International Journal of Bifurcation and Chaos, Vol. 20, No. 11 (2010) 3585–3616

DOI: 10.1142/S0218127410027854

INFLUENCES ON OTTO E. RÖSSLER'S EARLIEST PAPER ON CHAOS

CHRISTOPHE LETELLIER and VALÉRIE MESSAGER
CORIA — Université de Rouen, Avenue de l'Université, BP 08,
F-76801 Saint-Etienne du Rouvray Cedex, France

Received June 25, 2010

Dedicated to Otto E. Rössler for his 70th birthday

Otto E. Rössler is well-known in "chaos theory" for having published one of the most often used benchmark systems producing chaotic attractors. His contribution is mostly reduced to this simple chaotic system published in 1976. Our aim is to show that a slightly earlier paper contains, in fact, much more and reveals a deep topological understanding of how chaotic attractors are organized in phase space. Moreover it is shown that Otto had three main influences: Andronov, Khaikin and Vitt's textbook, the 1963 Lorenz paper and Li and Yorke's theorem "period-three implies chaos". In this paper, these three contributions are clearly identified as the main influences on Rössler's earliest paper on chaos. The content of the latter is briefly compared to other works that appeared (or were available as reprints) before its own publication.

Keywords: Chaos; Rössler system; topology of chaos.

1. Introduction

Chaos theory corresponds to the tool box according to which solutions to nonlinear dynamical systems that are sensitive to initial conditions are investigated. Like most scientific theories, the history of chaos as perceived by a broad audience has its leading contributors: Henri Poincaré (1854–1912) and Edward Lorenz (1913–2008) are commonly identified with such a status. The names of additional contributors depend on the sociological context: American mathematicians would quote Stephen Smale but Russian ones would honor Andreii Kolmogorov (1903–1987) or Yakov Sinai, French mathematicians would invoke David Ruelle, ecologists would mention Robert May, chemists would refer to Boris Belousov (1893–1970) and Anatol Zhabotinskii (1938–2008). The recognition of major contributions is mainly orientated by the point of view or the field invoked. Nevertheless, it is commonly admitted that Poincaré's seminal works [Poincaré, 1890, 1899] are considered as the foundation of chaos theory. If he did not exactly observe chaotic solutions, he emphasized the great sensitivity to initial conditions around homoclinic orbits which he discovered [Poincaré, 1899]. Since his early works, Poincaré introduced the phase space, the four types of fixed points in two-dimensional space — he did that a few years after Nikolaï E. Joukovsky (1847–1921) [Joukovsky, 1876] as shown by [Dobrovolsky, 1972] —, the Poincaré cross-section, the first-return map and bifurcations [Poincaré, 1881, 1885].

Most of these concepts were used by Lorenz while investigating a set of three ordinary differential equations producing a beautiful chaotic attractor [Lorenz, 1963] resulting from a severe truncation of the Navier–Stokes equations describing Rayleigh–Bénard convection. Lorenz applied to his dissipative system the techniques developed by his teacher David Birkhoff (1884–1944) in the context of conservative systems. A key point in Lorenz's 1963 paper is that a digital electronic computer was used to calculate the solutions to his set of differential equations representing the evolution of the system as a trajectory in the phase space. He thus provided for the first time a *phase portrait* of a chaotic attractor. A second example of aperiodic behavior investigated in the sixties should be mentioned: this is the galactic motion reduced to a dynamical systems with two degrees of freedom — close to the three-body problem — investigated by Michel Hénon and Carl Heiles using numerical simulations [Hénon & Heiles, 1964]. Their system was conservative and four dimensional. They did not show a trajectory in the phase space but investigated the solution by means of a cross-section. In particular, they observed "ergodic trajectories [that were] dense everywhere in the [chaotic] sea between [quasi-periodic] islands." This means that for some parameter values, the phase portrait is mainly filled by chaotic trajectories with but a few domains associated with quasi-periodic or periodic regimes. To our knowledge Lorenz thus remains the only one who published in the sixties a chaotic attractor in a plane projection of the phase space. Yoshisuke Ueda plotted one but did not publish it before the end of the seventies (see Appendix A).

As soon as difference or differential equations are considered, computers are necessary to figure out the structure underlying nontrivial solutions, that is, aperiodic solutions. Few scientists possibly faced up to aperiodic solutions before computers were used: Poincaré in his third volume of *New Methods in Celestial Mechanics* [Poincaré, 1899], Balthazar van der Pol (1889–1959) and van der Mark while investigating the triode [van der Pol & van der Mark, 1927], Mary Lucy Cartwright (1900–1998) and John Edensor Littlewood (1885–1977) while studying properties of the aperiodic solution to the forced "van der Pol equation" [Cartwright & Littlewood, 1947]. But they had no possibility to obtain a global — and accurate — view of the trajectory in phase space, thus missing the structure underlying aperiodic solutions. Only Poincaré started to have a clue about the complexity of such a structure by focusing his attention on the (unstable) periodic orbits as first used by George William Hill (1838–1914) in his Lunar theory [Hill, 1878].

Introduced by Poincaré due to their simplicity compared to the difficulty presented by differential equations, David Birkhoff studied diffeomorphisms of the cross-section associated with differential equations [Birkhoff, 1927] and Smale studied the global orbit structure of some diffeomorphisms of the cross-section [Smale, 1967]. Strongly influenced by Smale's ideas, David Ruelle and Floris Takens pushed the idea that "strange" as used in the 1970s — behavior may arise from relatively simple systems [Ruelle & Takens, 1971]: in other words, it was no longer necessarily required to invoke high dimensional systems to explain complex behaviors. But in spite of their crucial contribution in the development of "chaos theory", their abstract mathematical language and the lack of beautiful pictures did not let them have an obvious impact on a broad audience as chaos theory has today. In fact, Ruelle became popular for speaking about the Lorenz attractor and the "Japanese attractor" discovered by Ueda [Ruelle, 1980]. Nevertheless, for scientists, and especially for mathematicians [Aubin & Dalmedico, 2002], Ruelle (and in a less visible way, Takens) is an important contributor to the early development of chaos theory. But Ruelle and Takens did not have a school case (or even a picture) until they became aware of the Lorenz paper [Ruelle, 1976].

The first paper by Rössler was about a theoretical system for "Biogenesis" [Rössler, 1971]. Most of his following papers were about some kinds of chemical reactions interpreted by electronic circuits [Rössler & Seelig, 1972; Rössler, 1972a, 1974b]. The multivibrator [Rössler, 1972b, 1975] and the Bonhoeffer–van der Pol oscillator [Rössler, 1972a; Rössler & Hoffmann, 1972] were very often invoked. Rössler started to publish about chaos in 1976. He then flooded the "chaos market" with various types of chaos with suggestive names like "spiral chaos [Rössler, 1976a], "screw type chaos" and "funnel chaos" [Rössler, 1977c], "sandwich chaos" [Rössler, 1976d], "walking-stick map" [Rössler, 1977e], "folded-towel map" [Rössler, 1979b], "superfat attractor" [Kube *et al.*, 1993] among others. We choose to focus our attention, not on the most often quoted paper [Rössler, 1976c] but on Rössler's earliest paper on chaos [Rössler, 1976a], not so widely known but far richer. With this latter paper, Rössler published the second chaotic attractor displayed in the phase space. Due to his use of informal (nontechnical) terms and very suggestive pictures — he even demonstrated "the sound of chaos" in a lecture by connecting a loudspeaker to his computer, Otto E. Rössler quickly attracted a broad audience and he remains today as the one who introduced one of the two most investigated chaotic flows.

But his use of seemingly inadequate terms for a scientific paper led him to be not so widely recognized by his peers. Using a too elliptical way of writing, Rössler never spent time to introduce the firm background he used. As a consequence, what is commonly retained from his contribution to chaos theory reduces to the so-called Rössler attractor and, to a more restricted extent, to the first example of a hyperchaotic attractor. Our aim in the present paper is to detail what influenced the structure of this paper and to revisit its implicit content.

2. Short Biography up to 1980

Otto E. Rössler was born in 1940. His father, Otto Rössler (1907–1991), was a linguist recognized for having introduced a new system of Egypto–Semitic consonant correspondences and the term "Afro-semitic" languages [Rössler, 1971]. Strongly impressed by the "open mind" of his father and his religious mother, he tried to find his own way. As an adolescent, he built his own radio-transmitter and thus got acquainted with electronics while still in highschool at Tübingen. In 1957, he got an individual licence (DL9 KF) from the Deutscher Amateur Radio Club. He then studied medicine up to 1966 at the University of Tübingen. In 1966, he defended his inaugural dissertation — supervised by Erich Letterer (1895–1982) — for getting his grade of doctor in medicine [Rössler, 1966]. Deeply interested how Life could come from a "chemical soup", he exchanged letters with Carl-Friedrich von Weizsäcker (1912–2007) and met him. Under his recommendation, he spent one year (1966–1967) at the Max-Planck Institute for the Physiology of Behavior (Seewiesen) supervised by Konrad Lorenz (1903–1989) and Erich von Holst (1908–1962). Otto then spent two years at the University of Marburg where he was a medical assistant under the supervision of Reimara Waible who became his wife one year later. During that period, Otto wrote — in German — a first paper entitled "Contributions to the theory of spontaneously evolving systems I: a simple model class" to the *J. Theoretical Biology*. The editor, Robert Rosen (1934–1998), who was reading German, accepted the paper for publication but required a translation in English. Not yet fluent in English, Otto never made it and the paper remained unpublished. Interested enough by this first paper, Rosen honored Otto's application for a one-year position at the Center of Theoretical Biology (State University of New York at Buffalo). At this center, a very stimulating atmosphere was present, as Vahe Bedian reported from one of his stays (slightly after Rössler visited) [Bedian, 2001]:

> In the early 1970s, the temporary Ridge Lea campus of SUNY/Buffalo was home to the Center for Theoretical Biology and the Department of Biophysical Sciences, where I was a graduate student. It was a stimulating and supportive place to think and learn from some of the best in the field: Robert Rosen, Fred Snell, Robert Spangler, Robert Rein and Howard Pattee. In front of chalkboards and in the hallways, we discussed everything from the uncertainty principle, to von Neumann's automata, to neural networks, to Stuart Kauffman's binary switch networks, to the complexity of quantum mechanical computations.

Bedian also mentioned that Spangler was the one to go "beyond iterative simulations [to] formalize the model as a nonlinear dynamical system." This is exactly what Spangler and Snell did with the oscillating chemical reation they simulated in 1961 with a digital computer [Spangler & Snell, 1961] and in 1967 with an analog computer [Spangler & Snell, 1967]. In the latter, they showed a few periodic oscillations and a phase portrait of a limit cycle, which they identified with the synonymous concept in the textbook *Nonlinear Oscillations* published in 1962 by Nicholas Minorski (1885–1970) [Minorski, 1962]. Although Rössler did not meet Spangler and Snell during his stay, he later quoted their 1967 paper in [Rössler, 1975]. Rössler started to investigate some differential equations during his stay at Buffalo.

Friedrich-Franz Seelig who had a chair ("Lehrstuhl") for Theoretical Chemistry at the University of Tübingen offered Otto to join his new group with a stipend from the Deutsche Forschungsgemeinschaft (DFG). In the early 60s, Seelig had done his diploma work with Hans Kuhn and Fritz-Peter Schäfer to build an analog computer consisting of a network of electrical oscillators, connected to capacitors to solve the two-dimensional Schrödinger equation [Seelig *et al.*, 1962]. This system was excited by means of a radio frequency generator. In 1965, Seelig solved a two-dimensional Schrödinger equation with a digital computer (IBM 7090) [Seelig, 1965]. Rössler had met Seelig via Hans Kuhn who was working on the origin of Life, Otto's first research topic. Kuhn handed down

Rössler to Seelig as it were. Sharing an interest for the origin of Life, in differential equations and electronics (computers), they agreed that nonlinear systems like Otto's evolutionary soup and electronic systems (without self-induction and without coupling condensers) were virtually isomorphic [Rössler, 2010]. This triggered a cooperation project between Seelig — a quantum chemist — and Rössler — a medical physiologist — to look for reaction-kinetic analogs to electronic circuits.

Rössler joined Seelig at Tübingen returning from Buffalo in 1970. After being sent by the Division of Theoretical Chemistry to attend an EAI (Enterprise Application Integration) course on analog computing, he had to teach that topic, for which his radio-amateur past was useful. With the founding money obtained with his new position at Tübingen, Seelig bought (with 80 000 DM) an analog computer — a Dornier DO 240 (Fig. 1) — equipped with digital potentiometers, a digital clock and two function generators. With this computer, Seelig obtained limit cycles — plotted in phase space — with computer simulation of a linear chemical model [Seelig, 1971] and of a model for a spike oscillator [Karfunkel & Seelig, 1972]. As a "Stipend-holder" of the DFG, Rössler was free in his research and, started to study few-variable systems with Seelig. He started by investigating a chemical multivibrator [Rössler, 1972a]. To learn about the dynamics of such an electronic circuit,

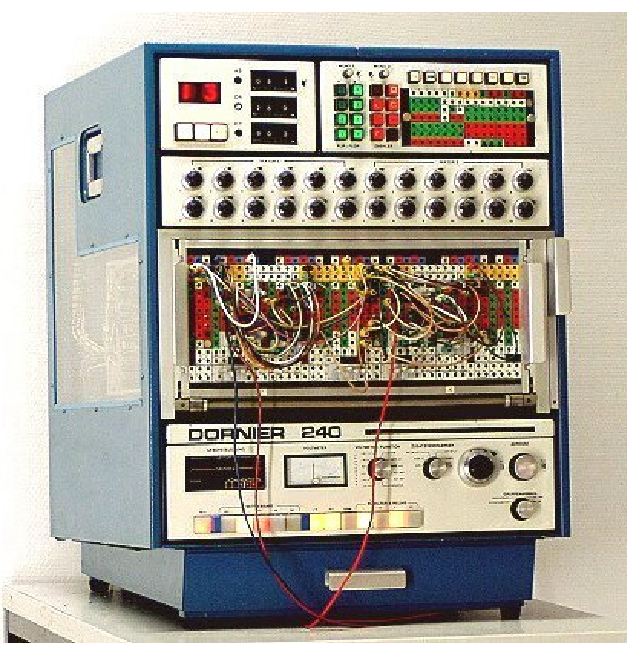
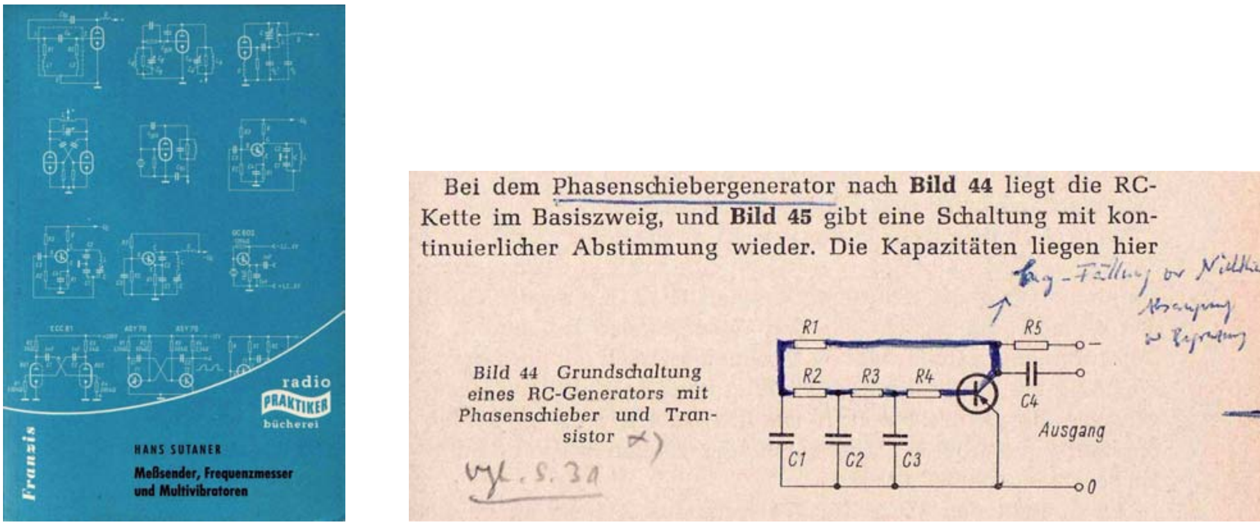


Fig. 1. Analog computer Dornier DO 240 as bought by Seelig in 1970.

he read the textbook by Aleksandr Andronov, S. E. Khaikin and Aleksandr Vitt in its 1966 English edition [Andronov *et al.*, 1966], that is, in a deeply revised version by N. A. Sjelstov since it contains more than 400 additional pages compared to the original version edited by Mandel'shtam (see [Pechenkin, 2002]).

Inspired by a little book entitled *Measuring-signal generators*, *Frequency Measuring Devices and Multivibrators* from the Radio-Amateur Library [Sutaner, 1966], Rössler and Seelig began to "translate" electronic systems into nonlinear chemical reaction systems (among them the RC-oscillator of Fig. 44 of that book as shown in Fig. 2). Morphogenetic reaction systems, devised by [Rashevsky, 1940] and [Turing, 1952], fitted in, enabling the design of a chemical oscillator based on a chemical flip-flop, that is, a bistable multivibrator that has two stable states in a subsystem and hence can be used as one bit of memory. The latter had been invented by William Henry Eccles (1875–1966) and Franck Wilfred Jordan [Eccles & Jordan, 1918, 1919]. Rössler remained fascinated by the multivibrator [Rössler, 1975] and the electronic Eccles–Jordan trigger as he called it in [Rössler, 1974b]. This had led to the "flip-flop" studied with Seelig [Seelig & Rössler, 1971; Rössler & Seelig, 1972]. Rössler necessarily associated the multivibrator with the universal circuit introduced by [Khaikin, 1930] and its description in phase space as in [Andronov *et al.*, 1966] (see Sec. 3.1). Most of Rössler's early papers — say between 1972 and 1975 — were devoted to chemical reactions that reproduce the dynamics underlying some electronic circuits, and many of them explicitly discussed the multivibrator [Seelig & Rössler, 1972; Rössler & Seelig, 1972; Rössler, 1972a, 1972b, 1975]. In 1972, with Dietrich Hoffmann, Rössler provided "a first evidence that the Belousov–Zhabotinsky reaction is a Bonhoeffer oscillator, i.e. a special type of chemical hysteresis oscillators" [Rössler & Hoffmann, 1972]. A link was explicitly made with relaxation oscillations as done by Bonhoeffer when he investigated a model for the excitation of nerves [Bonhoeffer, 1948]. The model proposed by Bonhoeffer has all characteristics of the so-called van der Pol equations, and the limit cycle drawn in the phase space by Bonhoeffer is very similar to the one published in [van der Pol, 1926]. This is why biologists sometimes speak about the "Bonhoeffer–van der Pol" oscillator. Rössler was exactly in the spirit of Bonhoeffer's works, trying out analogies



(a) (b)

Fig. 2. (a) Cover and (b) Fig. 44 with Rössler's handwriting regarding Hans Sutaner's book published in 1966.

between physiological or chemical problems and electronic circuits.

In this vein, Rössler and Seelig proposed a two-cellular homogeneous chemical multivibrator [Seelig & Rössler, 1971; Rössler & Seelig, 1972]. An example of a homogeneous system involving a two-variable bistable system (switch) found on the analog computer by Otto was

$$\begin{cases} \dot{A} = -k_2 A - k_3 B \dfrac{A}{K+A} + k_1 + k_6 C \\ \dot{B} = -k_2 B - k_3 A \dfrac{B}{K+B} + k_1 + \beta_B \\ \dot{C} = k_4 B - k_5 C. \end{cases} \tag{1}$$

This system is still quoted as one of the very first chemical reaction systems designed to implement logic circuits [Zauner, 2005]. Computer output of this abtract chemical reaction (Fig. 2 in [Rössler, 1975]) was compared to an electronic multivibrating device in [Chrilian, 1971]. Then starting to investigate the subsystem A–B by replacing the term $k_6 C$ in the first equation with a constant term β_A, Rössler commented in the paper submitted in 1971 [Rössler, 1975]:

> The equations of this partial system are well-known in electronics where they apply to the usual symmetrical RS flip-flop: the so-called Eccles–Jordan trigger [Eccles & Jordan, 1919]; only the nonlinear terms [...] are normally replaced by a more generally formulated class of functions (see [Andronov *et al.*, 1966, p. 309, Equation (5.61)]. However, the very system [A–B] is obtained, even in the electronic case, if n-channel field-effect transistors are employed as the active elements [Rössler, 1974a].
>
> If the standard analytical techniques used in electronics [Andronov *et al.*, 1966, p. 310]) are applied to the present special case, it is again found (a) that either equation, when set equal to zero, yields a curved nullcline; (b) that both nullclines intersect each other in either 1 or 3 steady states; (c) that the intermediate steady state is a saddle-point and the other ones (or the remaining one, respectively) are stable nodes; and (d) that the presence of additional limit sets (limit cycles) is excluded.

It is thus clear that Rössler was deeply influenced by the contribution of Andronov's group. This not only framed his early studies on chemical reactions but also his first studies on chaos as we will show in this paper.

Rössler thus started to design some three-variable oscillator based on a two-variable bistable system coupled to a slowly moving third-variable. The resulting three-dimensional system was only producing limit cycles at the time. In this period, Rössler also introduced *dynamical automata* as components for the building up of complex chemical reaction systems: in other words, he had in mind to build chemical reaction systems as complex as electronic circuits are [Rössler, 1972a]. At an international congress on *Rhythmic Functions in*

Biological Systems held on September 8–12, 1975 in Vienna, he met Art Winfree (1942–2002) again — a theoretical biologist who started his career as an engineering physicist and studied chemical waves [Winfree, 1972], circadian rhythm [Winfree, 1980] and cardiac arrhythmia [Winfree, 1989]. Winfree — also an expert in computers — was regularly exchanging letters with Rössler about oscillating chemical reactions or dynamical systems. Winfree was looking for a "kinetics with a source and no limit cycle" (Letter from Art to Otto, May 25, 1975). The concepts invoked in these letters were nullclines and bistability (Otto to Art, June 23, 1975), Lorenz equations (Art to Otto, September 17, 1975) with the comment "Guckenheimer, Li and Yorke are doing a long job on this Eq. now; not yet ready for press", differential systems, saddle and stable foci, damped oscillations, chemical monoflop, limit cycles, self-oscillations with the comment "I had already seen such a behavior of monoflops on the analog computer: when a chemical monoflop was just above the threshold of being self-oscillating, irregular spikes of differing amplitudes occurred" (Otto to Art, September 30, 1975). At this conference in Vienna, Winfree challenged Rössler in 1975 to find a biochemical reaction reproducing the Lorenz attractor. To stimulate Otto to the task, Art sent a collection of reprints and preprints with a letter dated October 7, 1975. The paper sent were:

(1) Lorenz, E. N. [1963] "Deterministic nonperiodic flow," *J. Atmospheric Sciences* **20**, 130–141.
(2) May, R. & Oster, G. F. "Bifurcations and dynamic complexity in simple ecological models" (preprint later published [May & Oster, 1976]).
(3) Hoppensteadt, F. C. & Hyman, J. M. "Periodic solutions of a logistic difference equation" (preprint later published [Hoppensteadt & Hyman, 1977]).
(4) Li, T. Y. & Yorke, J. A. "Period-three implies chaos" (preprint later published [Li & Yorke, 1975]).
(5) Guckenheimer, J., Oster G. F. & Ipaktchi, A. "Dynamics of density-dependent population models (preprint later published [Guckenheimer *et al.*, 1976]).

Otto was strongly impressed by Lorenz's paper: he thus proposed Winfree to write a "joint paper, entitled *Chemical Nonperiodic Flow*, 3 *examples*" in a letter dated October 15, 1975 (Winfree denied this offer, Art to Otto, October 22, 1975). Lorenz's paper influence is confirmed by the explicit quotation in the abstract of [Rössler, 1976a]. As it will be shown, Li and Yorke's paper [1975] also had a strong influence on Rössler's mind and was crucial for providing a numerical proof of chaos. During these times, Rössler failed to find a chemical or biochemical reaction producing the Lorenz attractor but he instead found a simpler type of chaos in a paper he wrote during the 1975 Christmas holidays [Rössler, 1976a]. This paper will be the core of the present work and we will carefully investigate its contents as well as the style in which it was written. It is only much later that Otto discovered jointly with Peter Ortoleva a biochemical reaction scheme producing a Lorenz-like dynamics [Rössler & Ortoleva, 1978]. The obtained attractor does not have the rotation symmetry of the Lorenz attractor, but it is characterized by a map equivalent to the one published by Lorenz [1963]. This type of chaos was later designated as "unimodal cut chaos" in [Letellier *et al.*, 2006]. Between 76 and 82, there were many other different types of chaos that were also proposed by Rössler (see [Letellier *et al.*, 2006] for a review).

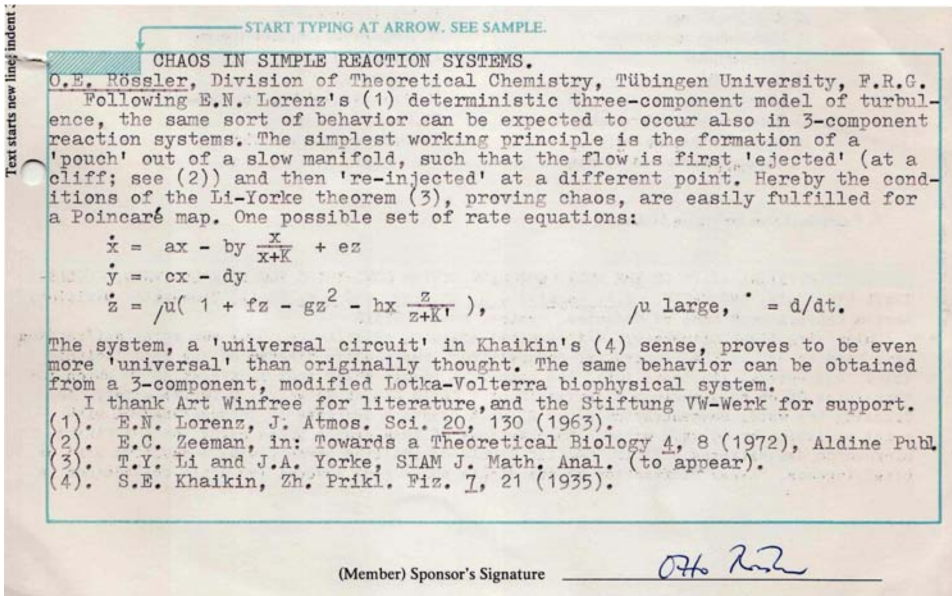
3. Otto E. Rössler's Main Influences

We will briefly review the three main infuences on Rössler while discovering his first chaotic system. According to an abstract submitted on 1 December 1975 for the 1976 *Biological Society Meeting* (Fig. 3), these influences are (i) Lorenz' 63 paper, (ii) Li–Yorke theorem and (iii) the universal circuit for which a paper by S. E. Khaikin — whose content is discussed in [Andronov *et al.*, 1966] — is quoted (see [Khaikin, 1930]).[1] But let us follow the chronology.

3.1. *The multivibrator by Andronov, Khaikin and Vitt*

As mentioned in the short biography, Rössler was quite acquainted with electronics. Since he was also attracted by dynamical systems, the textbook by Andronov, Khaikin and Vitt [Andronov *et al.*, 1966] became one of his favorite books in the early 70s. Indeed as soon as he realized that equations for

[1] This Khaikin's paper is erroneously dated by Rössler as 1935, and not as 1930 as it should have been.



Text starts new line; indent

START TYPING AT ARROW. SEE SAMPLE.

CHAOS IN SIMPLE REACTION SYSTEMS.

O.E. Rössler, Division of Theoretical Chemistry, Tübingen University, F.R.G.

Following E.N. Lorenz's (1) deterministic three-component model of turbulence, the same sort of behavior can be expected to occur also in 3-component reaction systems. The simplest working principle is the formation of a 'pouch' out of a slow manifold, such that the flow is first 'ejected' (at a cliff; see (2)) and then 're-injected' at a different point. Hereby the conditions of the Li-Yorke theorem (3), proving chaos, are easily fulfilled for a Poincaré map. One possible set of rate equations:

$$\dot{x} = ax - by\frac{x}{x+K} + ez$$

$$\dot{y} = cx - dy$$

$$\dot{z} = \mu\left(1 + fz - gz^2 - hx\frac{z}{z+K'}\right), \qquad \mu \text{ large}, \quad \dot{} = d/dt.$$

The system, a 'universal circuit' in Khaikin's (4) sense, proves to be even more 'universal' than originally thought. The same behavior can be obtained from a 3-component, modified Lotka-Volterra biophysical system.

I thank Art Winfree for literature, and the Stiftung VW-Werk for support.

(1). E.N. Lorenz, J. Atmos. Sci. 20, 130 (1963).
(2). E.C. Zeeman, in: Towards a Theoretical Biology 4, 8 (1972), Aldine Publ.
(3). T.Y. Li and J.A. Yorke, SIAM J. Math. Anal. (to appear).
(4). S.E. Khaikin, Zh. Prikl. Fiz. 7, 21 (1935).

(Member) Sponsor's Signature Otto Rössler

Fig. 3. Abstract submitted by Otto E. Rössler on 1 December 1975 for the Biological Society Meeting planned for 1976.

describing life would be too complicated as an exclusive object of research, he concentrated his interest on the basic chemical elements that could be used to build complex chemical reactions. The very first elementary circuit he investigated was a two-variable multivibrator [Rössler, 1972b]. The latter corresponds to an electronic circuit investigated by Henri Abraham (1868–1943) and Eugène Bloch (1880–1944) [Abraham & Bloch, 1919a, 1919b]. This circuit was then studied by Stephen Butterworth (1885–1958) [Butterworth, 1920], Edward Victor Appleton (1892–1965) and his pupil Balthazar van der Pol [Appleton & van der Pol, 1921; Appleton, 1922]. In fact, according to Andronov himself, the Russian scientist and his co-workers started to investigate the multivibrator in 1929 (see Appendix 5). They quickly focussed their interest on an intermediary circuit between a double *RC* circuit and a multivibrator [Andronov *et al.*, 1966]. This so-called *universal circuit* — a circuit as simple as possible with a wide variety of behaviors — was described by three differential equations [Andronov *et al.*, 1966]:

$$\begin{cases} \mu\dot{u} = E_a - Ri_a(u) - \left(1 + \dfrac{R}{\beta r}\right)u \\ \qquad + (1-\beta)\dfrac{R}{\beta r}z - v_1 \\ \dot{v}_1 = z \\ \dot{z} = \dfrac{C_1}{\beta(1-\beta)C_2}n - \left(1 + \dfrac{C_1}{\beta C_2}\right)\dfrac{z}{1-\beta} \end{cases} \tag{2}$$

where $i_a(u)$ describes the characteristic equation of the circuit. This set of equations is three-dimensional, thus requiring three variables to describe the motion in a phase space (Fig. 4). This is one of the most important methodological breakthroughs introduced by Andronov and co-workers. They thus described the trajectory drawn in the three-dimensional space spanned by two potentials measured on the circuit, potential u measured at one of the two triodes and potential v_1 measured at a condenser located between the two

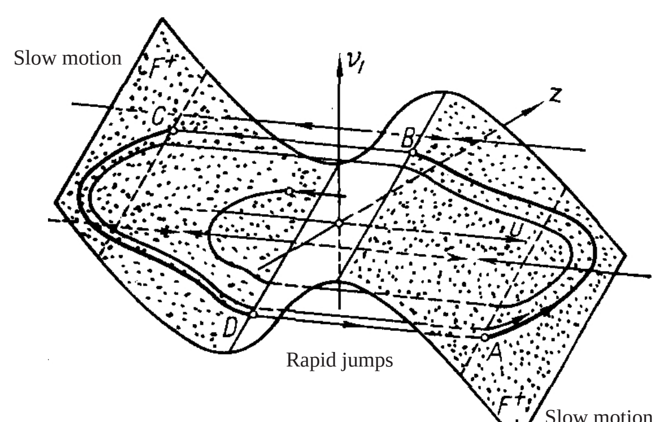


Fig. 4. Sketch used by Andronov and his co-workers to describe the trajectory produced by the universal circuit. What is noticeable is that they used a qualitative description in a three-dimensional phase space. The trajectory is organized around the "S"-shaped surface F where slow and fast motions are distinguished. (Adapted from [Andronov *et al.*, 1966].)

triodes, and the derivative $\dot{v}_1 = z$ of the second potential.

Oscillations were described as follows [Andronov *et al.*, 1966]:

> [...] the phase paths of "rapid" motion (jump) in the u, z, v_1 phase space recede away from the region $|u| \leq u_*$ of the surface F [...]. For $|u| \leq u_*$ only jumps of the voltage u are possible [...]. On the remaining part of the surface F (for $|u| > u_*$) [...] the paths of "rapid" motion approach the surface F [...]. On the portion F^+ of F where $|u| \leq u_*$ there are "slow" motions along paths [...]. Outside F^+ $u \to \infty$, for $\mu \to +0$ but $\dot{z}$ and $\dot{v}_1$ remain finite, therefore there are "rapid" motions along the paths $z = \text{const.}$, $v_1 = \text{const.}$ which lead to the surface F^+ where they pass into paths of "slow" motions. In due course all paths of "slow" motion pass into discontinuous jumps at $u = +u^*$ or at $u = -u^*$. It can easily be shown that all phase paths tend to a unique and stable limit cycle for $t \to +\infty$. Thus [...], whatever the initial conditions, discontinuous oscillations build up in the system.

In this description, Andronov and co-workers used a three-dimensional space to clearly distinguish "slow" and "fast" motions. They also explained why "jumps of the voltage" cannot be avoided. This therefore represents a dynamical analysis of the universal circuit.

The description provided by Andronov combines analytical computations and qualitative properties of the trajectory in the phase space. The figure drawn was thus deeply used to reach the conclusion that a stable limit cycle must exist. The unusual character of the description lies in combining physical properties of the system on the one hand and a representation of its evolution in the abstract phase space on the other. Strictly speaking, they should have been led to observe oscillations more complicated than periodic ones. But only periodic behaviors were discussed in the literature.

This contribution was impressive enough to frame Otto E. Rössler's mind in two ways: (i) it kept his attention focussed on the multivibrator and (ii) it introduced an S-shaped two-dimensional surface to explain how to produce nontrivial oscillating solutions. By "trivial" it is here meant a limit cycle with a nearly constant speed. From a dynamical point of view, the background provided by Andronov and his co-workers was the most important influence on Rössler's mind before 1975.

3.2. *The Lorenz paper*

3.2.1. *Phase space*

Lorenz's paper also strongly influenced the way in which Rössler's first paper on a chaotic system was written. The best testimony to this influence is provided by the structure of the paper. Edward Lorenz started his paper [Lorenz, 1963] by an introduction on tuburlent flows and weather forecasting. Focusing on "deterministic equations which are idealizations of hydrodynamical systems", Lorenz concentrated on "solutions which never repeat their past history exactly". The second section was devoted to general definitions about trajectories in phase space. One clear breakthrough in the study of dynamical systems reintroduced by Lorenz was the use of projection of phase space. Not in an empirical way, but he stated rather clearly that a system governed by the set of equations

$$\dot{X}_i = F_i(X_1, X_2, \ldots, X_M), \quad (i = 1, \ldots, M) \qquad (3)$$

"may be studied by means of *phase space* — an M-dimensional Euklidean space Γ whose coordinates are $X_1, \ldots, X_M$". Lorenz then was very clear about what was represented here and who introduced the concept:

> Each *point* in phase space represents a possible instantaneous state of the system. A state which is varying in accordance with (3) is represented by a moving *particle* in phase space, traveling along a *trajectory* in phase space. For completeness, the position of a stationary particle, representing a steady state, is included as a trajectory.
>
> Phase space has been a useful concept in treating finite systems, and has been used by such mathematicians as Gibbs [1902] in his development of statistical mechanics, Poincaré [1881] in his treatment of the solutions of differential equations, and Birkhoff [1927] in his treatise on dynamical systems.

No doubt that Lorenz was acquainted to such background through Birkhoff's work. Birkhoff was Dean of the Faculty of Arts and Science at Harvard University where he taught since 1912. Lorenz got his MA in mathematics from Harvard University in 1940 and, he attended Birkhoff's lectures. Birkhoff is well known to be one of the continuators of Poincaré's work as told by Veblen [2001]:

> As remarked by Marston Morse "Poincaré was Birkhoff's true teacher". I remember well how frequently, in the walks we used to take together during his sojourn in Princeton, Birkhoff used to refer to his reading in Poincaré's *Les Méthodes Nouvelles de la Mécanique Céleste*, and I know that he was intensively studying all of Poincaré's work on dynamics. In a very literal sense Birkhoff took up the leadership in this field at the point where Poincaré laid it down.

There is therefore a clear bridge between Poincaré and Lorenz.

The use of phase space was one of the very key points in Lorenz's paper. In contrast to this, many other contributions using electronic or analog computers published around the 60's did not use that concept. Those we identified were

(1) Tsuneji Rikitake (1921–2004) in investigating earth magnetic field reversals in 1958 [Rikitake, 1958];
(2) Arkadii Grasiuck and Anatoly Oraevsky who investigated the dynamics of a laser system in 1964 [Grasiuk & Oraevsky, 1964];
(3) Derek Moore and Edward Spiegel who studied a simple model for pulsating stars in 1966 [Moore & Spiegel, 1966].

But these other three contributions only showed excerpts from time series as reported in Fig. 5. Note that all these systems are quadratic — including linear and nonlinear terms up to the second degree — with a symmetry property. Rikitake's model is a set of three ordinary differential equations with a rotation symmetry — as also the Lorenz system has — and the Moore and Spiegel system is a set of three differential equations with an inversion symmetry. The Grasiuk and Oraevsky's model is four-dimensional with a rotation symmetry. All of them produce very similar time series. At first sight, one could conclude that the underlying dynamics are equivalent but a topological analysis — in phase space — reveals that only the attractor solution to the Rikitake model is topologically equivalent to that of the Lorenz system. The Moore and Spiegel attractor has a much more complex topology, and the attractor produced by the Grasiuk and

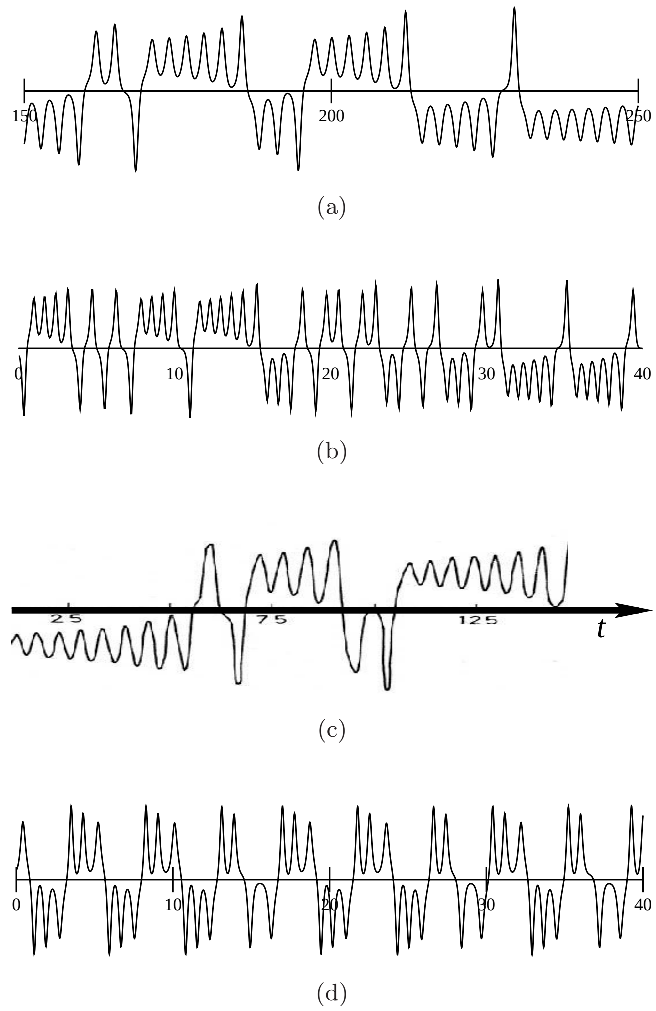


Fig. 5. Time series observed around the 60s by numerical integration of ordinary differential equations modeling the respective systems investigated. (a) Rikitake [1958]: Earth magnetic field reversals. (b) Lorenz [1963]: Rayleigh–Bénard convection. (c) Grasiuk and Oraevsky [1964]: laser system. (d) Moore and Spiegel [1966]: pulsating star.

Oraevsky model looks like a Lorenz attractor rotating around its rotation axis [Letellier & Ginoux, 2009]. Another system is discussed separately (see Appendix 6) since it was discovered in the early 60s but only published by the end of the 70s.

3.2.2. *The stability of periodic solutions*

The third section of Lorenz' paper introduced some definitions about the stability of "nonperiodic flow." This was the mathematical background inherited from Birkhoff since Lorenz wrote explicitly that his work was "influenced by the work of Birkhoff [1927] on dynamical systems, but differs in that Birkhoff was concerned mainly with conservative systems."

Nemytsky and Stepanov's book [1960] was also quoted but Lorenz himself wrote that this quotation was added because it was required by one of the referees of his paper [Lorenz, 1993]. Some definitions about stable and unstable points, periodic, quasi-periodic and nonperiodic solutions were provided. He also stated that "two states differing by imperceptible amounts may eventually evolve into two considerably different states." As a consequence, "an acceptable prediction of an instantaneous state in the distant future may well be impossible." Such sensitivity to initial conditions was one of the relevant points highlighted by David Ruelle by the mid 70s to distinguish chaos from other qualitative types of dynamical behavior [Ruelle, 1976].

3.2.3. *Numerical integration and application of linear theory*

The procedure for integrating numerically nonconservative systems was then discussed in Section IV of Lorenz's paper and, Section V was devoted to the convection equations for the Rayleigh–Bénard convection introduced by Saltzman [1962]. Lorenz reduced them to the set of three ordinary differential equations

$$\begin{cases} \dot{x} = -\sigma x + \sigma Y \\ \dot{y} = -xz + Rx - y \\ \dot{z} = xy - bz. \end{cases} \tag{4}$$

Then Lorenz applied linear theory to these equations in Section VI. It was shown that the trajectory was always bounded, and that "each small volume shrinks to zero" as the time goes to infinity, that is, that the system was dissipative (nonconservative). The stability of the fixed points was then studied and Lorenz showed that the solution oscillates around the two fixed points defined by $x_\pm = y_\pm = \pm\sqrt{b(R-1)}$ and $z = R-1$. Section VII was devoted to the numerical integration of differential equations (4) with parameter values $R = 28$, $\sigma = 10$ and $b = 8/3$. Computations were performed on a Royal McBee LGP-30 electronic computer. Lorenz provided 6000 iterations. Since one second was required per iteration, each run took roughly one hour and forty minutes. Lorenz showed a short portion of the trajectory, typically 500 iterations, chosen after the first 1400 iterations from the initial conditions $x_0 = 0$, $y_0 = 0.1$ and $z_0 = 0.0$. A recomputed trajectory with a modern computer and the same parameters leads to the trajectory shown in Fig. 6.

3.2.4. *Topological analysis*

To provide an idea of how the trajectory was organized in three-dimensional phase space $\mathbb{R}^3(x, y, z)$, Lorenz introduced "isopleths" that return the value of x as a smooth single-valued function of y and z. Isopleths allow to represent the "surface" on which the trajectory evolves [Fig. 7(a)]. Lorenz was thus able to show that the trajectory "passes back and forth from one spiral to the other without intersecting itself." This surface was topologically equivalent to what is now called a branched manifold — or a template — on which all trajectories can be drawn [Fig. 7(b)]. Such a manifold was used since the mid 1970s by Franck Williams

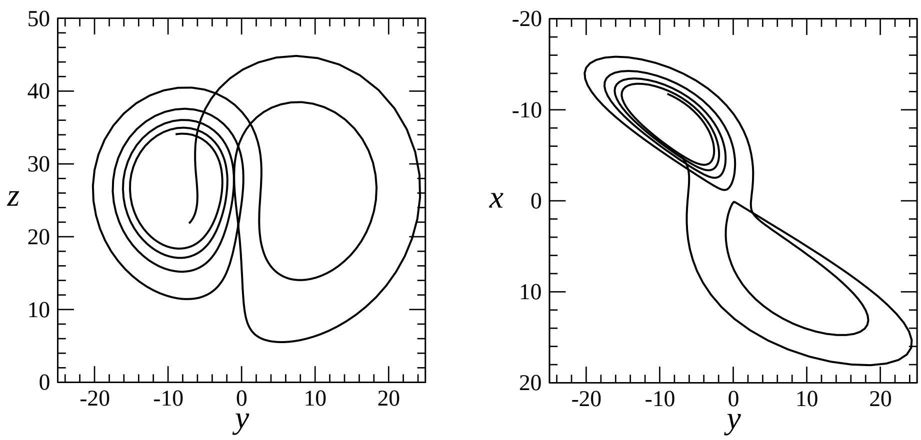
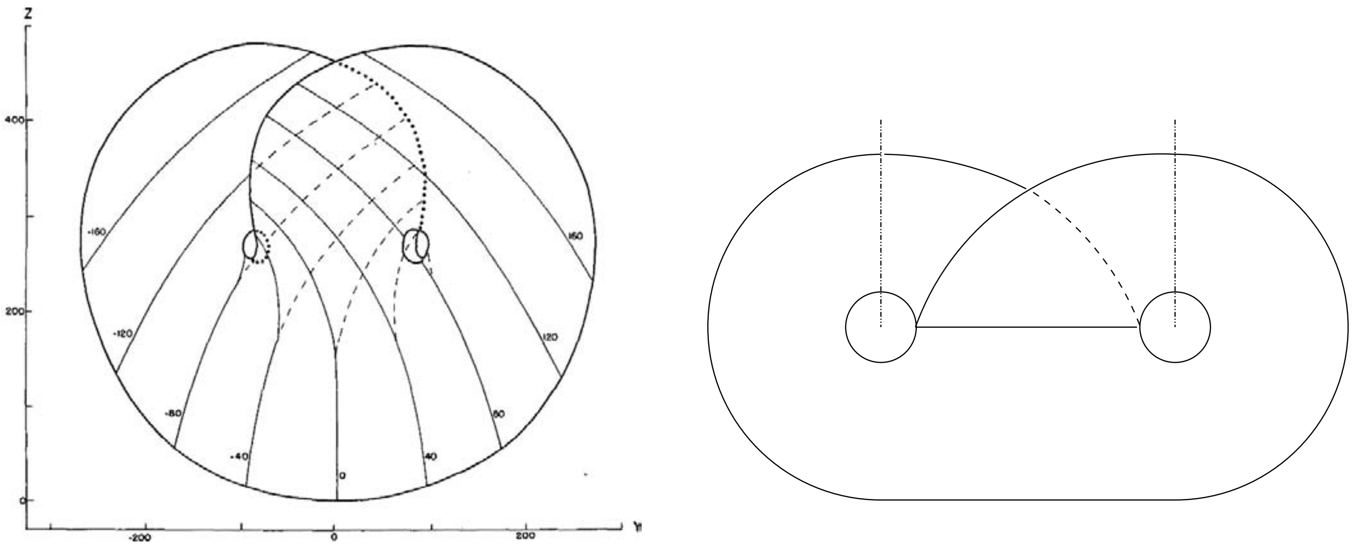


Fig. 6. Numerical solution to the Lorenz equations (4). Projections on the x–y and the y–z planes in phase space of the segment of the trajectory extending from iteration 1400 to 1900. These segments slighlty differ from those obtained by Lorenz in 1963.



(a) Isopleths plotted by Lorenz

(b) Birman's branched manifold

Fig. 7. (a) Isopleths of x as a function of y and z (thin solid curves). Where two values of x exist, the dashed lines are isopleths of the lower value. Heavy solid curve, and extensions as dotted curves, indicate natural boundaries of surfaces. (b) Representation of the associated branched manifold drawn by Williams. The two-component Poincaré section associated with the maxima of variable z is also drawn.

for describing the Lorenz attractor [Williams, 1977, 1979] as "a picture already present in Lorenz' paper (compare Figs. 7(a) and 7(b)). As Williams wrote [Williams, 1977], "a computer gives the same picture up to a smooth deformation when programmed to find the attractor of the system". The branched manifold was important as a knot holder, that is, to synthetize the relative organization of unstable periodic orbits embedded within the attractor, as later shown by Birman and Williams [1983].

3.2.5. *First-return map to maxima*

Then Lorenz proposed a first-return map to maxima of variable z in order to identify the possible periodic sequences that can be produced. It helped him to conclude that

> the periodic trajectories, whose sequences of maxima form a denumerable set, are unstable, and only exceptional trajectories, having the same sequences of maxima, can approach them asymptotically. The remaining trajectories, whose sequences of maxima form a nondenumerable set, therefore represent deterministic nonperiodic flow.

This argument was used to show that trajectories were actually nonperiodic since unstable periodic orbits were "exceptional", that is, the probability to have a trajectory remaining in the neighborhood of a periodic orbit was nearly zero.

Lorenz then used a first-return map to describe the dynamics governing the transitions from one spiral to the other:

> ... the trajectory apparently leaves one spiral only after exceeding some critical distance from the center. Moreover, the extent to which this distance is exceeded appears to determine the point at which the next spiral is entered; this in turn seems to determine the number of circuits to be executed before changing spirals again.

In order to investigate that feature carefully, Lorenz used the successive maximum values of z. He thus plotted the value of the $(n+1)$th maximum value of z versus the nth maximum (Fig. 8). This is what is now called a *first-return map* to a Poincaré section. Lorenz introduced that tool for having "an empirical prediction scheme" allowing to predict the number of "circuits" (oscillations around one of the focus fixed points) described by the trajectory between two successive transitions from one spiral to the other. With such a map, it is possible to follow through how the trajectory visits the attractor using a simple geometric construction (Fig. 8). The increasing branch (left part of the map) corresponds

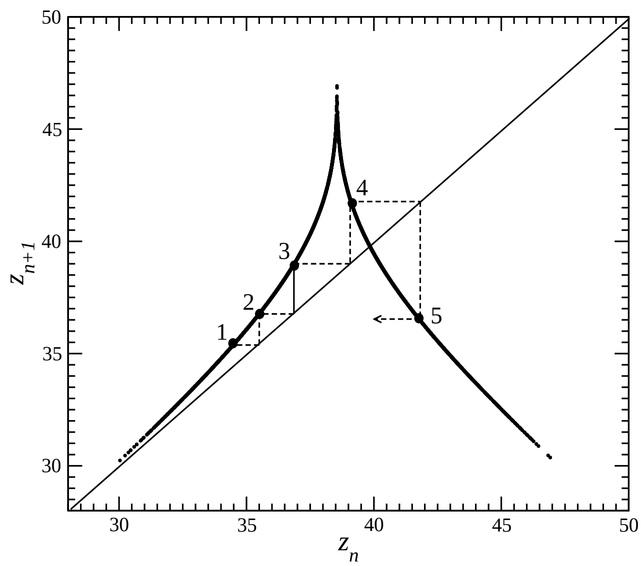


Fig. 8. Successive values of relative maximum plotted as z_{n+1} versus z_n as shown in [Lorenz, 1963]. The bisecting line has been added to make explicit the geometric construction (dashed line) that allows to track the evolution of the trajectory within the attractor.

to the successive oscillations around the same focus and the decreasing branch (right part) is associated with a transition from one spiral to the other. For instance, as shown in Fig. 8, starting from point 1, there are thus two oscillations in the initial spiral (points 2 and 3), then one transition in the other spiral (point 4) and, finally, a return to the initial spiral (point 5) before new oscillations in the initial spiral, and so on.

To conclude, the most important points used by Lorenz were (i) plotting the trajectory in plane projections of the phase space, (ii) showing that the trajectory can be described as evolving on a surface and (iii) using a first-return map (or a Poincaré map) to show that the trajectory is nonperiodic with the help of periodic sequences.

3.3. *Main results of Li and Yorke's paper*

The paper published in 1975 by Li and Yorke [1975] remains highly reputed for (i) having introduced the term chaos and (ii) providing a theorem that can be understood as follows: as soon as a system has a period-3 orbit for solution, then there is chaos. The term chaos was introduced for designating "complicated phenomena [that] may sometimes be understood in terms of simple model" [Li & Yorke, 1975]. In that sense, chaos was used in a quite adequate way since it traditionally designates the "*indescribable* state of Earth before creation". A simple difference equation (a second-order polynomial) may have surprisingly complicated dynamic behavior", complicated meaning here: not actually understood. Similar conclusions could be obtained from the adverb "chaotically" used once in Ruelle and Takens' paper published in 1971 [Ruelle & Takens, 1971]. James Yorke himself recently conceded that defining clearly what is chaos remains an open problem, particularly because it depends on the context in which it is used [Yorke, 2009]. It has to be noted that the word "chaos" also appeared in the title of a paper published by May in 1974 in which he quotes Li and Yorke's preprint, and in the subsequent paper by Guckenheimer, Oster and Ipaktchi [Guckenheimer *et al.*, 1976], all of them in the collection sent by Winfree to Rössler.

Second, the most important theorem proved by Li and Yorke was

> Let J be an interval and let $F : J \mapsto J$ be continuous. Assume there is a point $a \in J$ for which the points $b = F(a)$, $c = F^2(a)$ and $d = F^3(a)$, satisfy
>
> $$d \leq a < b < c \quad (\text{or } d \geq a > b > c).$$
>
> Then
>
> (1) For every $k = 1, 2, \ldots$ there is a periodic point in J having period k.
>
> (2) Furthermore, there is an uncountable set $S \subset J$ (containing no periodic points), which satisfies the following conditions:
>
> (a) for every $p, q \in S$, with $p \neq q$,
>
> $$\lim_{n\mapsto\infty} \sup|F^n(p) - F^n(q)| > 0$$
>
> and
>
> $$\lim_{n\mapsto\infty} \inf|F^n(p) - F^n(q)| = 0.$$
>
> (b) for every $p \in S$ and periodic point $q \in J$,
>
> $$\lim_{n\mapsto\infty} \sup|F^n(p) - F^n(q)| > 0.$$

Consequently, when $q = a$, that is, there is a periodic point with period 3, there is period-k point for any $k = 1, 2, \ldots$ Furthermore, there is an uncountable subset of points x in J which are not even asymptotically periodic. This only means that there is an uncountable subset of unstable periodic orbits. This is a proof of the argument previously quoted from [Lorenz, 1963] as the letter by Winfree to

Rössler dated on September 17, 1975 seems to confirm. On the other hand, it has to be noted that the Li–Yorke theorem is in fact included in the Sharkovsky's theorem [Sharkovsky, 1964].

Chaotic behaviors will be only encountered when there is no stable periodic point. This last property is quite hard to prove and, for instance, a computer assisted proof for the chaoticity of the Lorenz attractor was only obtained in 1999 [Tucker, 1999]. On the other hand, the Lozi map [Lozi, 1978]

$$\begin{cases} \dot{x}_{n+1} = y_n + 1 - a|x_n| \\ \dot{y}_{n+1} = bx_n \end{cases} \tag{5}$$

widely known for providing a two-dimensional chaotic attractor was recently proved as only having "giga-periodic orbits" when iterated with finite precision [Lozi, 2006]. For instance, for parameter values ($a = 1.7$ and $b = 0.5$) for which a chaotic attractor was proved as corresponding to a chaotic solution [Misiurewicz, 1980], a giga-periodic limit cycle of period 436 170 188 959 was obtained after 19 hours of computation. Two different limit cycles were obtained for the Hénon map ($a = 1.4$ and $b = 1.3$) with period equal to 3 800 716 788 and 310 946 608, respectively. These two limit cycles were obtained from different initial conditions. It is nearly impossible to actually obtain an aperiodic orbit using numerical simulation. Consequently, when a map of the interval has a period-3 orbit, one can conclude that there is at least one orbit of each period and these orbits are nondenumerable. But it remains to prove that there is no stable periodic point embedded, something that remains nontrivial to be shown in most cases.

Strictly speaking, showing that there is a period-3 orbit in a uni-dimensional map is not enough to prove that the behavior is chaotic. But it is sufficient to show that you have an infinite number of periodic orbits and that any periodicity is realized as a periodic orbit. In addition to that, if a proof for the underlying determinism is obtained (trivial when the trajectory results from iterations of a set of deterministic equations) as well as a proof of the boundedness of the behavior (numerically, it is a rather good approximation to wait for a long time and to check whether there is a bounded surface that is never crossed again once the trajectory is inside) and a proof for the sensitivity to initial conditions, it may quite confidently be concluded that the behavior under investigation is chaotic. (But remember that the Lozi map is a good counter-example.) For scientists working with an experimental data set or with "computer experiments", there is no rigorous way to distinguish an arbitrarily long periodic orbit from a chaotic solution. Thus, numerical experimentalists often used the existence of a period-3 orbit as a proof for an uncountable subset of nonperiodic points and implicitly assumed that the studied solution was chaotic.

4. Earliest Rössler's Paper on Chaos

4.1. *Phase space and chaotic attractor*

One of the most surprising points made at the beginning of first Rössler's paper about chaotic systems is that he claimed that "chaos is known for a long time", referring to Poincaré's work on two coupled oscillators, Arnold's map, Smale's Horseshoe map and Ruelle and Takens' strange attractor introduced in the context of the route to turbulence. As far as we know, Poincaré did not investigate explicitly two coupled oscillators but made his important contribution on complex behavior with a sensitivity to initial conditions while working on the three-body problem [Poincaré, 1899] (which involves two conservative oscillators, as explained below). Indeed, Poincaré investigated in the first volume of his *New Methods for Celestial Mechanics* a problem corresponding to a simplified case of a three-body problem — A, B and C — interacting according to a gravitation law [Poincaré, 1892]:

> The simplicity is yet larger if we assume that the mass of C is much larger than the mass of A and that the distance AC is very large (this is the case in the Lunar theory). If we assume AC infinitely large and the mass of C infinitely large, in such way that the angular velocity of C on its orbit remains finite; if at the same time, mass B is related to two moving axes, that is, to an axis Aξ corresponding to AC and an axis Aη perpendicular to the first, the equations of motion become, as M. Hill showed:
>
> $$\begin{cases} \dfrac{\mathrm{d}^2\xi}{\mathrm{d}t^2} - 2n\dfrac{\mathrm{d}\eta}{\mathrm{d}t} + \left(\dfrac{\mu}{r^3} - 3n^2\right)\xi = 0 \\ \dfrac{\mathrm{d}^2\eta}{\mathrm{d}t^2} + 2n\dfrac{\mathrm{d}\xi}{\mathrm{d}t} + \dfrac{\mu}{r^3}\eta = 0 \end{cases} \tag{6}$$
>
> [where] n designates the angular velocity of C.

Using $\xi = x, \dot{x} = y, \eta = u$ and $\dot{\eta} = v$, these equations take the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = 2n\,v - \left(\dfrac{\mu}{r^3} - 3n^2\right)x \\ \dot{u} = v \\ \dot{v} = -2n\,y - \dfrac{\mu}{r^3}u \end{cases} \tag{7}$$

that may be decomposed into two oscillators

$$\begin{cases} \dot{x} = y \\ \dot{y} = \left(\dfrac{\mu}{r^3} - 3n^2\right)x \end{cases} \quad \text{and} \quad \begin{cases} \dot{u} = v \\ \dot{v} = -\dfrac{\mu}{r^3}u. \end{cases} \tag{8}$$

Poincaré never considered explicitly that problem as a two coupled oscillator problem. In fact, May — who was quoted by Rössler when referring to Poincaré's problem with two coupled oscillators — just mentioned as a "simple mechanical example, a double pendulum" that is "capable of extraordinarily bizarre motions if the springs are nonlinear, a fact well known to Poincaré" (see Fig. 9) [May & Oster, 1976]. To the best of our knowledge, there is no such problem investigated in Poincaré's collected works.

Arnold as well as Smale investigated the global structure of periodic orbits. Ruelle and Takens [1971] were mainly concerned with showing that a torus T^4 — that is, a torus resulting from four different oscillators — was enough, under nonlinear coupling, to have a "strange attractor which is locally the product of a Cantor set and a piece of two-dimensional manifold" [Ruelle & Takens, 1971]. From a practical point of view, this corresponds to a behavior that can be decomposed into an infinite number of frequencies. All these examples dealt with nonperiodic solutions. In that sense, for sure, chaos was known before but it was certainly not very broadly known although the papers by May [1974] and Li and Yorke [1975] were already published with that word in the title. This term was also used by Guckenheimer, Oster and Ipaktchi in the preprint Rössler received and later published [Guckenheimer *et al.*, 1976]. The preprint was mentioned in a letter written by Rössler: "There is much enchantment (and solid results) in the Oster–May–Guckenheim (sic) analysis" (Otto to Art, October 15, 1975). In their paper, Guckenheimer and co-workers also wrote

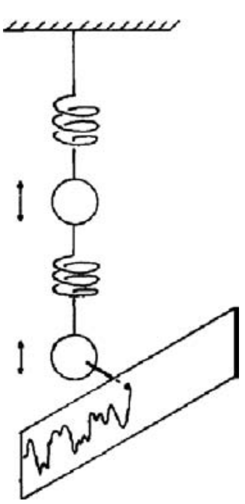
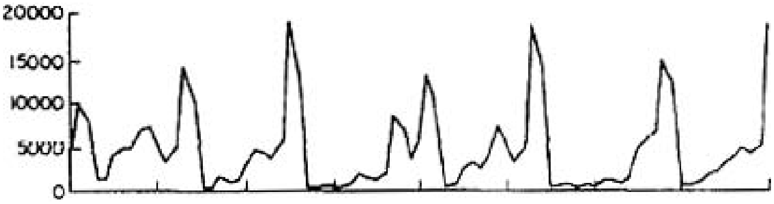


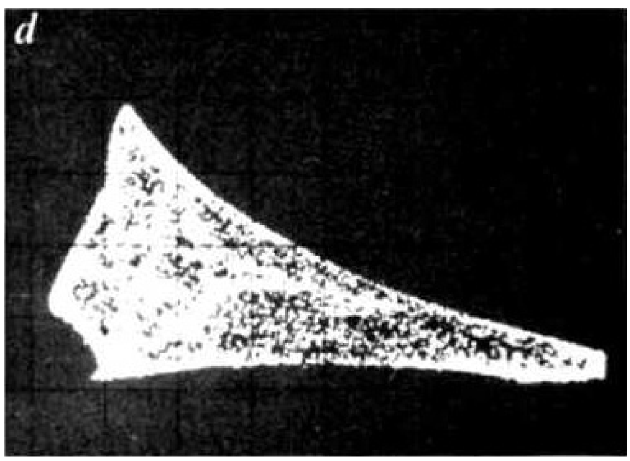
Fig. 9. Schematic illustration of a double pendulum with nonlinear springs and consequently, an apparent chaotic strip-chart record of its motion. (After [May & Oster, 1976].)

> Several authors have pointed out recently that even simple deterministic models can exhibit apparently chaotic behavior which is essentially indistinguishable from a random process [Li & Yorke, 1975; May, 1974, 1975]. This blurs the distinction between deterministic and stochastic effects in models. The capacity of familiar dynamical systems to display complicated behavior has been known since Poincaré's discussion of "homoclinic points" in Hamiltonian systems. During the **last decade**, significant progress has been made toward understanding the nature of such complex behavior.

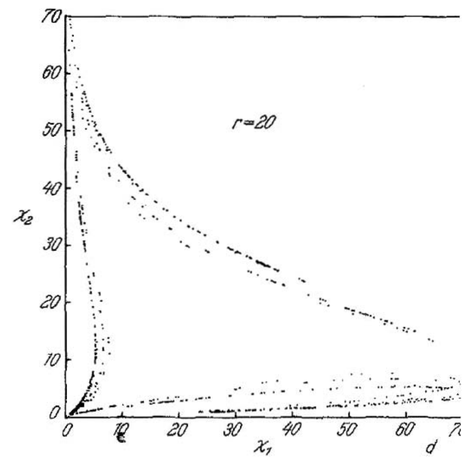
Here Guckenheimer and co-workers did not only quote papers by scientists already mentioned above, but also two others far less often quoted. The oldest paper is the one published in 1968 by Pennycuick, Compton and Beckingham [Pennycuick *et al.*, 1968]. They investigated the growth of a population divided into age-groups, with the aid of a "computer model", and observed irregular oscillations [Fig. 10(a)] which they just described as "alternating peaks of large and small amplitude." The second paper was published in [Beddington *et al.*, 1975]. In it the word "chaos" was used for "cycles of any integral period or complete aperiodicity, depending on the initial conditions" [Beddington *et al.*, 1975]. In this paper, many further works [Lorenz, 1963; May, 1974; Li & Yorke, 1975] about "chaos" were quoted, justifying not only the use of the term "chaos" but also giving a representation in phase space [Fig. 10(b)] to reveal the structure underlying the aperiodic solution described. Their chaotic



(a)



(b)



(c)

Fig. 10. Chaotic behaviors obtained by computer simulations in (a) Chaotic time series observed by Pennycuick *et al.* [1968], (b) Beddington and co-workers [Beddington *et al.*, 1975] and (c) Guckenheimer and co-workers [Guckenheimer *et al.*, 1976] (but only published in 1977, the manuscript was received by the editor on September 3, 1976). Rössler only knew the presentation given by Guckenheimer and his co-workers.

attractor was a solution to a two-dimensional map modeling a host-parasite system.

At that time, the word chaos designated aperiodic solutions that could not be described more accurately. The terms "strange" or "chaotic" were used equivalently at least up to 1984, that is, up to the article "strange attractors that are not chaotic" by Celso Grebogi, Edward Ott, Steve Pelikan and James Yorke [Grebogi *et al.*, 1984]. These scientists chose to use these two words as follows:

> *Chaotic* refers to the dynamics on the attractor, while *strange* refers to the geometrical structure of the attractor. [...]
>
> *Definition.* A chaotic attractor is one for which typical orbits on the attractor have a positive Lyapunov exponent.[2]

Chaotic means "sensitive to initial conditions" and strange is associated with the property announced in [Ruelle & Takens, 1971], that is, an "attractor which is locally the product of a Cantor set and a piece of two-dimensional manifold". Thus, although Guckenheimer and co-workers used strange, the formal definition can be taken for chaotic. It was [Guckenheimer *et al.*, 1976]:

> By a "strange attractor", for a map $f(\cdot)$ we mean an infinite set Λ with the following properties:
>
> (1) Λ is invariant under $f(\cdot)$, i.e. $f(\Lambda) = \Lambda$.
> (2) Λ has an orbit which is dense in Λ.
> (3) Λ has a neighborhood a consisting of points whose orbits tend asymptotically to $\Lambda : \lim_{t\to\infty} f^{(t)}(a) \subset \Lambda$.
>
> The requirement that Λ be infinite guarantees that Λ consists of more than one single periodic orbit.
>
> The central — and startling — fact about strange attractors is that orbits on or near them may behave in an essentially

[2]From this single definition, many wrong conclusions were published about "chaotic" experimental time series since such a positive Lyapunov exponent is not sufficient to ensure the chaoticity of the dynamics underlying a given data set.

chaotic and unpredictable fashion. Thus, despite the fact that the model is completely deterministic, the dynamical behavior of trajectories can only be predicted statistically!

Although having one clear definition, Guckenheimer and his co-workers confessed that only statistical predictions were possible. Anyway they performed an analysis in different steps:

(1) argue that the map has a strange attractor,
(2) examine the topology of the attractor,
(3) examine the nature of orbits in the attractor,
(4) discuss the "statistical mechanics" of the attractor,
(5) etc.

For the first item, the problem consists in determining whether the map in question has an orbit that is dense in Λ, that is, that there is no limit cycle present. But, as recently shown by Lozi [2006], this is an illusory task, and Guckenheimer *et al.* were forced to conclude "from our simulations, it does not appear to have one. If there are stable periodic orbits their domains of attraction are quite small and their periods very long. On a reasonable time scale the dynamics are very similar to that expected in a strange attractor." At a second step, they used the Horeshoe map onto the square introduced by Smale [1967] to conclude that "at each iterate of f the square is stretched, twisted and folded onto itself. [$\ldots\Lambda$] is a one-dimensional set composed of a Cantor set of segments which run longitudinally on the square. These segments fold and join in a pairwise manner. As they fold they must cross, so some of the points of Λ lie at the intersection of two segments." They were thus able to show that the Cantor set introduced by Ruelle and Takens is an ingredient of a strange attractor. Rössler was deeply interested in this part of the paper: "I will have to study the folding trick in detail" (Otto to Art, October 15, 1975).

As Guckenheimer and co-workers had to admit, "the topological structure of Λ is enormously complicated; moreover, the details of the topology are quite fragile (structurally unstable)." Since it appeared hopeless to fully describe the topology of Λ, they switched to item # 3, that is, to describe the orbits. In order to do so, they used a symbolic dynamics and a Markov transition matrix to detail how a point is mapped into its image under f. In the end, they also investigated the statistical mechanics of the attractor using a measure μ, that is, the proportion of time (or the fraction of phase points) that the orbit spends in different regions of Λ. Then they computed the Markov transition matrix between these different regions.

As can be seen from the example of the paper by Guckenheimer and his co-workers, investigating the topology of chaos was not an easy task. Facing the complicated structure of chaotic attractors, they switched to the study of periodic orbits, the "sole keyhole, so to speak, that we can try to penetrate into a place that had been impossible to enter up to the present", to quote Poincaré [1892] at this point. Chaotic behavior was more or less well defined but its characterization remained a difficult task. As Poincaré, Birkhoff, Lorenz, Smale, Li and Yorke, Guckenheimer, Oster and Ipaktchi showed, periodic orbits do open up a possibility to attack the problem.

Thus — as can be seen by now — to use the word "chaos" as a well known concept was a little bit too strong, but it enabled Rössler to avoid writing too much about it, by just quoting previous works. In spite of the appearances, he was right with this. By this time, only few scientists were already able to figure which type of solution was invoked by this word. He nevertheless assumed that it was not necessary to (re)write more about it, being afraid to write "trivial" things, trying not to bore his readers. In a certain sense, Otto thus implemented the common syndrome that what he understood was surely trivial. As a result, we can show that his writing is full of implicit contents, neither trivial nor (widely) understood before him.

From the previously quoted works, it has now become clear that the most relevant ingredient for investigating nonperiodic solutions is the concept of phase space. While Lorenz spent a full section on this requirement, Rössler in the context of chaotic behaviors continued to consider phase space as a natural concept associated with differential equations and just used it as an obvious concept. Lorenz, Andronov and co-workers, Guckenheimer and co-workers had already used it. So it was therefore well known!

When Rössler considered two coupled oscillators, this meant for him to use a space spanned by "four state variables." For him, the trajectory thus evolves on "a non-Euklidean metric". Note that he wrote "Euklidean" in the unusual spelling used in [Lorenz, 1963]. Rössler pointed out the fact that "a 2-torus can be re-embedded in Euklidean

3-space was somehow not exploited". This remark later became a task completed with a set of three ordinary differential equations producing a trajectory inscribed within a torus T^2 embedded in $\mathbb{R}^3$ [Rössler, 1977a]. In fact his 77-model only produces quasiperiodic motion inscribed within a torus T^2. He thereby failed to obtain what is today called "toroidal chaos." Such behavior remains an open problem from the flow point of view (many papers were devoted to toroidal chaos only from the Poincaré section point of view [Curry & Yorke, 1978; Afraimovich & Shilnikov, 1983; Anishchenko *et al.*, 1993]), mainly so because toroidal chaos is quite rarely observed in three-dimensional phase space [Arnéodo *et al.*, 1983; Deng, 1994; Li, 2008; Letellier & Gilmore, 2009]). All these remarks by Rössler indicate that he was used to thinking in terms of a structure embedded in phase space. No doubt that he was re-encouraged using phase space by his reading of [Lorenz, 1963] and [Andronov *et al.*, 1966].

4.2. *First-return map to a Poincaré section*

Much as Lorenz did [1963], Rössler understood that a Poincaré map was a most useful tool to investigate dynamical systems. At first sight, one could be surprised to see a quotation of Hirsch and Smale [1974] to specify that Poincaré maps can be considered as a "transition law from one amplitude to the next" since it was exactly what Lorenz did with his own map. But there is a deep departure occurring at this point; Hirsch and Smale were referring to a surface of section transverse to the flow. For a rigorous viewpoint, a return-map to a Poincaré section is not built with the maxima of a given state variable only, but rather is obtained using the trajectorial intersections with a surface of section transverse to the flow. Indeed, some examples can be constructed in which maximum values of a variable do not provide a safe Poincaré map [Letellier, 1994]. Moreover, there is an underlying problem due to the rotation symmetry observed in the Lorenz system. Rössler did not refer to these differences other than saying "the mode of action of this system [...] was apparently too complicated" [Rössler, 1976a]. But he questioned the equivalence between the Lorenz map (Fig. 8) and a Poincaré map. This can be motivated by the fact that retaining the maxima of variable z corresponds to a two-component Poincaré section [Letellier, 1994], with one surface of section in each spiral [Fig. 7(b)]. This led Rössler to write [1976c] that:

> Unexpectedly, the qualitative behavior of [the Lorenz equations] is still insufficiently understood, mainly because the usual technique for analyzing oscillations — to find a (Poincaré) cross-section through the flow which is a (auto-) diffeomorphism [Smale, 1967] is not applicable. A trick which exploits the inherent (although imperfect) symmetry between the two leaves of the [Lorenz] flow, so that in effect only a single leaf needs to be considered, has yet to be found.

This sentence is a sign that he had already set out to find a flow "with a single leaf" (or spiral) and associated with a "Lorenz map". Indeed this desire motivated Otto to later find a system with a Lorenz map but without symmetry, that is, with a single "leaf" [Rössler & Ortoleva, 1978]. It was shown much later that the symmetry can indeed be removed using the so-called *image* system obtained by a coordinate transformation [Miranda & Stone, 1993], and that in the case of an order-n symmetry, an n-component Poincaré section has to be used [Letellier *et al.*, 1994; Tsankov & Gilmore, 2004]. The question asked by Rössler as

$$\text{“Lorenz map (return map to maxima)} \stackrel{?}{=} \text{Poincaré map”}$$

was therefore very welcome but was not made explicit enough to capture sufficient attention from dynamicists. In a similar manner, Rössler touched on the difference between a global Poincaré section, that is, a surface of section transverse to the whole flow, and a local Poincaré map in the neighborhood of a limit cycle as it was used by mathematicians like Poincaré in his early works [Poincaré, 1881]. The main departure is that the latter can be computed everywhere along the trajectory, unlike the former since regions where folding occurs need to be avoided [Letellier, 1994].

In spite of this theoretical question left open in 1976, Rössler understood that an important role has to be played by the so-called "cap-shaped" map. "Cap" meaning "a soft, flat hat without a rim and usually with a peak." Such flowering vocabulary is quite typical of Rössler's style. He could have used a "parabola", as often employed to designate the "bell-shaped" curve. But Rössler was still hung-up by mathematics. He always claimed that he was not a mathematician — and he was right in the sense that he never acquired a formal background in mathematics — but was always fascinated by mathematics, from his first self-posed task (to write down the differential equation for Life), or as revealed by the main influences in his first paper on chaos, namely that of Lorenz (who graduated in mathematics), of Smale, of Li and Yorke and of Andronov and co-workers... Being afraid of misrepresenting his mathematical concepts, he avoided mistakes by using words like "cap-shaped" and many others, as can be found in all of his papers on chaos. This undoubtedly prevented his writing from being considered too seriously by other active workers and consequently, to be investigated further from a conceptual point of view. The side-effect was that his fuzzy (he would say "geometric") way of writing prevented his work from obtaining an even wider recognition. Today his contributions to chaos are mostly identified with his "simple equation for continuous chaos" [Rössler, 1976c] and his (four-dimensional) equation for hyperchaos [Rössler, 1979a]. When scientists are pushed about their opinion on Rössler's contributions, they sometimes confess that they are at a loss on what to say about his works.

In his main paper, Lorenz first plotted the trajectory of his system in the corresponding phase space $\mathbb{R}^3(x, y, z)$, and then computed a first-return map built on the maximum values of variable z. This is the natural way for investigating a system. But Rössler understood that reversing the Lorenz procedure, that is, starting from a given map to obtain a flow — this was designated by Smale [1967] as getting a *suspension* of a map — would be useful to design various types of chaos. It is important to note that Rössler had already in mind the tools to look for different types of chaos as done in [Rössler, 1976d, 1977c, 1977d], for instance. There is no rigorous — analytical — general way to obtain a suspension from a given map, and Rössler did that by trial and error, that is, on a long journey spent in front of his computer. For instance, the hyperchaotic attractor was found after three months, day and night, spent in front of his digital computer (HP 9845B) [Rössler, 2010]. In his first paper, as he commented himself, "the particular three-dimensional flow [...] was not found in this way" [Rössler, 1976a], that is, he did not start out from an expected map but rather from the S-shaped surface shown in Fig. 11(d).

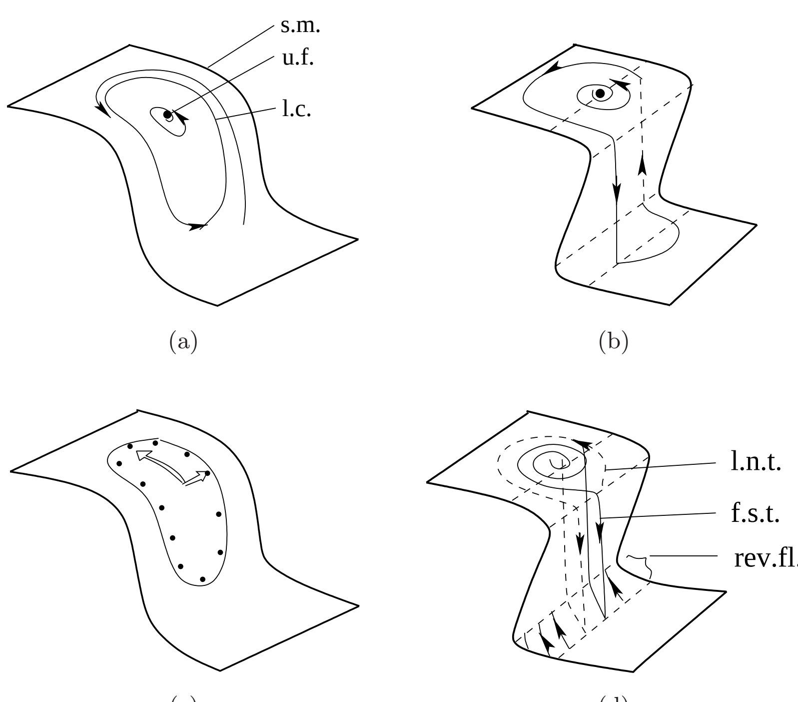


Fig. 11. Main trajectorial flow of a universal circuit. s.m. = slow manifold, u.f. = unstable focus, l.c. = limit cycle, the intermediate part of slow manifold in (b) and (d) is unstable, f.s.t. "first switched trajectory", l.n.t. = "last nonswitched trajectory", rev.fl. = reversed direction of flow "downstairs." (a) Nearly linear mode (= limit cycle). (b) Relaxation mode (= limit cycle). (c) Analogous "Soft Watch" (after Salvador Dali's synonymous painting, 1933). (d) Chaos-producing mode (see text).

4.3. *Qualitative properties of the expected dynamics*

In the second section of his paper, Rössler explained the way in which he came to his first set of equations. He revealed how he was influenced by [Andronov *et al.*, 1966]. He started out with the Khaikin's universal circuit [Khaikin, 1930] and the corresponding S-shaped surface (Fig. 4) which he then modified [Fig. 11(d)] to get a dynamics different from the one investigated in [Andronov *et al.*, 1966]. The key was to modify (bend-over more and more) the S-shaped surface in order to have an additional "orientation of flowing" on the other half, that is, eventually a motion with a "twist" would form. Again, the way in which this was expressed was not so clear [Rössler, 1976a]:

> . . . a slight modification is sufficient to turn the device into chaos generating machine: by simply introducing a different orientation of flowing on the other [lower] stable branch of the slow manifold (with the consequence of a "reinjection" of part of the flow after its having passed through a twisted round-about loop).

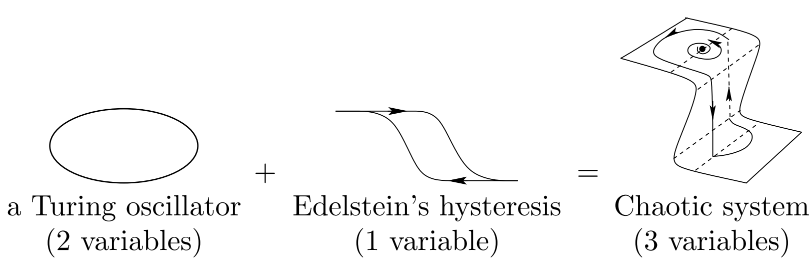
The original picture he added [Fig. 11(d)] showed a "reversed direction of flow" reinjecting the trajectory (upstair) in a nonlinear way, thereby allowing for an aperiodic trajectory [see Fig. 11(d) with its original caption] [Rössler, 1976a].

With this collection of figures, Rössler certainly reached the paroxysm of his way of presenting things since he introduced in one of the panels the "soft watch" painted by Salvador Dali. The single objectively existing connection is the analogy with the S-shaped surface. We asked Otto many times for a justification of this inclusion of the soft watch in the original paper. Among the answers we got:

- he did not want to be taken too seriously! (He was possibly impressed by his strict mother and promised himself not to emulate this trait [Rössler, 2007]).
- "it was the same shape of a surface as I had just drawn under the influence of Andronov, remembering having seen that before in art" [Rössler, 2010].
- "I had a knack for watches ever since I dismantled my grandmother's big table watch, with its S-shaped curves to the left and right on the top, at age three. And never being punished for not having been able to reassemble it from the little cogwheels and springs that I had retrieved from it" [Rössler, 2010].

For Dali, the "soft watch" is a representation of the perception of time and space, and of the behavior of the memory, acquiring soft forms that adjust themselves to the circumstances [Dali, 1952], but Otto said "the distortion of memory did not come to my mind" [Rössler, 2010]. Dali's soft watch could serve to speak to a broad audience, to stimulate the reader. Indeed, as René Descartes (1596–1650) — one of Otto's preferred scientists — wrote in the beginning of his *Discours de la Méthode* (1637), he wanted to speak to everybody using familiar vocabulary. But if Dali's soft watch can provoke smiles during a talk for a broad audience, it would surely leave strict scientists perplexed.

Returning to the way in which equations were obtained, Rössler introduced them as resulting from an oscillating Turing cell [Turing, 1952] and an Edelstein switch [Edelstein, 1970]. Note that Edelstein quoted Spangler and Snell [1961, 1967]. In a graphical way, this could be expressed as



Two variables were required to obtain a periodic oscillator and a third one to introduce the switching mechanism. This was equivalent to Andronov and co-workers' description of the universal circuit where they decomposed these behaviors into a two-dimensional slow motion surface F (Fig. 4) that can be described by the two equations in u and z and, "rapid jumps" made in the third dimension. The S-shape surface can only be described in a three-dimensional phase space.

4.4. *The equations and their chaotic solution*

Now there is a gap between the S-shaped surface proposed in Fig. 11(d) and the obtained equations. The intermediary step consists in a reaction scheme (Fig. 12) where each arrow stands for a source or a sink of the concentration of the substances A, B or C. Arrows directed toward other arrows indicate catalytic rate control. This scheme results from a two-dimensional chemical multivibrator published in 1972 [Rössler, 1972b] and a third variable, the so-called Edelstein switch. Rössler described it as follows [Rössler, 1976a]:

> The following reaction scheme (Fig. 12) constitutes **one possible way** to realize the principle by chemical means. It combines a 2-variable chemical oscillator (variables a, b) with a single-variable chemical hysteresis system (c), as prescribed by the recipe. The system obeys, under the usual assumptions of well-stirredness and isothermy as well as an appropriate concentration range, the following set of rate equations:
>
> $$\begin{cases} \dot{a} = k_1 + k_2 a - \dfrac{(k_3 b + k_4 c)a}{a+K} \\ \dot{b} = k_5 a - k_6 b \\ \mu\dot{c} = k_7 a + k_8 c - k_9 c^2 - \dfrac{k_{10} c}{c+K'} \end{cases} \tag{9}$$
>
> where a denotes the concentration of substance A, etc., $\dot{} = \mathrm{d}/\mathrm{d}t$, $k_{10} = k'_{10} e_0$, $e_0 =$ const., and K, K' are Michaelis constants.

As René Lozi remarked [Lozi, 2010], this set of rate equations is not "exact" in the sense that it does not

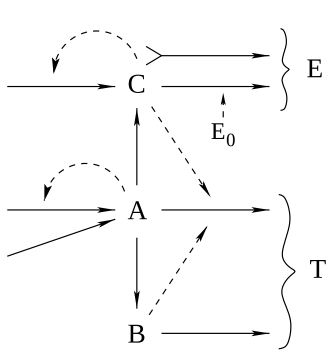


Fig. 12. Combination of an Edelstein switch with a Turing oscillator in a reaction system producing chaos. E = switching subsystem, T = oscillating subsystem; constant pools (sources and sinks) have been omitted from the scheme as usual. (Adapted from [Rössler, 1976a].)

correspond exactly to the scheme shown in Fig. 12. It rather should be [Lozi, 2010]:

$$\begin{cases} \dot{a} = k_1 + k_2 a - \dfrac{(k_3 b + k_4 c)a}{a+K} \\ \dot{b} = k_5 a - k_6 b - \dfrac{k_3 ab}{a+K} \\ \mu\dot{c} = k_7 a + k_8 c - k_9 c^2 - \dfrac{k_{10} c}{c+K'} - \mu \dfrac{k_4 ac}{a+K}. \end{cases} \tag{10}$$

In fact, once the "ideal" reaction scheme — which only represents "a possible way" — is drawn, there is a lot of intuition and time spent on the analog and/or digital computer varying the coefficients to get the result. This is confirmed by the system which was proposed in the abstract sent on 1 December, 1975 (Fig. 3), here rewritten with the symbols as used in (10) for simplifying the comparison

$$\begin{cases} \dot{a} = k_2 a - \dfrac{k_3 ab}{a+K} + c \\ \dot{b} = k_5 a - k_6 b \\ \dot{c} = \mu \left(1 - k_8 c - k_9 c^2 - \dfrac{k'_4 ac}{c+K} \right). \end{cases} \tag{11}$$

Thus, term $k_4 ac/(a+K)$ in the first equation was reduced to a linear term $k_4 c$, the term $k_3 ab/(a+K)$ was removed from the second equation, and the linear term $k_7 a$ was removed from the third equation where the two nonlinear terms were mixed together. This system is a variant of the exact form (10) proposed by Lozi and of the system actually published in [Rössler, 1976a].

The S-shaped surface therefore served to design the general structure of the equations and then the parameters were determined by a manual "maieutic" technique. In other words, the "principle for generating chaos" can be summed up into a procedure to predefine some qualitative properties of the expected behavior and to then use different components introduced as chemical automata [Rössler, 1972a] to design roughly the structure of the equations (or exactly in a limit). The final part of the work is just… time and patience in the front of a computer (Fig. 13).

This means that many parameters were tried and only those leading to the expected dynamical behavior were retained. In other words, during his search for appropriate parameter values, a few terms were set to zero. In the present case, the two rational terms recovered by Lozi were in fact removed. Such empirical step was not described in the original paper.

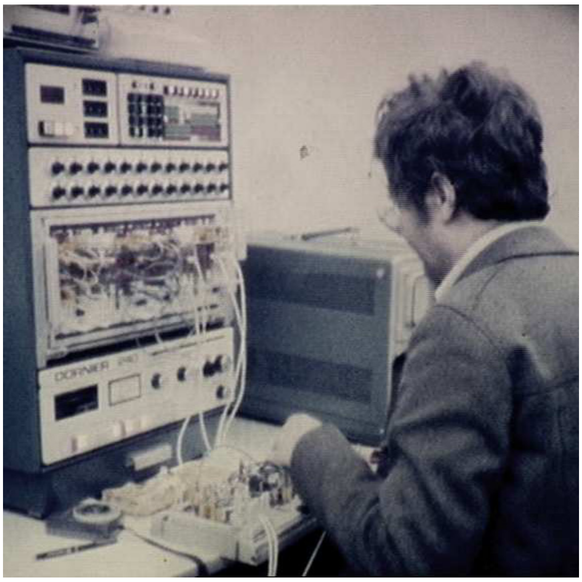


Fig. 13. Otto E. Rössler in the front of his computer in 1979.

The first chaotic system thus obtained by Rössler corresponds to the parameter values as follows: $k_1 = 37.8, k_2 = 1.4, k_3 = 2.8, k_4 = 2.8, k_5 = 2, k_6 = 1, k_7 = 8, k_8 = 1.84, k_9 = 0.0616, k_{10} = 100, K = 0.05, K' = 0.02, \mu = 1/500; a_0 = 7, b_0 = 12, c_0 = 0.2, t_0 = 0$ and $t_{\text{end}} = 43.51$. The value of parameter μ was changed here compared to the one published (1/25) to recover the picture printed. Such misprint is quite rare in Rössler's papers.

Few years ago, Otto claimed that his simple equations for continuous chaos were derived from the first chemical reaction scheme (Fig. 12). As usual, this fact was implicitly expressed in [Rössler, 1976c]:

> Therefore, a simpler equation which directly generates a similar flow and forms only a single spiral may be of interest, even if this equation has, as a "model of a model", no longer an immediate physical interpretation.
>
> Equation (9) incidentally illustrates a more general principle for the generation of "spiral type" chaos [Rössler, 1977c]: combining a two-variable oscillator (in this case x and y) with a switching-type subsystem (z) in such a way that the latter is being switched by the first while the flow of the first is dependent on the switching state of the latter. Equation (9) has in fact been derived from a more complicated equation for which this "building-block principle" has been shown to apply strictly [Rössler, 1976a].

The simpler system was obtained after nights and days spent in front of his computer. His objective was actually to simplify his set of original equations (9) to obtain a set of simpler equations that "contains just one (second-order) nonlinearity in one variable" as mentioned in the abstract in [Rössler, 1976c]. If one starts from model (9), it is impossible to reach the simpler equations by removing some terms. But if one starts from the "exact" equations (10) as those proposed by Lozi, then it is possible to obtain the so-called Rössler equations. Setting

$$\begin{cases} k_1 = k_2 = K = 0 \quad \text{and} \quad k_3 = k_4 = 1 \\ k_5 = 0, \ \ k_5 = 1 \quad \text{and} \quad k_6 = -a \\ k_9 = k_{10} = 0, \ \ k_4 = k_8 = -1, \ \ k_7 = c, \\ K = \mu = 1 \end{cases} \tag{12}$$

and removing a from the denominator of the last term, then applying the coordinate transformation $(a, b, c) \mapsto (x, y, z)$, one gets

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = b + z(x - c) \end{cases} \tag{13}$$

where the constant term b seems to be added. But this system can easily be rewritten in such a way that the third equation is replaced with

$$\dot{z} = \tilde{b}x + z(x - \tilde{c}), \tag{14}$$

that is, setting $k_7 = \tilde{b}$ and $k_8 = -\tilde{c}$.

The reaction scheme (Fig. 12) was thus used by Rössler as a starting point to predefine the algebraic structure of the set of differential equations he wanted to stimulate on his computer. Switching from the general model (10) to the reduced one is actually obtained in an empirical way. What was lost with this empirical stage? This is explained by Lozi [2010]:

> In the case of singular perturbation, the last equation of system (9), leads to
>
> $$\mu = 0 \Rightarrow \mu\dot{c} = k_7 a + k_8 c - k_9 c^2 - \frac{k_{10} c}{c + K'}. \tag{15}$$
>
> Hence the slow manifold is given by
>
> $$\dot{c} = \frac{1}{k_7}\left[-k_8 c + k_9 c^2 + \frac{k_{10} c}{c + K'}\right] \tag{16}$$

which is an homographic function possessing one singularity for $c = -K'$. This function is a coarse approximation of the S-shaped (or double-folded) manifold used in Fig. 11 in order to obtain the "chaos-generating machine" of Andronov.

But when the numerical simulations are performed to plot the trajectory with the slow manifold (Fig. 15), only the lower half of the S-shaped surface is obtained. This was in some sense pointed out by Rössler who wrote: "It may be noted that due to the asymmetry of the slow manifold, only one of its two thresholds is effective at the assumed, relatively low value of μ [Rössler, 1976a].

Once he got a chaotic solution to his equation (Fig. 14), Rössler came to the conclusion that "qualitative properties cannot be deduced from simulation alone" and, as Lorenz had done, the dynamics would be better understood by using a Poincaré map. In contrast to what Lorenz did, Rössler did not compute the Poincaré map he had provided. He rather drew them after many visual inspections of the dynamics using stereoscopic projections as used in many of his papers (to verify the analytical limiting result he had obtained first). Thus, and in agreement with his combining an oscillating circuit with a switch, the map was made up from two branches. One branch was associated with the relaxation process induced by the switching nonlinear mechanism. At a certain threshold value, the linear process was interrupted by the switch, thereby limiting the diverging spiral. The crucial point in this part of the paper is that Rössler drew the map with a qualitatively correct curvature (Fig. 16). For instance, he could have idealized his map by a tent

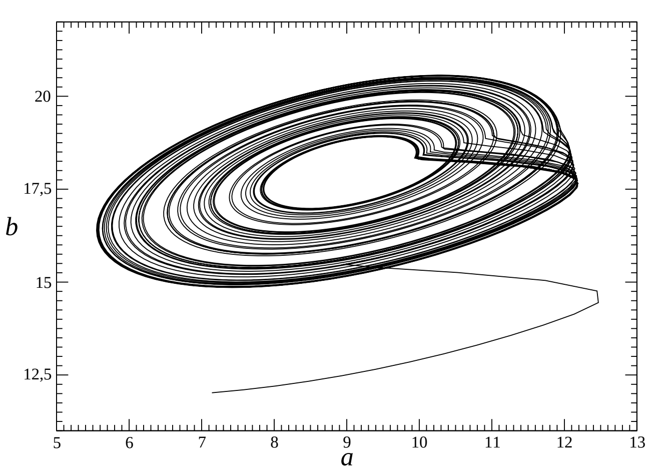


Fig. 14. Numerical simulations of Eq. (9).

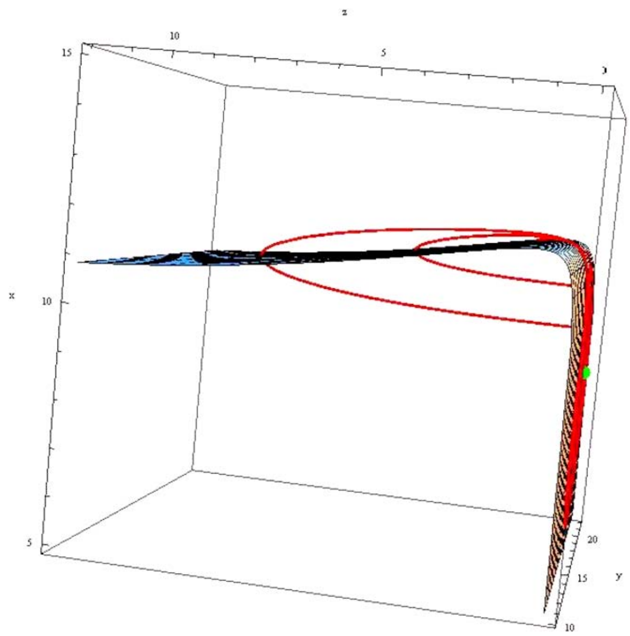


Fig. 15. Chaotic trajectory solution to system (9) plotted with the slow manifold (courtesy of Jean-Marc Ginoux).

map as Lorenz did at the end of his paper. The general shape of the map was checked by computing a first-return map to a Poincaré section (Fig. 17). It reveals that Otto's ability to read phase space was accurate.

He thus used the opportunity to show that the trajectory was bounded in phase space by showing that the "cut", that is, the threshold at which the relaxation mechanism cuts the linear expansion, induces a "quadratic box" — the term "quadratic box" has no clear meaning and was replaced by

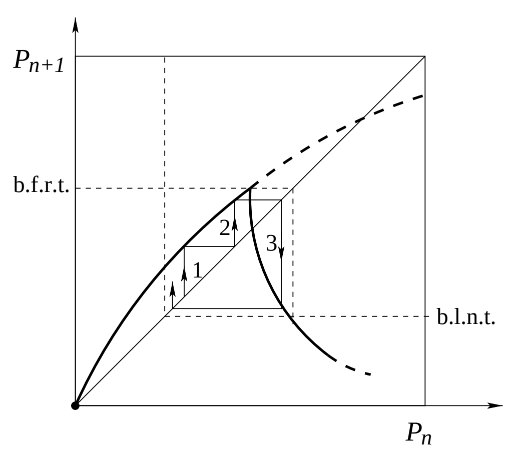


Fig. 16. Poincaré map of a universal circuit in the chaotic mode [see Fig. 11(d)], supposed that $\mu \to 0$. b.f.r.t. = borderline determined by first reinjected trajectory; b.l.n.t. = borderline determined by last nonreinjected trajectory; 1, 2, 3 = steps proving chaos.

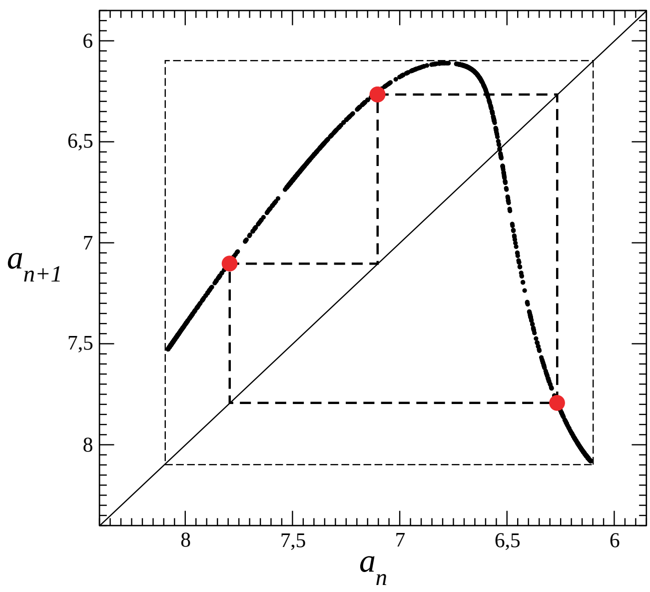


Fig. 17. First-return map to Poincaré section P for system (9) with the same parameter values as reported in Fig. 14. The box within which the period-3 orbit is observed is drawn with a thin dashed line. The three periodic points are linked by thick dashed lines.

"Li–Yorke box" in [Rössler, 1977c] — whose edges bound the behavior. This results from a common geometric construction for first-return map. Such a proof can be considered as a numerical proof for a bounded trajectory although drawn by hand. Slightly later, Rössler clarified his interpretation of the Li–Yorke theorem by stating: this "path [is] proving that period-3, and hence chaos, is possible within the box" [Rössler, 1977c]. Rössler then showed that there is a period-3 orbit within his map (Fig. 16). Using the Li and Yorke theorem, he was thus able to deduce the "existence of an uncountable set of repelling periodic attractors of measure zero". The conclusion was nearly the same as Li and Yorke's one.

Rössler here used the surprising terminology "repelling periodic attractors." This is to compare with the concept of "semi-attractor" introduced by Kantz and Grassberger [1985]. "Repelling periodic attractors" correspond in fact to periodic orbits of saddle type. An attractor was used for the stable manifold W^s and "repelling" for the unstable manifold W^u (Fig. 18). At first sight, the terminology is confusing (but no more nor less than using "semi-attractor"). It does result from the understanding of the actual property of unstable periodic orbits. Thus a fuzzy terminology hides here a good understanding of the underlying concepts. Unstable periodic orbits were not explicitly described in the

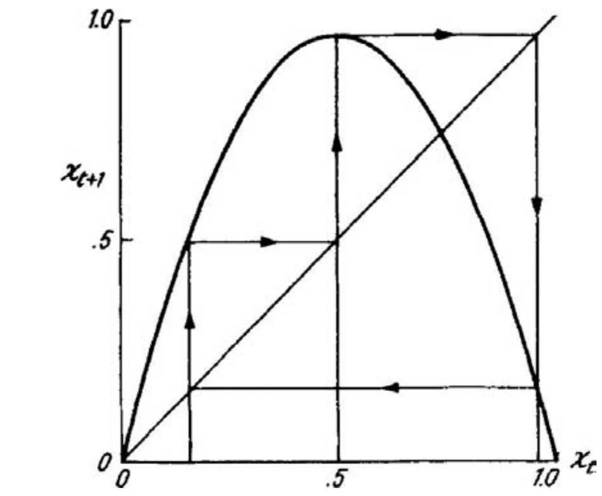


Fig. 18. The map $x_{n+1} = \mu x_n(1 - x_n)$ folds the interval [0,1] onto itself ($\mu = 3.832$) as shown in [Guckenheimer *et al.*, 1976].

Lorenz paper, either. They had been called "doubly asymptotic orbits" in Poincaré's works [Poincaré, 1891].

Thus in agreement with what Lorenz wrote, Rössler led to the conclusion that "all solutions in between [periodic orbits] are nonperiodic" [Rössler, 1976a]. The ingredients injected in the proof for chaos were thus quite similar to those introduced by Lorenz, that is, the trajectory had to be bounded and to live in the neighborhood of an infinite number of unstable periodic orbits. For Rössler, showing that the trajectory was bounded and that a period-3 orbit was identified within the first-return map was sufficient to prove the existence of chaos (according to Li and Yorke's theorem). It actually remains to be proven that there is no attracting (very long) periodic orbit — as pointed out by Guckenheimer and co-workers [Guckenheimer *et al.*, 1976], an utopian task as recently revealed by Lozi [2006].

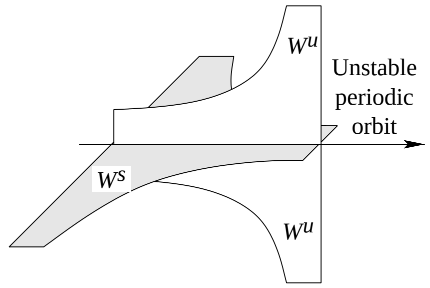


Fig. 19. Periodic orbits of saddle type (today called "unstable periodic orbits") are at the intersection of a stable (attracting) manifold W^s and an unstable (repelling) manifold W^u.

4.5. *Topological analysis*

To complete his study, much as Lorenz had with "isopleths" (Fig. 7), Rössler drew a surface, but he did that without any quantitative argument. He just drew it by interpreting his stereoscopic representation of the trajectory. The sketch of the flow (Fig. 20(a) adapted from [Rössler, 1976a]) had a finite thickness in order to be easily interpreted in terms of a Smale's horseshoe map as the latter is usually represented [Fig. 20(b)]. It contained a lot of details — not clearly explained — that reveal (once they are understood) how deep was Rössler's understanding. This was justified as

> the "folded pancake" does not display the trajectories themselves, but only an "envelope" (made up of surfaces without contact, cf. [Andronov *et al.*, 1966]) which is entered by trajectories (as depicted), but never left. The picture is directly derived from Figs. 11(d) (turned upside down) and 14, respectively, displaying the principal properties only. The rectangular cross-section on the left-hand side is seen to be mapped diffeomorphically onto a subset of itself, as required from a two-dimensional map. The "horseshoe" which is formed upon reinjection is also clearly visible.

First, Lorenz' "isopleths" became a "folded pancake" in Otto's paper. This has to be compared to the "branched manifold" introduced by Williams [1977] that became Otto's "envelope" for the chaotic trajectory. Envelope only means that it contains and frames the trajectory in phase space. Williams explicitly wrote that "the study of the attractor can be reduced to the study of the branched manifold with a semiflow on it" [Williams, 1977]. Rössler implicitly (and correctly) used his "envelope" for characterizing different types of chaotic attractor he observed (see for instance [Rössler, 1979b]).

The relation to the Horseshoe map was important in order to make a strong connection with periodic orbits and symbolic dynamics, as done by Guckenheimer and his co-workers [Guckenheimer *et al.*, 1976]. But this was not used here. What Rössler wanted to highlight was how the attractor can be split up into two parts as usually done in the Horseshoe map. Typically, when the latter is investigated, everything is done in the original square (top of Fig. 20(b)), that is, the curved part is cut to avoid mathematical complications. Practically speaking, this means that the two parts of the original square can be split and labeled by "1" and "2", respectively. This leads to the "allowed slit" that Rössler introduced to guarantee that "no

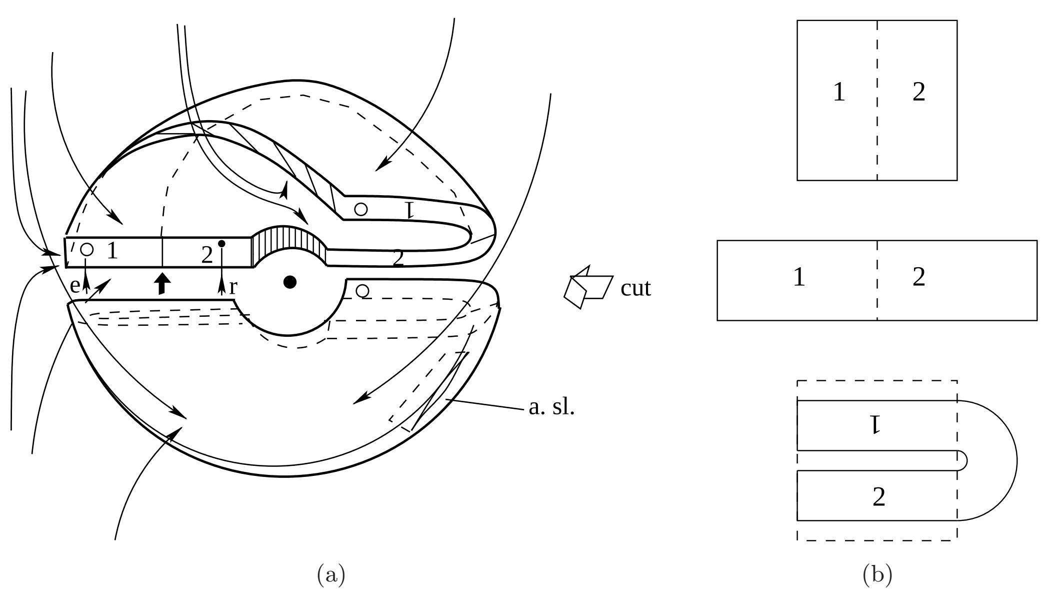


Fig. 20. The "three-dimensional blender." [cf. Fig. 14(a)] $\rightarrow$ = trajectories entering the structure from the outside; 1, 2 = half cross-section (demonstrating the "mixing transformation" that occurs), e = entry point of some arbitrary chosen trajectory, r = reentry point of the same trajectory after one cycle. $\uparrow$ = "horseshoe map", a.sl. = allowed slit (see text). In (b), do not forget that the folding is associated with a rotation by π around the central fixed point $\bullet$, thus sending the folding at right and not at left, as symbols "1" and "2" would have suggested otherwise. This right-hand sketch of an iteration of the Horseshoe map was not published in [Rössler, 1976a]. (a) Sketch of the flow. (b) Horseshoe map.

trajectories are damaged". He thus used a similar cutting process as used by Smale. Such a slit was required to obtain the next figure shown by Rössler whose purpose was to exhibit its two different domains having different topological properties. These two domains were needed to make explicit the mechanism responsible for producing chaos. Rössler thus split his "blender" into two strips as shown in Fig. 21. This was definitely a significant step, because the "blender" was not only an "envelope" to describe the structure of the attractor, but was also used to exhibit a possible partition of the attractor, based on topological properties. A partition was also used, for example, by Guckenheimer and his co-workers to introduce a symbolic dynamics to encode trajectories [Guckenheimer *et al.*, 1976].

Unfortunately, in this first paper, Rössler did not detail this too much. In the main text, he stressed the relevant role played by the "central core." It is important because it contains the two required ingredients for producing chaotic behavior. The "allowed slit" was a metaphor for the "splitting chart" responsible for the stretching as later introduced by Gilmore [l998]. And the uppper part of the central core contains the "squeezing mechanism" where the two strips are "glued" together. Stretching and squeezing are the two relevant mechanisms for producing chaos. In Fig. 21, he clearly showed one strip corresponding to a "normal loop" and a second one associated with a "Möbius loop".

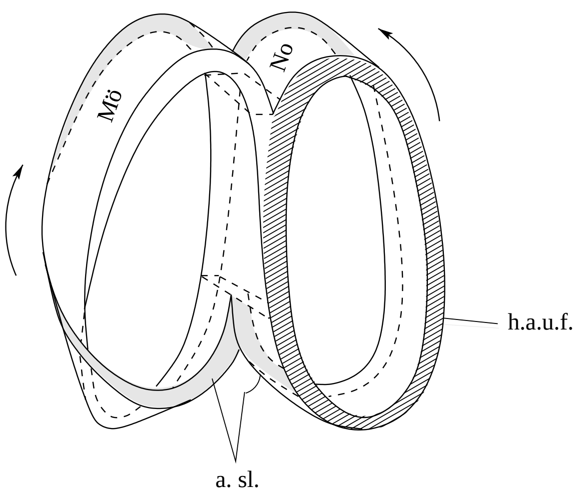


Fig. 21. A structure equivalent to that shown in Fig. 20(a). Mö = Möbius loop, No = normal loop; h.a.u.f. = hole around the unstable focus in Fig. 20(a); a.sl. = boundaries of the allowed slit in Fig. 20. The two arrows — added by the authors — show the direction of the flow.

The only difference between normal and Möbius loop is a "twist". There is thus a connection with the "twist" mentioned at the beginning of his paper while describing Fig. 11(d). But this was not done explicitly.

5. Conclusion

With his first paper on chaos, Otto E. Rössler continued the "tour de force" — already achieved by Lorenz in an uncompleted form — to provide a rather extended analysis of his system. For sure, among the list of papers (or preprints) available when Rössler did his work, the paper by Guckenheimer, Oster and Ipaktchi [Guckenheimer *et al.*, 1976] was the most complete. From a content point of view, the departure between Lorenz's paper and Rössler's is not too wide. We showed comparatively which points were actually addressed in the two papers. The main results obtained by Rössler were: (i) a second dissipative continuous systems investigated in the phase space. It was also the second example (after Lorenz) where the underlying structure of the phase portrait was interpreted using a topological analysis. By distinguishing two topologically inequivalent domains, no doubt that Rössler provided a more advanced understanding of his chaotic attractor. Hénon and Heiles were not able to push the topological analysis of their system due to two main reasons: (i) the system was four-dimensional — it is still an open problem to perform topological analysis in four dimensional spaces — and (ii) conservative, that is, not relaxing on a branched manifold.

But the reason why Rössler's paper does not always leave the reader with a positive feeling is for sure the writing style. With his mathematical presentation, Lorenz left the reader with the impression that most of the concepts he introduced — inherited from Birkhoff — to investigate his chaotic system were well under control. Rössler, with his use of a flowery style, seemingly left the impression that he was not too conversant with the concepts. We believe that the detailed analysis that we provided shows that the concepts were rather deeply understood and that he touched on the remaining key points some of which he later completed himself: like distinguishing between different types of chaos from the topological point of view by the use of paper models, providing suspension of different Poincaré maps, laying the ground of hyperchaos... But his fascination for mathematics and his displayed lack of self-confidence while writing mathematics pushed him toward a writing style quite rarely found — not to say unusual — in mathematical papers. It should be mentioned that such a way of writing was not so pregnant in his earlier papers where he combined chemical reactions and electronic circuits, two fields where he had a strong background.

The inclusion of Dali's soft watch strikes us as an advertisement to say "dear reader I am not comfortable with writing mathematics, consequently, please, forgive me if I am not rigorous". As we showed above, Rössler's use of the mathematical concepts was quite correct. But his presentation, full of implicit allusions combined with a too informal vocabulary, contributed to leave a mixed feeling in some of his scientific readers. As a consequence, Rössler was successful according to one of his aims: not to be too seriously considered by less pictorially oriented colleagues but, at the same time he thereby was not able to attract enough scientists to stimulate deep investigation into his own contributions so far. This was already pointed out in a letter from Winfree to Rössler (April 1976) (Fig. 22):

> "Many thanks for your marvelous preprint on chaos which is too compact; you need to expand, spell out more explicitly. Diagrams especially are a wonder of richness, but few will take time to study with the needed care. [...] I love the sense of humor latent in your writing. But wish you would write more explicitly, more detail so I can fully understand".

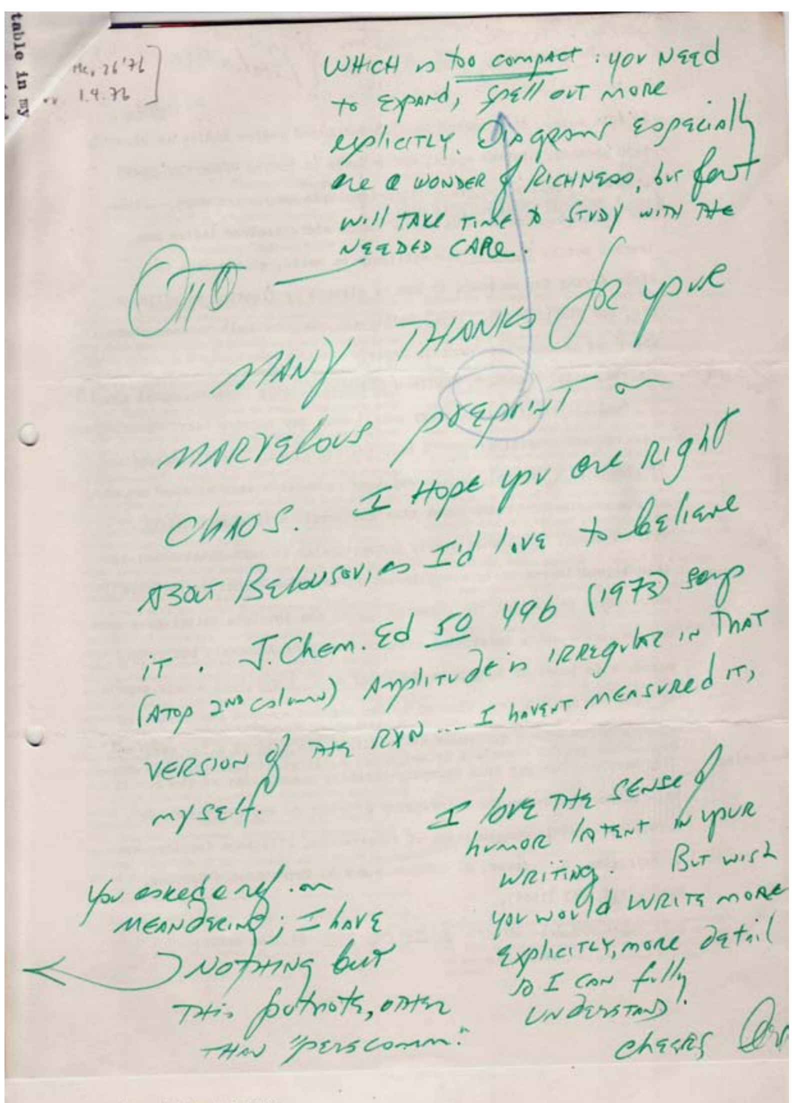


Fig. 22. Scan of the letter sent by Art Winfree to Otto Rössler on April 1976.

As a consequence, his most often quoted papers are the two published in *Physics Letters A*, which is, the paper titled "a simple equation for continous chaos" [Rössler, 1976c] and the paper "An equation for hyperchaos" [Rössler, 1979a], respectively. Unfortunately, these papers were clearly less rich than the first paper investigated here, including others published in the *Zeitschrift für Naturforschung A*. Thus, Rössler's contribution is often reduced to these two sets of differential equations. The large set of systems he proposed — recently systematically collected along with the corresponding branched manifolds [Letellier *et al.*, 2006] — still remains widely unknown.

Rössler wanted to be reachable, so as to speak to everybody. As a side effect always encountered in vulgarization, many points had to remain implicit including key details. Another problem is that he often jumped too quickly from one idea to the next, which renders the actual content of his papers very dense. His writings are therefore sometimes quite difficult to read and a "decoder" is required — as we did in this paper regarding his first contribution to chaos theory.

Rössler did not provide enough detail to be fully understood by nonspecialist readers and perhaps by numerous specialists not yet focussing on his type of questions. Most likely, his pictorial writing style came from his unsystematic mathematical background — remember that his education was in medicine and his PhD thesis in physiology. He had the advantage of not being intimidated by mathematical difficulties but, unfortunately, Rössler was not rigorous enough in introducing a new terminology: Rather than using Greek or Latin roots for his new notions as recommended by Louis Guyton de Morveau (1737–1816) [Guyton de Morveau, 1782] for neologisms, Rössler used informal terms from daily life — like "pancake", "cap-shaped", "veined pattern", "blender", "walking stick", "folded towel",... The same ideas dressed with a more mathematical clothing and with clear statements of the concepts used would have led to a reference paper for many years already.

In spite of this, this first paper on chaos triggered at least two important contributions. One paper comes from biochemistry and was devoted to an enzyme reaction. A chaotic behavior was identified based on the Li–Yorke theorem as used by Rössler [Olsen & Deng, 1977]. The second important result triggered by Otto's first paper was the chaotic behavior observed in the Belousov–Zhabotinsky reaction by the John Hudson's group [Schmitz *et al.*, 1977]. The chaotic nature of their experimental data was explained "taking the geometric representation used in a paper by Rössler", then reproducing pictures much inspired by Otto's original drawings (Fig. 11) but omitting the soft watch!

Acknowledgments

We wish to thank Otto E. Rössler for many stimulating enriching discussions. René Lozi and Jean-Marc Ginoux are thanked for their helpful comments and suggestions during the preparation of this manuscript.

References

Abraham, H. & Bloch, E. [1919a] "Sur la mesure en valeur absolue des périodes des oscillations électriques de hautes fréquences," *Comptes-Rendus de l'Acad. des Sci.* **168**, 1105–1108.

Abraham, H. & Bloch, E. [1919b] "Mesure en valeur absolue des périodes des oscillations électriques de hautes fréquences," *Annales de Physique* **9**, 237–302.

Afraimovich, V. S. & Shilnikov, L. P. [1983] "On invariant two-dimensional tori, their breakdown and stochasticity," *American Math. Soc. Trans.* **149**, 201–212.

Andronov, A. A., Vitt, A. A. & Khaikin, S. E. [1966] *Theory of Oscillators* (Pergamon Press: Oxford).

Anishchenko, V. S., Safonova, M. A. & Chua, L. O. [1993] "Confirmation of the Afraimovich-Shilnikov torus-breakdown theorem via a torus circuit," *IEEE Trans. Circuits Syst.-I*, **40**, 792–799.

Appleton, E. & van der Pol, B. [1921] "On a type of oscillation-hysteresis in a simple triode generator," *Philos. Mag.*, Series 6, **43**, p. 177.

Appleton, E. [1922] "The automatic synchronisation of triode oscillators," *Proc. Cambridge Philos. Soc.* **21**, 231.

Arnéodo, A., Coullet, P. & Spiegel, E. A. [1983] "Cascade of period doublings of tori," *Phys. Lett. A* **94**, 1–6.

Aubin, D. & Dalmedico, A. D. [2002] "Writing the history of dynamical systems and chaos: Longue durée and revolution, disciplines and cultures," *Historia Mathematica* **29**, 273–339.

Beddington, J. R., Free, C. A. & Lawton, J. H. [1975] "Dynamic complexity in predator-prey models framed in difference equations," *Nature* **255**, 58–60.

Bedian, V. [2001] "Self-description and the origin of the genetic code," *Biosystems* **60**, 39–47.

Birkhoff, G. D. [1927] *Dynamical Systems* (American Mathematical Society, NY).

Birman, J. S. & Williams, R. F. [1983] "Knotted periodic orbits in dynamical systems I: Lorenz's equations," *Topology* **22**, 47–82.

Bonhoeffer, K. F. [1948] "Activation of passive irons as a model for the excitation of nerve," *J. Gen. Physiol.* **32**, 69–91.

Butterworth, S. [1920] "The maintenance of a vibrating system by means of a triode valve," *Proc. Physical Soc. London* **32**, 345–360.

Cartwright, M. L. & Littlewood, J. E. [1947] "On non-linear differential equations of the second-order," *Ann. Math.* **48**, 472–494.

Chrilian, P. M. [1971] *Electronic Circuits: Physical Principles, Analysis and Design* (McGraw-Hill, NY).

Curry, J. H. & Yorke, J. A. [1978] "A transition from Hopf bifurcation to chaos: Computer experiments with maps on $\mathbb{R}^2$," *Lecture Notes in Mathematics* **668**, 48–66.

Dali, S. [1952] *The Disintegration of the Persistence of Memory.*

Deng, B. [1994] "Constructing homoclinic orbits and chaotic attractors," *Int. J. Bifurcation and Chaos* **4**, 823–841.

Dobrovolsky, A. [1972] "Sur l'histoire des points singuliers des équations différentielles," *Revue d'Histoire des Sciences* **25**, 3–11.

Eccles, W. H. & Jordan, F. W. [1918] *Improvements in Ionic Relays*, British patent number: GB 148582, (filed June 21, 1918 and published August 5, 1920).

Eccles, W. H. & Jordan, F. W. [1919] "Sustaining the vibration of a tuning fork by a triode valve," *The Electrician* **82**, 704.

Edelstein, B. B. [1970] "Biochemical model with multiple steady states and hysteresis," *J. Theoret. Biol.* **29**, 57–62.

Gear, C. W. [1971] *Numerical Initial Value Problems in Ordinary Differential Equations* (Prentice-Hall, Englewood Cliffs).

Gibbs, J. W. [1902] *Elementary Principles in Statistical Mechanics* (Scribner NY).

Gilmore, R. [1998] "Topological analysis of chaotic dynamical systems," *Rev. Mod. Phys.* **70**, 1455–1529.

Grasiuk, A. Z. & Oraevsky, A. N. [1964] "The dynamics of quantum oscillators," in *Quantum Electronics and Coherent Light* (Academic Press, NY), pp. 192–197.

Grebogi, C., Ott, E., Pelikan, S. & Yorke, J. A. [1984] "Strange attractors that are not chaotic," *Physica D* **13**, 261–268.

Guckenheimer, J., Oster, G. F. & Ipaktchi, A. [1976] "Periodic solutions of a logistic difference equation," *J. Math. Biol.* **4**, 101–147.

Guyton de Morveau, L. [1782] "Mémoire sur les dénominations chimiques, la nécessité d'en perfectionner le système, et les règles pour y parvenir," *Obervations sur la Physique, Sur l'histoire Naturelle et sur les Arts* **19**, p. 370.

Hayashi, C. [1953] *Nonlinear Oscillations in Physical Systems* (McGraw-Hill).

Hayashi, C., Ueda, Y. Akamatsu, N. & Itakura, H. [1970] "On the behavior of self-oscillatory systems with external force," *Trans. Instit. Electroni. Commun. Engin. A* **53**, 150–158.

Hénon, M. & Heiles, C. [1964] "The applicability of the third integral of motion: Some numerical experiments," *The Astrophys. J.* **69**, 73–79.

Hill, G. W. [1878] "Researches on the Lunar theory," *Amer. J. Math.* **1**, 5–26.

Hirsch, M. W. & Smale, S. [1974] *Differential Equations, Dynamical Systems, and Linear Algebra* (Academic Press, NY).

Hoppensteadt, F. C. & Hyman, J. M. [1977] "Periodic solutions of a logistic difference equation," *SIAM J. Appl. Math.* **32**, 73–81.

Joukovsky [1876] "Kinematika jidkgo tela (Kinematics of liquid bodies)," *Mathematisch Sbornik* (Moscou), **8**, 1–79, 163–238 (In Russian).

Kantz, H. & Grassberger, P. [1985] "Repellers, semi-attractors, and long-lived chaotic transients," *Physica D* **17**, 75–86.

Karfunkel, H. R. & Seelig, F. F. [1972] "Reversal of inhibition of enzymes and the model of a spike oscillator," *J. Theoret. Biol.* **36**, 237–253.

Khaikin, S. E. [1930] "Continuous and discontinuous oscillations," *Zhurnal Prikladnoi Fiziki* **7**, 21.

Kube, M. C., Rössler, O. E. & Hudson, J. L. [1993] "A superfat chaotic attractor," *Chaos Soli. Fract.* **3**, 141–148.

Letellier, C. [1994] *Caractérisation Topologique et Reconstruction D'attracteurs Étranges*, PhD Thesis, University of Rouen.

Letellier, C., Dutertre, P. & Gouesbet, G. [1994] "Characterization of the Lorenz system taking into account the equivariance of the vector field," *Phys. Rev. E* **49**, 3492–3495.

Letellier, C., Roulin, E. & Rössler, O. E. [2006] "Inequivalent topologies of chaos in simple equations," *Chaos Soli. Fract.* **28**, 337–360.

Letellier, C. & Gilmore, R. [2009] "Poincaré sections for a new three-dimensional toroidal attractor," *J. Phys. A* **42**, 015101.

Letellier, C. & Ginoux, J.-M. [2009] "Development of the nonlinear dynamical systems theory from radio engineering to electronics," *Int. J. Bifurcation and Chaos* **19**, 2131–2163.

Li, T. Y. & Yorke, J. [1975] "Period-three implies chaos," *Amer. Math. Monthly* **82**, 985.

Li, D. [2008] "A three-scroll chaotic attractor," *Phys. Lett. A* **372**, 387–393.

Lorenz, E. N. [1963] "Deterministic nonperiodic flow," *J. Atmosph. Sci.* **20**, 130–141.

Lorenz, E. N. [1993] *The Essence of Chaos* (UCL Press, London).

Lozi, R. [1978] "Un attracteur étrange du type attracteur de Hénon," *J. Physique* (France) **39**, C5.

Lozi, R. [2006] "Giga-periodic orbits for weakly coupled tent and logistic discretized maps," *Modern Mathematical Models, Methods and Algorithms for Real World Systems*, eds. Siddiqi, A. H., Duff, I. S. & Christensen, O. (Anamaya Publishers, New Delhi), pp. 80–124.

Lozi, R. [2010] Private communication, January 2010.

Matsumoto, T., Chua, L. O. & Komuro, M. "The double scroll," *IEEE Trans. Circuits Syst. CAS* **32**, 798–818.

May, R. [1974] "Biological populations with nonoverlapping generations; stable points, limit cycles, and chaos," *Science* **186**, 645–647.

May, R. [1975] "Deterministic models with chaotic dynamics," *Nature* **256**, 165–166.

May, R. & Oster, G. F. [1976] "Bifurcations and dynamic complexity in simple ecological models," *The Amer. Natural.* **110**, 573–599.

Minorski, N. [1962] *Nonlinear Oscillations* (Van Nostrard, Amsterdam).

Miranda, R. & Stone, E. [1993] "The proto-Lorenz system," *Phys. Lett. A* **178**, 105–113.

Misiurewicz, M. [1980] "Strange attractors for the Lozi mappings," *Ann. NY Acad. Sci.* **375**, 348–358.

Moore, D. W. & Spiegel, E. A. [1966] "A thermally excited nonlinear oscillator," *Astrophys. J.* **143**, 871–887.

Nemytskii, V. V. & Stepanov, V. V. [1960] *Qualitative Theory of Differential Equations* (Princeton University Press, Princeton).

Olsen, L. & Deng, H. [1977] "Chaos in an enzyme reaction," *Nature* **267**, 177–178.

Pechenkin, A. [2002] "The concept of self-oscillations and the rise of synergetics ideas in the theory of nonlinear oscillations," *Studies in History and Philosophy of Modern Physics*, Vol. 33, pp. 269–295.

Pennycuick, C., Compton, R. M. & Beckingham, L. [1968] "A computer model for simulating the growth of a population or of two interacting populations," *J. Theoret. Biol.* **18**, 316–329.

Poincaré, H. [1881] "Mémoire sur les courbes définies par une équation différentielle,' *J. Mathématiques*, Serie 3, **7**, 375–422.

Poincaré, H. [1885] "Mémoire sur les courbes définies par une équation différentielle," *J. Mathématiques*, Serie 4, **1**, 167–244.

Poincaré, H. [1890] "Sur le problème à trois corps et les équations de la dynamique," *Acta Mathematica* **13**, 1–270.

Poincaré, H. [1891] "Le problème des trois corps," *Revue Générale des Sciences* **2**, 1–5.

Poincaré, H. [1892] *Les Méthodes Nouvelles de la Mécanique Céleste* (Gauthier-Villars, Paris); Reprinted by Blanchard, A., Paris, 1987, Tome I.

Poincaré, H. [1899] *Les Méthodes Nouvelles de la Mécanique Céleste* Gauthier-Villars, Paris; [1897] Reprinted by Blanchard, A. Paris. Tome III.

Rashevsky, N. [1940] "An approach to the mathematical biophysics of biological self-organization and a cell polarity," *Bull. Math. Biophys.* **2**, 65–67.

Rikitake, T. [1958] "Oscillations of a system of discs dynamos," *Proc. Cambr. Philos. Soc.* **54**, 89.

Rosen, R. [1970] *Dynamical System Theory in Biology* (Wiley, NY).

Rössler, O. E. [1966] *Dauerimmunisierung von Albinomäusen Mit Rinder-γ-Globulin* [Long-term Immunization of Albino Mice with Bovine-γ-Globulin], University of Tübingen.

Rössler, O. E. [1971] "Das Ägyptische als semitische Sprache," *Christentum am Roten Meer Part I*, eds. Altheim, F. & Stiehl, R. (de Gruyter, Berlin), pp. 263–326.

Rössler, O. E. [1971] "Ein systemtheoretisches Modell zur Biogenese," *Zeitschrift für Naturforschung B* **26**, 741–746.

Rössler, O. E. [1972a] "Basic circuits of fluid automata and relaxation systems," *Zeitschrift für Naturfoschung B* **27**, 333–343.

Rössler, O. E. [1972b] "A principle for chemical multivibration, *J. Theoret. Biol.* **36**, 413–417.

Rössler, O. E. & Hoffmann, D. [1972] "Repetitive hard bifurcation in a homogeneous reaction system," in *Analysis and Simulations of Biochemical Systems*, eds. Hemkers, H. C. & Hess, B., *Proc. the 8th FEBS-Meeting* (Amsterdam, North-Holland), pp. 91–102.

Rössler, O. E. & Seelig, F. F. [1972] "A Rashevsky-Turing system as a two-cellular flip-flop," *Zeitschrift für Naturforsch B* **27**, 1445–1448.

Rössler, O. E. [1974a] "A synthetic approach to exotic kinetics (with examples)," *Lecture Notes in Biomathematics* **4**, 546–582.

Rössler, O. E. [1974b] "Chemical automata in homogeneous and reaction-diffusion kinetics," *Lecture Notes in Biomathematics* **4**, 399–418.

Rössler, O. E. [1975] "A multivibrating switching network in homogeneous kinetics," *Bull. Math. Biol.* **37**, 181–192.

Rössler, O. E. [1976a] "Chaotic behavior in simple reaction system," *Zeitschrift für Naturforschung A* **31**, 259–264.

Rössler, O. E. [1976c] "An equation for continuous chaos," *Phys. Lett. A* **57**, 397–398.

Rössler, O. E. [1976d] "Different types of chaos in two simple differential equations," *Zeitschrift für Naturforsch A* **31**, 1664–1670.

Rössler, O. E. [1977a] "Toroidal oscillation in a 3-variable abstract reaction system," *Zeitschrift für Naturforschung A* **32**, 299–301.

Rössler, O. E. [1977c] "Chaos in abstract kinetics: Two prototypes," *Bull. Math. Biol.* **39**, 275–289.

Rössler, O. E. [1977d] "Continuous chaos — Four prototype equations," *Proc. Nat. Acad. Sci.* (USA) **316**, 376–392.

Rössler, O. E. [1977e] "Syncope implies chaos in walking stick maps," *Zeitschrift für Naturforschung A* **32**, 607–613.

Rössler, O. E. & Ortoleva, P. J. [1978] "Strange attractors in 3-variable reaction systems," *Lecture Notes in Biomathematics* **21**, 67–73.

Rössler, O. E. [1979a] "An equation for hyperchaos," *Phys. Lett. A* **71**, 155–157.

Rössler, O. E. [1979b] "Continuous chaos: Four prototype equations," *Ann. NY Acad. Sci.* **316**, 376–392.

Rössler, O. E. [2007] Private communication, Rouen.

Rössler, O. E. [2010] Private communication, January 2010.

Ruelle, D. & Takens, F. [1971] "On the nature of turbulence," *Commun. Math. Phys.* **20**, 167–192.

Ruelle, D. [1976] "The Lorenz attractor and the problem of turbulence," *Lecture Notes in Mathematics* **565**, 146–158.

Ruelle, D. [1979] "Sensitive dependence on initial conditions and turbulent behaviors of dynamical systems," *Ann. NY Acad. Sci.* **316**, 408–416.

Ruelle, D. [1980] "Strange attractors," *Math. Intell.* **2**, 126–137.

Saltzman, B. [1962] "Finite amplitude free convection as an initial value problem-I," *J. Atmosph. Sci.* **19**, 329–241.

Schmitz, R. A., Graziani, K. R. & Hudson, J. L. [1977] "Experimental evidence of chaotic states in the Belousov–Zhabotinsky reaction," *J. Chem. Phys.* **67**, 3040–3044.

Seelig, F. F., Huber, W. & Kuhn, H. [1962] "Analogiebetrachtungen und analogrechner zur behandlung der korrelation von elektronen," *Zeitschrift für Naturforschung A* **17**, 114–121.

Seelig, F. F. [1965] "Numerical solution of 2- and 3-dimensional Schrödinger equations for any molecular potential by iterative variation of numerical test functions with a digital computer — I. Theoretical principles: Description of a computer program for solution of 2-dimensional Schrödinger equation," *Zeitschrift für Naturforschung A* **20**, 416–427.

Seelig, F. F. [1971] "Activated enzyme catalysis as a possible realization of the stable linear chemical oscillator model," *J. Theoret. Biol.* **30**, 497–514.

Seelig, F. F. & Rössler, O. E. [1971] "A chemical reaction flip-flop with one unique switching input," *Zeitschrift für Naturforschung B* **27**, 1441–1444.

Sharkovsky, A. N. [1964] "Co-existence of cycles of a continuous mapping of the line into itself," *Ukrainian Math. J.* **16**, 61–71 (in Russian).

Smale, S. [1967] "Differentiable dynamical systems," *Bull. Amer. Math. Soc.* **73**, 747–817.

Spangler, R. A. & Snell, F. M. [1961] "Sustained oscillations in a catalytic chemical system," *Nature* **4787**, 457–458.

Spangler, R. A. & Snell, F. M. [1967] "Transfer function analysis of an oscillatory model chemical system," *J. Theoret. Biol.* **16**, 381–405.

Sutaner, H. [1966] *Meßsender Frequenzmesser und Multivibratoren*, Radio-Praktiker Bucherei **128/130** (Franzis Verlag, Munich).

Tsankov, T. D. & Gilmore, R. [2004] "Topological aspects of the structure of chaotic attractors in $\mathbb{R}^3$," *Phys. Rev. E* **69**, 056206.

Tucker, W. [1999] "The Lorenz attractor exists," *Comptes-Rendus de l'Académie des Sciences* **328**, 1917–1202.

Turing, A. M. [1952] "The chemical basis of morphogenesis," *Philos. Trans. Roy. Soc. B* **237**, 37–72.

Ueda, Y. [1965] *Some Problems in the Theory of Nonlinear Oscillations*, PhD Thesis from the University of Tokyo; [1992] Reproduced in English in *The Road to Chaos* (Aerial Press).

Ueda, Y. [1978] "Random phenomena resulting from non-linearity in the system described by Duffing equations," *Trans. Instit. Electri. Engin. Japan A* **98** (in Japanese); [1985] Translated in English in *Int. J. Non-Linear Mech.* **20**, 481–491.

Ueda, Y. [1980] "Explosion of strange attractors exhibited by Duffing equation," *Ann. NY Acad. Sci.* **357**, 422–434.

Ueda, Y. [1992a] "Strange attractors and the origin of chaos," *Nonlin. Sci. Today* **2**, 1–16.

Ueda, Y. [1992b] *The Road to Chaos* (Aerial Press).

van der Pol, B. [1926] "On relaxation oscillations," *Philos. Mag.* **2**, 978–992.

van der Pol, B. & van der Mark, [1927] "Frequency demultiplication," *Nature* **120**, 363–364.

Veblen, O. [2001] "George David Birkhoff: A biographical memoir," *Biograhical Memoirs* **80**, 45–58.

Williams, R. F. [1977] "The structure of lorenz attractors," *Lecture Notes in Mathematics* **615**, 94–112.

Williams, R. F. [1979] "The structure of Lorenz attractors," *Publ. Math. de l'IHES* **50**, 73–99.

Winfree, A. T. [1972] "Spiral waves of chemical activity," *Science* **175**, 624–636.

Winfree, A. T. [1980] *The Geometry of Biological Time* (Springer, Berlin).

Winfree, A. T. [1989] "Electrical instability in cardiac muscle: Phase singularities and rotors," *J. Theoret. Biol.* **138**, 353–405.

Yorke, J. [2009] "Response to David Ruelle, Letter to the editor," *Notices Amer. Math. Soc.* **56**, 1232.

Zauner, K.-P. [2005] "Molecular information technology," *Critical Rev. Solid State and Mat. Sci.* **30**, 33–69.

Appendix A

Equations for the Multivibrator

In the third edition of the textbook *Theory of Oscillations* [Andronov *et al.*, 1966], Andronov, Khaikin and Vitt investigated an example of multivibrator with one RC circuit but with an inductive anode load, whose ohmic resistance was neglected (Fig. 23). As reported in [Pechenkin, 2002], Andronov started to investigate Abraham and Bloch's multivibrator in 1929 [Abraham & Bloch, 1919b] but was quickly led to the conclusion that it had no continous periodic solution even though experimental realization of the multivibrator produced self-oscillations. Andronov discussed this problem with his old professor, Leonid Mandelstam (1879–1944), who replied "If it has been proved that there is no cycle, this is something. Since the system executes oscillations, either your idealized scheme is unsuitable, or you dont know how to work with it" [Pechenkin, 2002]. Thus, he concluded that his first model was "defective." Andronov together with Vitt came then to the complete equations

$$\begin{cases} L\dfrac{\mathrm{d}i}{\mathrm{d}t} = E_\alpha - (u+v) \\ i = i_\alpha(u) + C\dfrac{\mathrm{d}v}{\mathrm{d}t} + C_\alpha\dfrac{\mathrm{d}(u+v)}{\mathrm{d}t} \\ C\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{u-E_g}{R} \end{cases} \tag{A.1}$$

by taking into account the parasitic inductance of the anode circuit but neglecting the small capacitance C_α in the expression $C + C_\alpha$. Introducing the dimensionless variables

$$x = \frac{u-E_g}{u_0}, \quad y = \beta\frac{v-v_0}{u_0}, \quad z = \frac{R}{u_0}(i - i_\alpha^0) \tag{A.2}$$

where $2u_0$ is the width of the descending section of the characteristic (Fig. 23), and using the dimensionless time

$$t_{\text{new}} = \frac{1}{\sqrt{LC}}, \tag{A.3}$$

Andronov and Vitt obtained the set of three ordinary differential equations

$$\begin{cases} \mu\dot{x} = -x + z - k\varphi(x) \\ \dot{y} = x \\ \dot{z} = -2hx - y \end{cases} \tag{A.4}$$

where $\mu = (RC_\alpha/\sqrt{LC})$ is a small positive parameter characterizing the stray capacitance C_α. The dimensionless characteristic

$$\varphi(x) = \begin{vmatrix} +1 & & x < -1 \\ -x & \text{for} & |x| \leq 1 \\ -1 & & x > +1 \end{vmatrix} \tag{A.5}$$

was thus approximated by a piecewise linear function. Such a procedure was later used by Takashi Matsumoto, Leon Chua and Matomasa Komuro [Matsumoto *et al.*, 1985] to obtain the so-called Chua system producing a two-scroll attractor.

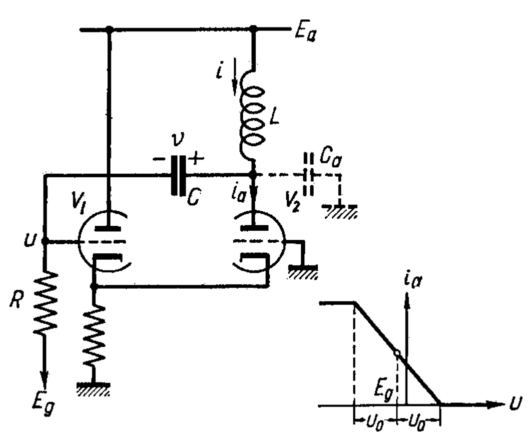


Fig. 23. Building block scheme of the multivibrator investigated in the textbook written by Andronov, Khaikin and Vitt (from [Andronov *et al.*, 1966]).

Appendix B

The Ueda Chaotic Attractor

In fact, Yoshisuke Ueda (born in 1936) worked at the beginning of the 1960s under the supervision of Chihiro Hayashi (1911–1987) in the department of electrical engineering at the University of Tokyo. Hayashi published in 1953 a reputed textbook on nonlinear (periodic) oscillations [Hayashi, 1953]. Framed in this context, Hayashi always considered nonperiodic oscillations as non-interesting transient regimes "that finally always end up — after a time that could be very long — by converging to a regular behavior" as reported in [Ueda, 1992a]. He thus disregarded all the results of Ueda showing nonperiodic behavior. These types of behavior were not mentioned in Ueda's papers before the 1970's only as "irregular and complicated" motions [Hayashi *et al.*, 1970] while describing some points in a stroboscopic section. But no representation in the phase space was yet included in this paper. It was only much later that Ueda published his first chaotic attractor represented in a projection of the phase space [Ueda, 1992a]. Ueda ensured us that, by November 27, 1961, he drew a trajectory in a plane projection of the phase space associated with the driven van der Pol equation

$$\ddot{v} - \mu(1 - \gamma v^2)\dot{v} + v^3 = B \cos \nu t \tag{B.1}$$

(Fig. 24). This is attested to by an old sheet of paper with a chaotic trajectory as shown in Fig. 24 and with the date hand written [Ueda, 1992b]. He later related that "it was nothing like the smooth oval closed curves in Fig. 24, but was more like a broken egg with jagged edges." Ueda mainly investigated the "irregular" behaviors using a Poincaré section and realized that the points were not distributed at random. He spent a long part of his PhD thesis to described how (unstable) periodic orbits were organized in the Poincaré section [Ueda, 1965]. In his thesis, there were few plane projections of phase space, they always showed periodic orbits or limit cycles. No chaotic attractor was shown in phase space.

In 1978, Ueda published a Poincaré section of the Duffing equation

$$\ddot{x} + k\dot{x} + x^3 = B \cos t \tag{B.2}$$

that led to what is now called chaos, or the "Japanese attractor" by Ruelle [1980]. But by this

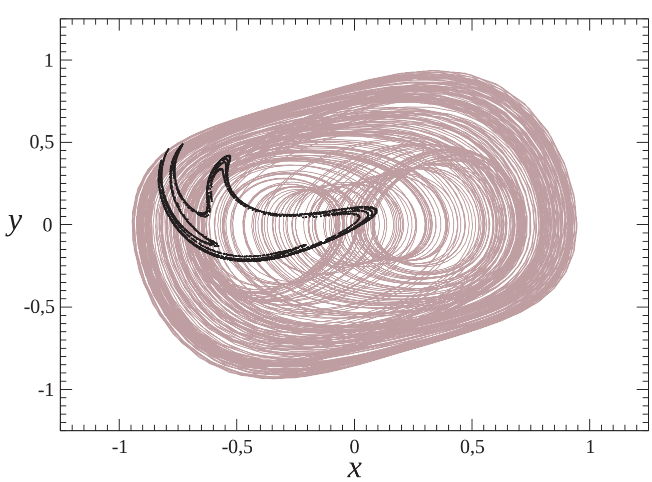


Fig. 24. Projection in the v–$\dot{v}$ plane of the phase space associated with the driven van der Pol equation (B.1) similar to the one obtained by Ueda on November 27, 1961. A continuous trajectory was drawn in light grey in the v–$\dot{v}$ plane and points in the Poincaré section at phase zero were plotted as heavy dots. Parameter values: $\mu = 0.2, \gamma = 8$ and $B = 0.35$. The original figure was reproduced in [Ueda, 1992b].

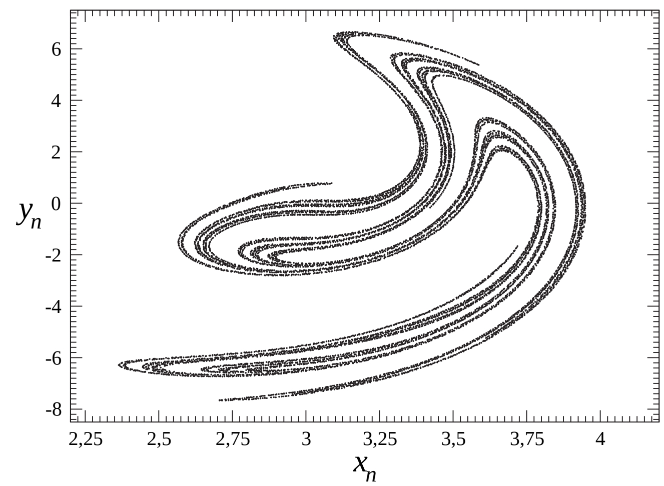


Fig. 25. The "Japanese attractor" as named by Ruelle [1980]. Poincaré section of the Duffing oscillator, as calculated by Ueda in 1978. Parameter values: $\omega = 1$, $\mu = 0, 1$ and $B = 12$ as used in the original work [Ueda, 1978].

time, Ueda was speaking about "random oscillations", introducing them as follows [Ueda, 1978].

> Simulation and/or calculation errors are unavoidable in the computer solutions for the differential equation. Therefore, random quantities are not introduced intentionally but these errors are regarded as uncertainties acting on the system. There errors seems to be sufficiently small compared with noises in the actual circuit. [...] Figure 25 shows a long-term orbit (a realization) of a random oscillation. The movement of images under iterations of f_λ is not uniquely determined even for the same initial point, but the general aspect (location, shape and size) of the orbit is reproducible, and further it seems stable in the Poisson sense. Therefore, a set of points as shown in Fig. 25 should be regarded as an outline of an attractor M representing the random oscillation.

It was only in 1980 that Ueda plotted a plane projection in the x–$\dot{x}$ plane associated with the Duffing system [Ueda, 1980]. If the resulting behavior was presented as an "orbitally stable, chaotic phenomenon", Ueda still believed that it was caused by the "uncertain factors in the real system." In fact, Ueda was still asking the question "Are computer solutions valid?" Lozi recently answererd "No" [Lozi, 2006], but for reasons that are exactly opposed to what Ueda was expecting!