x^+\right) = 0, \\ N_t^- + H^-\left(N_x^-\right) = 0 \end{cases} \quad \text{such that} \quad N = \min\left\{N^-, N^+\right\}$$

- Upstream demand advected by waves moving forward
- Downstream supply transported by waves moving backward

# Junction model

Optimization junction model (Lebacque's talk)
Lebacque, Khoshyaran (2005) [8]

$$
\max \left[ \sum_i \phi_i(q_i) + \sum_j \psi_j(r_j) \right]
\quad
\text{s.t.} \left| \begin{array}{ll}
0 \le q_i & \forall i \\
q_i \le \delta_i & \forall i \\
0 \le r_j & \forall j \\
r_j \le \sigma_j & \forall j \\
0 = r_j - \sum_i \gamma_{ij} q_i & \forall j
\end{array} \right. \tag{13}
$$

where $\phi_i$, $\psi_j$ are concave, non-decreasing

# Example of optimization junction models

- Herty and Klar (2003)
- Holden and Risebro (1995)
- Coclite, Garavello, Piccoli (2005)
- Daganzo's merge model (1995) [3]

$$\begin{cases} \phi_i(q_i) = N_{max} \left( q_i - \dfrac{q_i^2}{2 p_i q_{i,max}} \right) \\ \psi = 0 \end{cases}$$

where $p_i$ is the priority of flow coming from road $i$ and $N_{max} = \phi_i'(0)$

# Solution of the optimization model

Lebacque, Khoshyaran (2005)

Karush-Kuhn-Tucker optimality conditions:

- For any incoming road $i$

$$\phi_i'(q_i) + \sum_k s_k \gamma_{ik} - \lambda_i = 0, \quad \lambda_i \geq 0, \quad q_i \leq \delta_i \quad \text{and} \quad \lambda_i(q_i - \delta_i) = 0,$$

- and for any outgoing road $j$

$$\psi_j'(r_j) - s_j - \lambda_j = 0, \quad \lambda_j \geq 0, \quad r_i \leq \sigma_j \quad \text{and} \quad \lambda_j(r_j - \sigma_j) = 0,$$

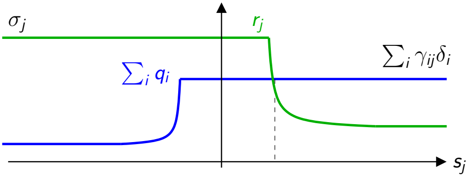
where $(s_j, \lambda_j)$ = Karush-Kuhn-Tucker coefficients (or Lagrange multipliers)

# Solution of the optimization model

Lebacque, Khoshyaran (2005)

$$\begin{cases} q_i = \Gamma_{[0,\delta_i]}\left( (\phi_i')^{-1}\left( -\sum_k \gamma_{ik} s_k \right)\right), & \text{for any} \quad i, \\ r_j = \Gamma_{[0,\sigma_j]}\left( (\psi_j')^{-1}(s_j)\right), & \text{for any} \quad j, \end{cases} \tag{14}$$

$\Gamma_{\mathcal{K}}$ is the projection operator on the set $\mathcal{K}$



# Model equations

$$
\begin{cases}
N^i_t + H_i(N^i_x) = 0, & \text{for any} \quad x \neq 0, \\
\\
\begin{cases} \partial_t N^i(t, x^-) = q_i(t), \\ \partial_t N^j(t, x^+) = r_j(t), \end{cases} & \text{at} \quad x = 0, \\
\\
N^i(t = 0, x) = N^i_0(x), \\
\\
\partial_t N^i(t, x = \xi_i) = \Delta_i(t), \\
\partial_t N^j(t, x = \chi_j) = \Sigma_j(t)
\end{cases}
$$

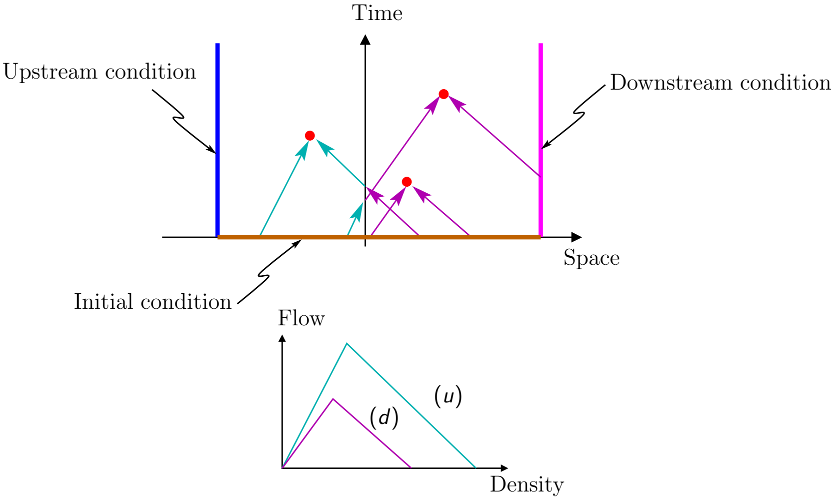
# Algorithm

Inf-morphism property: compute partial solutions for

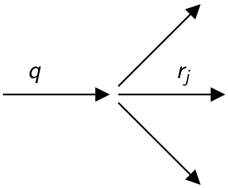
- initial conditions
- upstream boundary conditions
- downstream boundary conditions
- internal boundary conditions

1. Propagate demands forward
   - through a junction, assume that the downstream supplies are maximal
2. Propagate supplies backward
   - through a junction, assume that the upstream demands are maximal

# Spatial discontinuity



# Diverge




Junction model

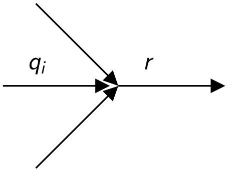
$$\max \left[ \phi(q) + \sum_j \psi_j(r_j) \right]$$

$$\text{s.t.} \left| \begin{array}{ll} 0 \leq q \leq \delta & \\ 0 \leq r_j \leq \sigma_j & \forall j \\ 0 = r_j - \gamma_j q & \forall j \end{array} \right.$$

whose solution is

$$\begin{cases} q = \Gamma_{[0,\delta]} \left( (\phi')^{-1} \left( -\sum_k \gamma_k s_k \right) \right), & \\ r_j = \Gamma_{[0,\sigma_j]} \left( (\psi_j')^{-1} (s_j) \right), & \text{for any } \quad j \end{cases}$$

# Merge




Junction model

$$\max \left[ \sum_i \phi_i(q_i) + \psi(r) \right]$$

$$\text{s.t.} \left| \begin{array}{ll} 0 \leq q_i \leq \delta_i & \forall i \\ 0 \leq r \leq \sigma & \\ 0 = r - \sum_i q_i & \end{array} \right.$$

whose solution is

$$\begin{cases} q_i = \Gamma_{[0,\delta_i]} \left( (\phi_i')^{-1} (-s) \right), & \text{for any} \quad i, \\ r = \Gamma_{[0,\sigma]} \left( (\psi')^{-1} (s) \right) & \end{cases}$$

# Final remarks

In a nutshell:

- Importance of the supply/demand functions
- General optimization problem at the junction
- No explicit solution right now

Perspectives:

- Estimation on networks
- Stability states (Jin's talk)
- Nash equilibria (Bressan's talk)

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Thanks for your attention

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