Converting Level-Set Gradients to Shape Gradients

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Introduction

Shape evolutions based on level-set representation :

- practical for topological changes
- common level-set representation : signed distance function

Minimizing an energy depending on the signed distance function :

- by gradient descent **with respect to the shape** (correct way)
- by gradient descent **w.r.t. the level-set representation**
- these 2 evolutions are **different !**

Aim :

- to express precisely the link between the two gradients
- to perform level-set evolutions based on the shape gradient

Shape evolutions by gradient descents

- **Shape** Γ, seen as a function $\Gamma : \mathbb{S}^1 \to \Omega \subset \mathbb{R}^2$
- Its associated **signed distance function** $\phi = \Phi(\Gamma) : \mathbf{x} \in \Omega \mapsto \pm d(\mathbf{x}, \Gamma)$.
- **Energy** to be minimized : $E(\Gamma) = F(\phi)$
- **Gradient descent :**

w.r.t. to the **shape** Γ :

$$\begin{cases} \Gamma(0) &= \Gamma_0 \\ \dfrac{\partial \Gamma(t)}{\partial t} &= -\nabla_\Gamma E(\Gamma(t)) \end{cases}$$

– correct shape optimization

w.r.t. to the **level-set representation** ϕ :

$$\begin{cases} \phi(0) &= \Phi(\Gamma_0) \\ \dfrac{\partial \phi(t)}{\partial t} &= -\nabla_\phi F(\phi(t)) \end{cases}$$

– no guarantee ϕ remains a signed distance function

– optimization space is different, larger

- Evolutions may **differ** significantly

Gradient definition

Directional derivative of energy E at shape Γ in direction $\delta\Gamma$: $DE(\Gamma)(\delta\Gamma)$

Choice of an inner product on normal shape variations :

$$\langle \delta\Gamma_1 | \delta\Gamma_2 \rangle_{L^2(\mathbb{S}^1 \to \mathbb{R})} = \int_\Gamma \delta\Gamma_1(s) \cdot \delta\Gamma_2(s) \; d\Gamma(s) \quad \text{where } d\Gamma(s) = \left\| \frac{d\Gamma}{ds} \right\|_{\mathbb{R}^2} ds$$

Shape gradient : is the unique deformation $\nabla_\Gamma E(\Gamma)$ of Γ s.t.:

$$\forall\, \delta\Gamma, \quad DE(\Gamma)(\delta\Gamma) = \langle \nabla_\Gamma E(\Gamma) | \delta\Gamma \rangle_{L^2(\mathbb{S}^1 \to \mathbb{R})}$$

Similarly for the **level-set gradient** $\nabla_\phi F(\phi)$:

$$\forall\, \delta\phi, \quad DF(\phi)(\delta\phi) = \langle \nabla_\phi F(\phi) | \delta\phi \rangle_{L^2(\Omega \to \mathbb{R})}$$

From shape gradients to signed-distance function evolutions

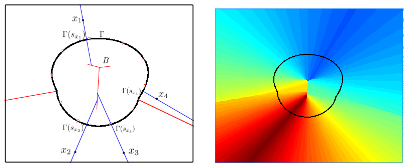
When is a signed-distance function evolution correct ?

- ϕ remains a s-d.f. under a variation $\delta\phi$

$\iff \; \exists\, \delta\Gamma, \;\; \forall \mathbf{x} \in \Omega, \;\; \delta\phi(\mathbf{x}) = -\delta\Gamma(s_\mathbf{x}) \cdot \boldsymbol{n}_\Gamma(s_\mathbf{x})$

where $s_\mathbf{x}$ is the projection of $\mathbf{x}$ on Γ : $\Gamma(s_\mathbf{x}) = \mathbf{x} - \phi(\mathbf{x}) \nabla_\mathbf{x}\phi(\mathbf{x})$

$\iff \; \delta\phi$ is **constant along projection lines** to Γ



- **Projection lines :** sets of points $L(s)$ of the domain Ω that share the same projection point $\Gamma(s)$ on Γ; they are segments whose extremities belong to the skeleton of Γ or to the image boundary $\partial\Omega$.
- One-to-one correspondence between admissible variations $\delta\phi$ and normal shape variations $\delta\Gamma$.

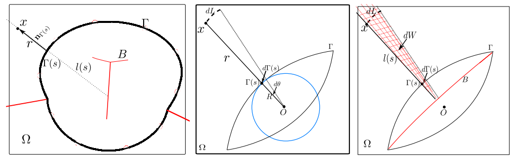
Relating shape gradient and level-set gradient :

Since $E(\Gamma) = F(\phi(\Gamma)) \;\; \forall \Gamma$, we have $DE(\Gamma)(\delta\Gamma) = DF(\phi) \left(\frac{d\phi}{d\Gamma}(\delta\Gamma) \right) \; \forall \delta\Gamma$ and thus :

$$\int_\Gamma \nabla_\Gamma E(\Gamma)(s) \cdot \delta\Gamma(s) \;\; d\Gamma(s) \;\; = \;\; -\int_\Omega \nabla_\phi F(\phi)(\mathbf{x}) \;\; (\delta\Gamma(s_\mathbf{x}) \cdot \boldsymbol{n}_\Gamma(s_\mathbf{x})) \;\; d\mathbf{x} \qquad \forall\, \delta\Gamma$$

Change of coordinate system : $\mathbf{x} \mapsto (s, r)$ where $\mathbf{x} = \Gamma(s) + r\, \boldsymbol{n}_{\Gamma(s)}$: $\int_\Omega \mapsto \int_{s \in \Gamma} \int_{L(s)}$:

$$\nabla_\Gamma E(\Gamma)(s) = -\int_{L(s)} \nabla_\phi F(\phi)(\mathbf{x}_{(s,r)}) \; \left| 1 - \kappa_{\Gamma(s)}\, r \right| \; dr \; \boldsymbol{n}_{\Gamma(s)}$$



Level-set variation associated to the shape gradient :

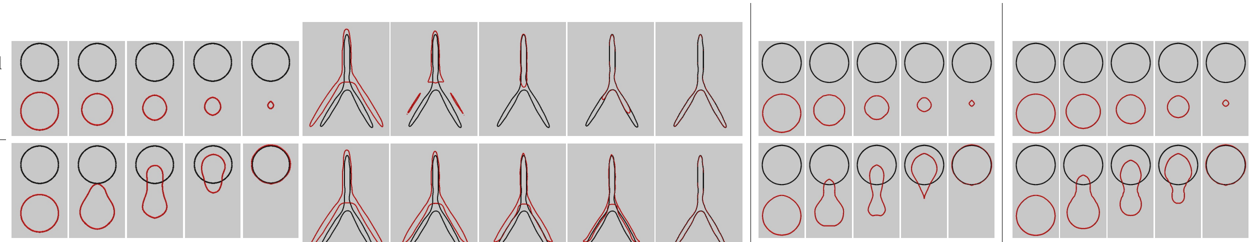
$$\delta\phi(\mathbf{x}) = \int_{L(s_\mathbf{x})} \left| 1 - \kappa(s_\mathbf{x})\, \phi(\mathbf{x}'_{(s_\mathbf{x},r)}) \right| \; \nabla_\phi F(\phi)(\mathbf{x}'_{(s_\mathbf{x},r)}) \; dr$$

$$= \lim_{ds \to 0} \frac{1}{d\Gamma(s)} \int_{dW(ds)} \nabla_\phi F(\phi)(\mathbf{x}') \; d\mathbf{x}'$$

Experimental results

Standard variational level-set method

Level-set evolution based on shape gradient



$$E(\Gamma) = F(\phi) = \|\phi - \phi_T\|^2_{L^2(\Omega \to \mathbb{R})}$$

$$F(\phi) = \int_\Omega (\phi - \phi_T)^2 H(-\phi_T) \; d\mathbf{x} + \int_\Omega (\phi - \phi_T)^2 H(-\phi) \; d\mathbf{x}$$

$$E(\Gamma) = \int_{\Gamma_T} \phi^2 \, ds + \int_\Gamma \phi_T^2 \, ds$$

Implementation details

- No need to find the skeleton
- Compute the projections on Γ of all points $\mathbf{y}$ in Ω by $\mathbf{y} - \phi(\mathbf{y}) \nabla_\mathbf{y}\phi(\mathbf{y})$
- In the simplistic case, just sum : $\delta\phi(\mathbf{x}) = \sum_{\mathbf{y} \text{ s.t. } s_\mathbf{x} = s_\mathbf{y}} \nabla_\phi F(\phi)(\mathbf{y})$
- In practice, discretize Γ (or sort projections into 1-pixel boxes W_i), assign weights $h_i^\mathbf{x}$ to express how much $\mathbf{x}$ belongs to W_i, and sum over points in the neighborhood of W_i : $\quad \delta\phi(\mathbf{x}) = \sum_i h_i^\mathbf{x} \sum_{\mathbf{y} \in \Omega} h_i^\mathbf{y} \; \nabla_\phi F(\phi)(\mathbf{y})$.

Discussion

- **dramatic difference** between shape gradient and level-set gradient
- level-set based shape priors for **thin structures** (roads, blood vessels...) are possible when using shape gradient
- can be extended to **any dimension** (3D shapes, etc.)
- recomputation of the signed-distance function from the shape is required when topological changes occur.
- low complexity but still searching for the most efficient implementation scheme