Hamilton-Jacobi Approach for Second Order Traffic Flow Models

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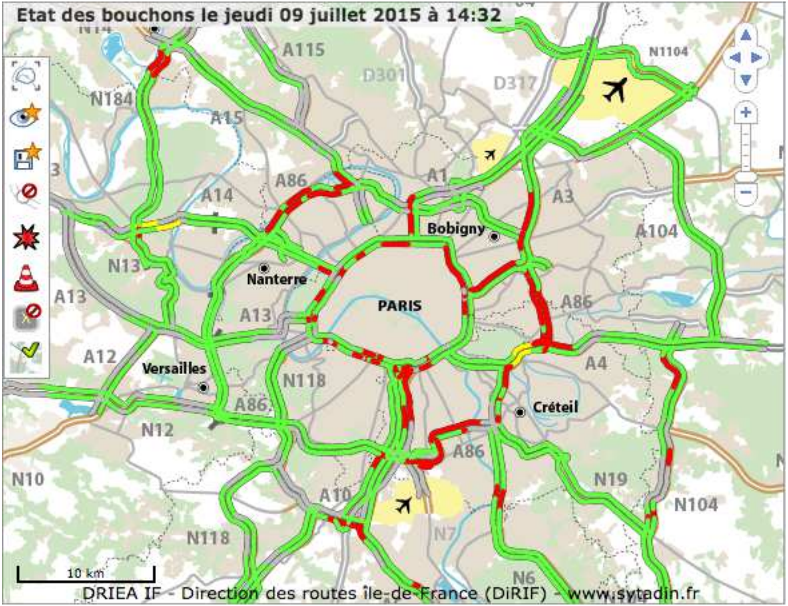
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Breakthrough in traffic monitoring

Traffic monitoring

- "old": loop detectors at fixed locations (Eulerian)
- "new": GPS devices moving within the traffic (Lagrangian)

State estimation with multiple data sources for real time monitoring

[Mobile Millenium, 2008]

Outline

1. Introduction to traffic
2. Variational principle applied to GSOM models
3. Numerical example

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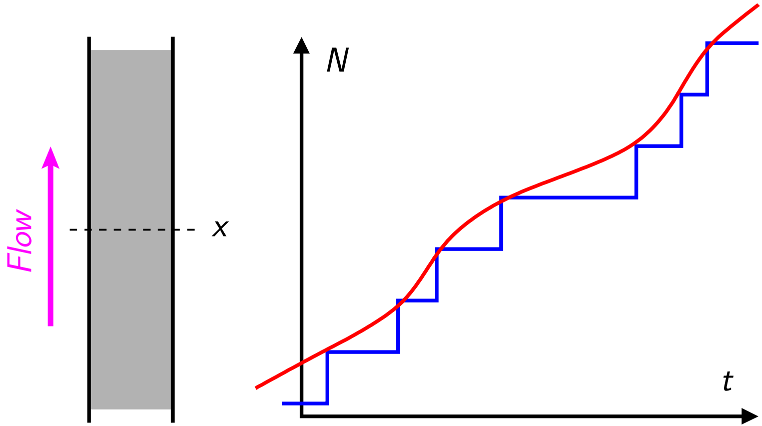
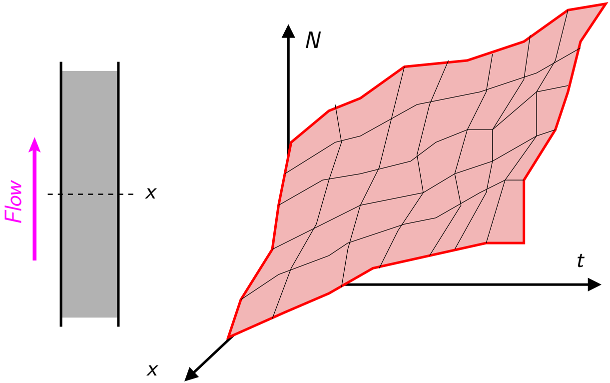
Convention for vehicle labeling



Three representations of traffic flow

Moskowitz' surface



See also [Makigami et al, 1971], [Laval & Leclercq, 2013]

Overview: conservation laws (CL) / Hamilton-Jacobi (HJ)

	Eulerian $t - x$	Lagrangian $n - t$
Variable (CL)	Density ρ	Spacing r
Equation (CL)	$\boxed{\partial_t \rho + \partial_x \mathfrak{F}(\rho) = 0}$	$\boxed{\partial_t r + \partial_n V(r) = 0}$
Variable (HJ)	Label N $N(t,x) = \int_x^{+\infty} \rho(t,\xi) d\xi$	Position $\mathcal{X}$ $\mathcal{X}(n,t) = \int_n^{+\infty} r(\eta,t) d\eta$
Equation (HJ)	$\boxed{\partial_t N + H(\partial_x N) = 0}$	$\boxed{\partial_t \mathcal{X} + \mathcal{V}(\partial_n \mathcal{X}) = 0}$
Hamiltonian	$H(p) = -\mathfrak{F}(-p)$	$\mathcal{V}(p) = -V(-p)$

LWR in Eulerian (t, x)

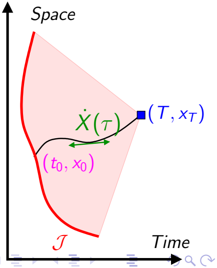
$$\begin{cases} \partial_t N + H(\partial_x N) = 0, & \text{on} \quad (0, +\infty) \times \mathbb{R}, \\ N = \mathbf{c}, & \text{on} \quad \text{Dom}(\mathbf{c}) \end{cases} \tag{1}$$

Lax-Hopf formula for HJ PDE (1)

- Traffic engineering [Newell, 1993], [Daganzo, 2006]
- Viability theory [Aubin, Bayen & Saint-Pierre, 2008], [Claudel & Bayen, 2010]

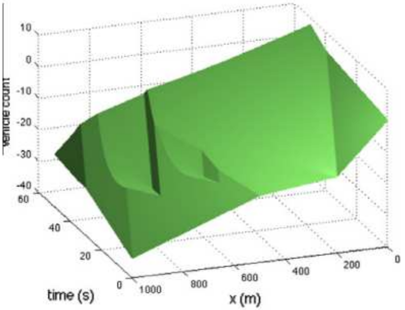
$$N(T, x_T) = \min_{u(.),(t_0,x_0)} \left\{ \mathbf{c}(t_0, x_0) + \int_{t_0}^{T} H^* \left(\dot{X}(\tau) \right) d\tau \right\}$$

$$\left| \begin{array}{l} \dot{X} \in \mathcal{U} \\ X(t_0) = x_0, \quad X(T) = x_T \\ (t_0, x_0) \in \mathcal{J} := \text{Dom}(\mathbf{c}) \end{array} \right. \tag{2}$$

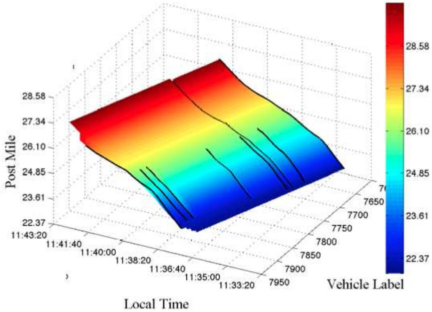


Examples of data assimilation

Eulerian coordinates (t, x)
[Mazaré et al, 2012]



Lagrangian coordinates (n, t)
[Han et al, 2012]



with initial, upstream, downstream and internal boundary conditions

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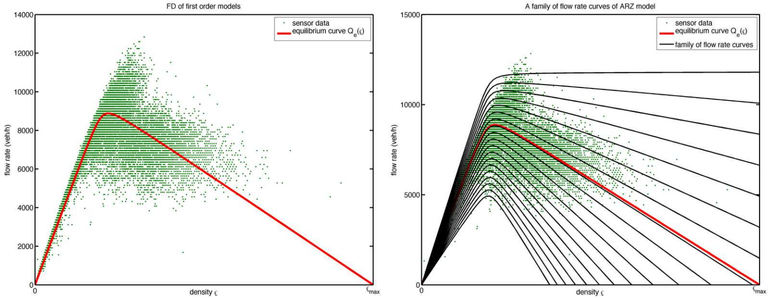
Motivation for higher order models

- Experimental evidence: multi-valuedness in congested case



[S. Fan, U. Illinois], NGSIM dataset

- Integrate measurements of: acceleration, fuel consumption, noise, etc.

GSOM family [Lebacque, Mammar, Haj-Salem 2007]

- Generic Second Order Models (GSOM) family

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 & \text{Conservation of vehicles,} \\ \partial_t(\rho I) + \partial_x(\rho v I) = \rho \varphi(I) & \text{Dynamics of the driver attribute } I, \\ v = \mathfrak{V}(\rho, I) & \text{Fundamental diagram,} \end{cases} \tag{3}$$

- Specific driver attribute I
  - the driver aggressiveness,
  - the driver origin / destination,
  - the vehicle class,
  - ...
- Flow-density fundamental diagram

$$\mathfrak{F} : (\rho, I) \mapsto \rho \underbrace{\mathfrak{V}(\rho, I)}_{=:v \text{ (speed)}} .$$

Some examples of GSOM models

- LWR model [Lighthill & Whitham, 1955], [Richards, 1956] = a GSOM model with no specific driver attribute
- ARZ model ([Aw, Rascle 2000] and [Zhang, 2002]) with driver attribute $I = v - V_e(\rho)$
- Colombo 1-phase model (from [Colombo, 2002]) with
  - no driver attribute in fluid situation
  - a non-trivial scalar attribute I in congested situation.

Other examples in [Lebacque, Khoshyaran 2013]

Classical solution method

- Kinematic waves or 1-waves:
  - similar to the seminal LWR model
  - density variations at speed $\nu_1 = \partial_\rho \mathfrak{V}(\rho, I)$
  - driver attribute I is continuous
- Contact discontinuities or 2-waves:
  - variations of driver attribute I at speed $\nu_2 = \mathfrak{V}(\rho, I)$
  - the flow speed v is constant.

[Lebacque et al., 2007]

GSOM in Lagrangian (n, t)

- From [Lebacque & Khoshyaran, 2013], GSOM in Lagrangian

$$\begin{cases} \partial_t r + \partial_n v = 0 & \text{Conservation of vehicles,} \\ \partial_t I = \varphi(n, t, I) & \text{Dynamics of } I, \\ v = \mathcal{W}(n, t, r) := \mathcal{V}(r, I(n, t)) & \text{Fundamental diagram.} \end{cases} \quad (4)$$

- Position $\mathcal{X}(n, t) := \int_{-\infty}^{t} v(n, \tau) d\tau$ satisfies the HJ equation

$$\partial_t \mathcal{X} - \mathcal{W}(n, t, -\partial_n \mathcal{X}) = 0, \quad (5)$$

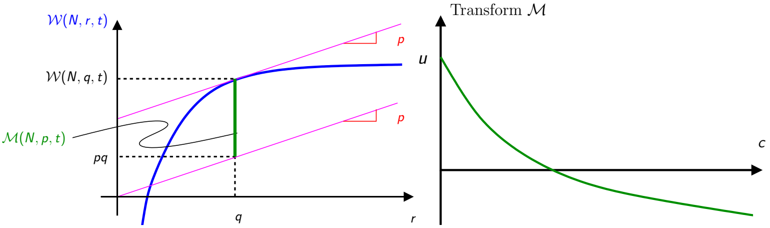
- And $I(n, t)$ solves the ODE

$$\left| \begin{array}{l} \partial_t I(n, t) = \varphi(n, t, I), \\ I(n, 0) = i_0(n), \qquad \text{for any} \quad n. \end{array} \right.$$

GSOM in Lagrangian (n, t)
(continued)

- Legendre-Fenchel transform of $\mathcal{W}$ according to r

$$\mathcal{M}(n,t,p) = \sup_{q \in \mathrm{Dom}(\mathcal{W}(n,t,\cdot))} \{\mathcal{W}(n,t,r) - pq\}$$



GSOM in Lagrangian (n, t)
(continued)

- Dirichlet problem

$$\begin{cases} \partial_t \mathcal{X} - \mathcal{W}(n, t, -\partial_n \mathcal{X}) = 0, & \text{in} \quad \mathbb{R} \times (0, +\infty), \\ \partial_t I = \varphi(n, t, I), & \text{on} \quad (0, +\infty), \\ \mathcal{X} = \mathbf{c}, & \text{on} \quad \mathrm{Dom}(\mathbf{c}). \end{cases} \tag{6}$$

- Lax-Hopf formula for (6)

$$\mathcal{X}(n_T, T) = \min_{(n_0, t_0)} \left\{ \mathbf{c}(n_0, t_0) + \int_{t_0}^{T} \mathcal{M}\left(n, \tau, \dot{n}(\tau)\right) d\tau \;\middle|\; \begin{array}{l} \dot{n} \in \mathcal{U} \\ n(t_0) = n_0, \quad n(T) = n_T \\ (n_0, t_0) \in \mathrm{Dom}(\mathbf{c}) \end{array} \right\} \tag{7}$$

GSOM in Lagrangian (n, t)
(continued)

- Optimal trajectories = characteristics

$$(\mathcal{U}) \begin{cases} \dot{n} = \partial_r \mathcal{W}(n, t, r), \\ \dot{r} = -\partial_N \mathcal{W}(n, t, r), \end{cases} \tag{8}$$

- System of ODEs to solve
- Difficulty: not straight lines in the general case

General ideas of [1]

First key element: Lax-Hopf formula

- Computations only for the characteristics

$$\mathcal{X}(n_T, T) = \min_{(n_0, r_0, t_0)} \left\{ \mathbf{c}(n_0, t_0) + \int_{t_0}^{T} \mathcal{M}(n, \tau, \partial_r \mathcal{W}(n, \tau, r)) d\tau \right\}$$
$$\left| \begin{array}{l} \dot{n}(t) = \partial_r \mathcal{W}(n, t, r) \\ \dot{r}(t) = -\partial_N \mathcal{W}(n, t, r) \\ n(t_0) = n_0, \quad r(t_0) = r_0, \quad n(T) = n_T \\ (n_0, t_0) \in \mathcal{K} := \mathrm{Dom}(\mathbf{c}) \end{array} \right. \tag{9}$$

General ideas
(continued)

Second key element: inf-morphism prop. [Aubin et al, 2011]

- Consider a union of sets (initial and boundary conditions)

$$\mathcal{K} = \bigcup_{l} \mathcal{K}_l,$$

- then the global minimum is

$$\mathcal{X}(n_T, T) = \min_{l} \mathcal{X}_l(n_T, T), \tag{10}$$

- with $\mathcal{X}_l$ partial solution to sub-problem $\mathcal{K}_l$.

Assumptions

- Piecewise affine value conditions
  - the initial condition: positions of vehicles at time $t = t_0$,
  - the "upstream" boundary condition: trajectory of the first vehicle $n = N_0$ traveling on the section,
  - and internal boundary conditions: cumulative vehicles counts at fixed location $\mathcal{X} = x_0$.
- Finite horizon problems $(n, t) \in [N_0, N_{max}] \times [t_0, t_{max}]$
- No relaxation $\varphi = 0$

PWA conditions

Definition (PWA conditions)

- *Initial conditions: let $t_0 \geq 0$ be fixed. Then for any $p \in \{1, ..., P\}$,*

$$\mathcal{X}^{ini}(N, t_0) = r_{0,p} N + \alpha_p, \quad \textit{for any} \quad N \in [n_p, n_{p+1}].$$

- *Upstream boundary condition: let $N_0 \in \mathbb{Z}$ be fixed. Then for any $q \in \{1, ..., Q\}$*

$$\mathcal{X}^{up}(N_0, t) = v_{0,q} t + \beta_q, \quad \textit{for any} \quad t \in [t_q, t_{q+1}].$$

- *Internal boundary condition: let $x_0 \geq 0$ be fixed. Then for any $p \in \{1, ..., P\}$*

$$\mathcal{X}^{int}(n, t) = x_0, \quad \textit{with} \quad \begin{cases} t = \mathfrak{f}_{0,p} n + \gamma_p, \\ n \in [n_p, n_{p+1}]. \end{cases}$$

Initial conditions

Assume

- Uniform discrete label step Δn
- Δn small enough such that initial data are piecewise constant

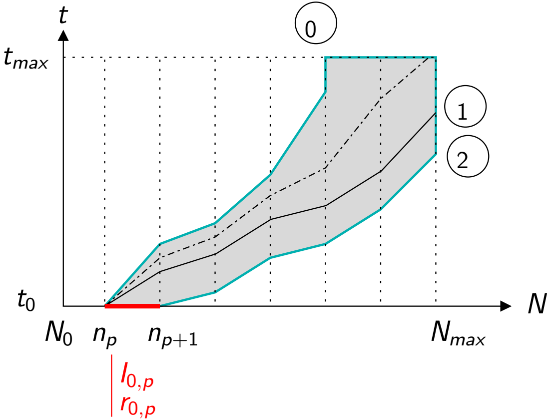
$$\begin{cases} I(n, t_0) = I_{0,p}, \\ r(n, t_0) = r_{0,p}. \end{cases}$$

Sub-algorithm for initial block $[n_p, n_{p+1}] \times \{t_0\}$

1. Initialize $\mathcal{X}$ to $+\infty$
2. Number of characteristics to compute
3. Compute $N(t)$ of each characteristic while $t \leq t_{max}$ and $N \leq N_{max}$
4. Calculate the (exact) solution $\mathcal{X}_p$ all along each characteristic
5. Compute the exact value at any point within the characteristics fan (simple translation)
6. In a rarefaction interpolate the value of $\mathcal{X}$ at each point within the influence domain.

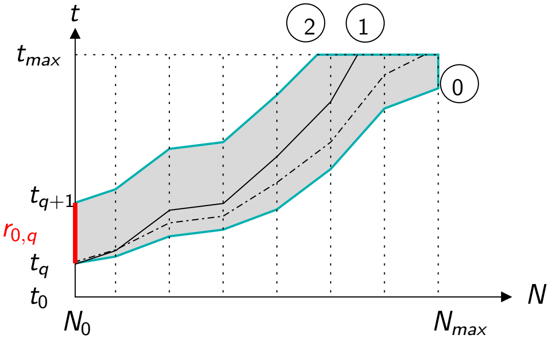
PWA initial conditions

- Domain of influence of the initial condition
- Couples for initial conditions $(n, r_0(n))$



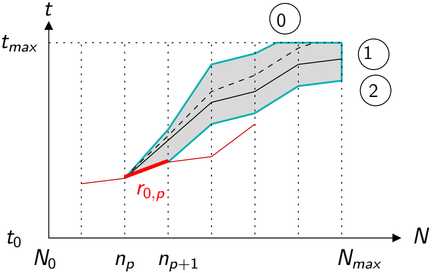
Upstream boundary conditions

- Domain of influence of the upstream boundary condition
- Couples for initial conditions $(N_0, r_0(t))$

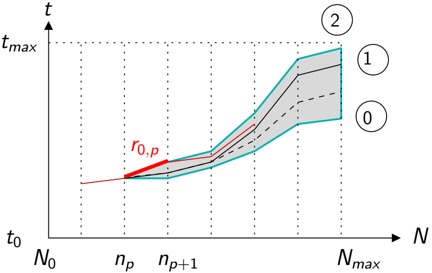


Internal boundary conditions

- Domain of influence of the internal boundary condition
- Triplet for initial conditions $(n(t), r_0(t), v_0(t))$



Under-critical case



Over-critical case

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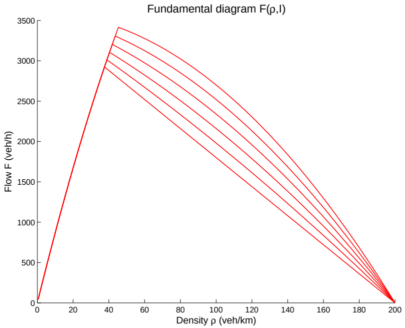
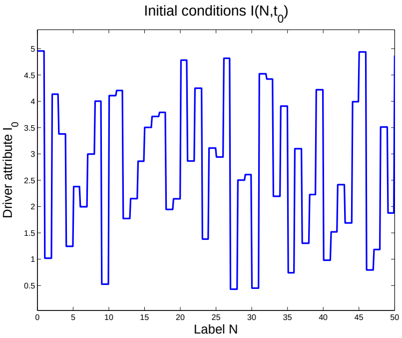
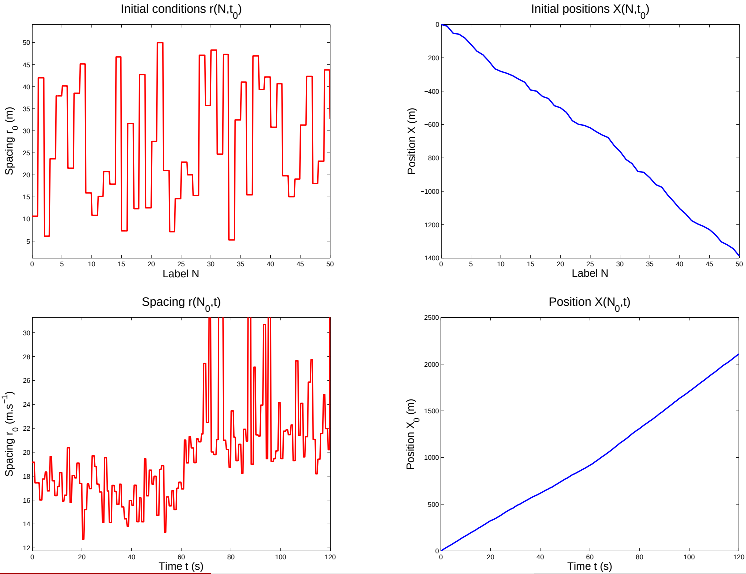
Fundamental Diagram and Driver Attribute

- Colombo 1-phase model ($I \in [0, 5]$)
- $\Delta n = 1$, $\Delta t = 1$

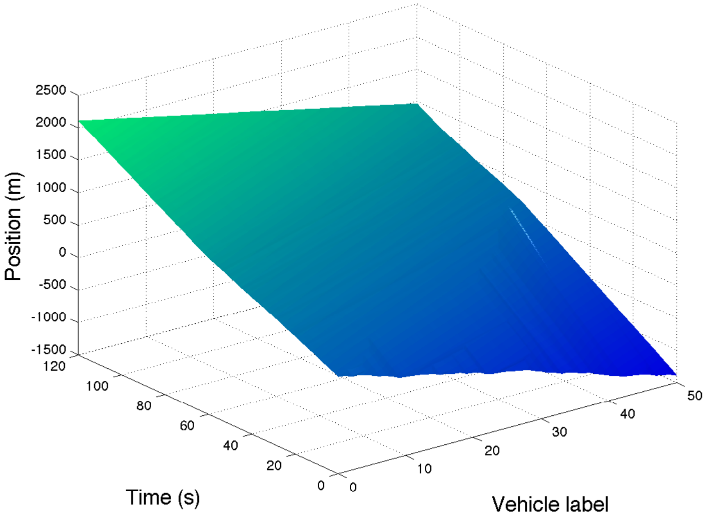




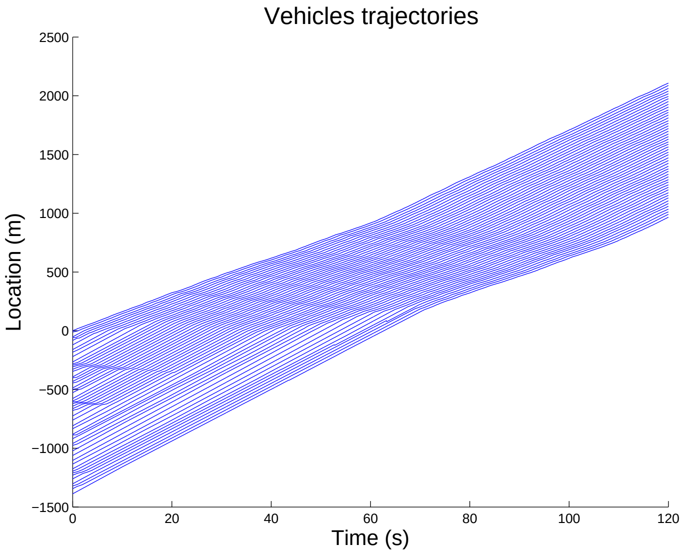
Initial and Boundaries Conditions



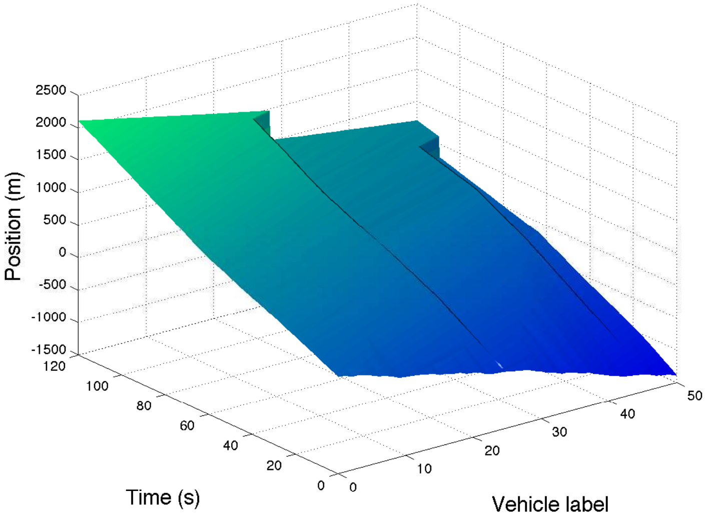
Numerical result (Initial cond. + first traj.)



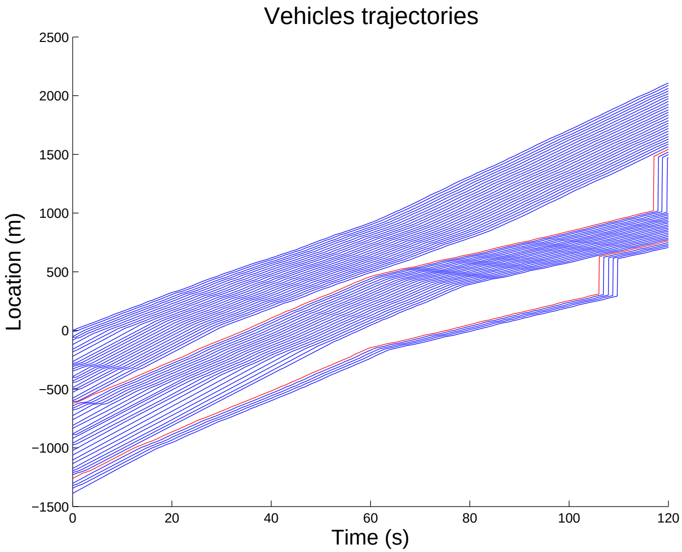
Numerical result (Initial cond. + first traj.)



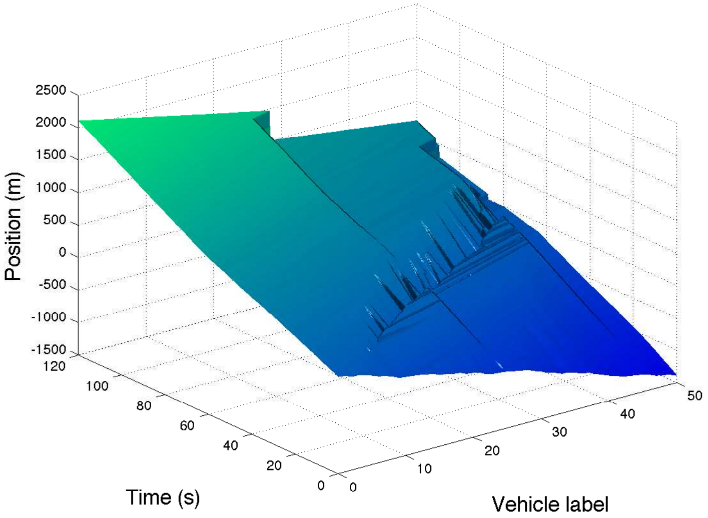
Numerical result (Initial cond.+ 3 traj.)



Numerical result (Initial cond. + 3 traj.)



Numerical result (Initial cond. + 3 traj. + Eulerian data)



Thanks for your attention

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Some references I

G. Costeseque and J.-P. Lebacque, *A variational formulation for higher order macroscopic traffic flow models: numerical investigation*, Transp. Res. Part B: Methodological, (2014).