Game Theory: introduction and applications to computer networks

Zero-Sum Games (follow-up)

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INRIA – EPI Maestro
20 January 2014

Part of the slides are based on a previous course with D. Figueiredo (UFRJ) and H. Zhang (Suffolk University)
Saddle Points main theorem

- The game has a saddle point iff
  \[
  \max_v \min_w u(v,w) = \min_w \max_v u(v,w)
  \]

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- Rose \( C \in \arg\max \min_w u(v,w) \)
  most cautious strategy for
  Rose: it secures the maximum
  worst case gain independently
  from Colin’s action
  (the game \textit{maximin value})

- Colin \( B \in \arg\min \max_v u(v,w) \)
  most cautious strategy for
  Colin: it secures the minimum
  worst case loss
  (the game \textit{minimax value})
Saddle Points main theorem

- Another formulation:
  - The game has a saddle point iff
    \[
    \text{maximin} = \text{minimax},
    \]

- This value is called the value of the game
Saddle Points main theorem

- The game has a saddle point iff
  \[
  \max_v \min_w u(v,w) = \min_w \max_v u(v,w)
  \]

N.C.

Two preliminary remarks

1. It holds (always)
  \[
  \max_v \min_w u(v,w) \leq \min_w \max_v u(v,w)
  \]
  because \( \min_w u(v,w) \leq u(v,w) \leq \max_v u(v,w) \) for all \( v \) and \( w \)

2. By definition, \((x,y)\) is a saddle point iff
   - \( u(x,y) \leq u(x,w) \) for all \( w \) in \( S_{Colin} \)
     - i.e. \( u(x,y) = \min_w u(x,w) \)
   - \( u(x,y) \geq u(v,y) \) for all \( v \) in \( S_{Rose} \)
     - i.e. \( u(x,y) = \max_v u(v,y) \)
Saddle Points main theorem

- The game has a saddle point iff
  \[ \max_v \min_w u(v,w) = \min_w \max_v u(v,w) \]

1. \[ \max_v \min_w u(v,w) \leq \min_w \max_v u(v,w) \]
2. if \((x,y)\) is a saddle point
   - \(u(x,y) = \min_w u(x,w), \quad u(x,y) = \max_v u(v,y)\)

N.C.
\[ u(x,y) = \min_w u(x,w) \leq \max_v \min_w u(v,w) \leq \min_w \max_v u(v,w) \leq \max_v u(v,y) = u(x,y) \]
The game has a saddle point iff
\[ \max_v \min_w u(v,w) = \min_w \max_v u(v,w) \]

**S.C.**
- \( x \) in argmax \( \min_w u(v,w) \)
- \( y \) in argmin \( \max_v u(v,w) \)

We prove that \((x,y)\) is a saddle-point

- \( w_0 \) in argmin\(_w\) \( u(x,w) \) (max\(_v\)min\(_w\)u(v,w)=u(x,w_0))
- \( v_0 \) in argmax\(_v\) \( u(v,y) \) (min\(_w\)max\(_v\)u(v,w)=u(v_0,y))
- \( u(x,w_0) = \min_w u(x,w) \leq u(x,y) \leq \max_v u(v,y) = u(v_0,y) \)

But \( u(x,w_0) = u(v_0,y) \) by hypothesis, then

\[ u(x,y) = \min_w u(x,w) = \max_v (v,y) \]
Saddle Points main theorem

- The game has a saddle point iff

\[ \max_v \min_w u(v,w) = \min_w \max_v u(v,w) \]

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This result provides also another way to find saddle points.
Properties

- Given two saddle points \((x_1, y_1)\) and \((x_2, y_2)\),
  - they have the same payoff (*equivalence property*):
    - it follows from previous proof:
      \[
      u(x_1, y_1) = \max_v \min_w u(v, w) = u(x_2, y_2)
      \]
  - \((x_1, y_2)\) and \((x_2, y_1)\) are also saddle points (*interchangeability property*):
    - as in previous proof

They make saddle point a very nice solution!
What is left?

- There are games with no saddle-point!
- An example?

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<td>1</td>
<td>0</td>
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maximin <> minimax
What is left?

- There are games with no saddle-point!
- An example? An even simpler one

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<td><strong>A</strong></td>
<td>2 → 0</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>-5 ← 3</td>
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<td>-5</td>
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\[ \text{maximin} \]

\[ \text{minimax} \]
Some practice: find all the saddle points

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<tbody>
<tr>
<td>A</td>
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<td>0</td>
<td>4</td>
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<td>C</td>
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<tr>
<th></th>
<th>A</th>
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<th>C</th>
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<tbody>
<tr>
<td>A</td>
<td>4</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>B</td>
<td>9</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>7</td>
<td>6</td>
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Games with no saddle points

<table>
<thead>
<tr>
<th></th>
<th>Colin</th>
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<tr>
<td>Rose</td>
<td>A</td>
</tr>
<tr>
<td>A</td>
<td>2→0</td>
</tr>
<tr>
<td>B</td>
<td>-5←3</td>
</tr>
</tbody>
</table>

- What should players do?
  - resort to randomness to select strategies
Mixed Strategies

- Each player associates a probability distribution over its set of strategies.
- Expected value principle: maximize the expected payoff.

Rose

<table>
<thead>
<tr>
<th></th>
<th>1/3</th>
<th>2/3</th>
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<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>-5</td>
<td>3</td>
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</table>

Rose’s expected payoff when playing A = 1/3*2+2/3*0=2/3
Rose’s expected payoff when playing B = 1/3*-5+2/3*3=1/3

How should Colin choose its prob. distribution?
Rose’s expected payoff

<table>
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<tr>
<th>Rose</th>
<th>Colin p</th>
<th>1-p</th>
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<tbody>
<tr>
<td>A</td>
<td>2p</td>
<td>0 p</td>
</tr>
<tr>
<td>B</td>
<td>-5p</td>
<td>3p</td>
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</table>

Rose’s exp. gain when playing A = 2p + (1-p)*0 = 2p
Rose’s exp. gain when playing B = -5p + (1-p)*3 = 3-8p

- How should Colin choose its prob. distribution?
  - Rose cannot take advantage of p=3/10
  - For p=3/10 Colin guarantees a loss of 3/5, what about Rose’s?
2x2 game

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<tbody>
<tr>
<td>Rose</td>
<td></td>
<td></td>
</tr>
<tr>
<td>q</td>
<td>2</td>
<td>0</td>
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<tr>
<td>1-q</td>
<td>-5</td>
<td>3</td>
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Colin’s exp. loss when playing A = 2q -5*(1-q) = 7q-5

Colin’s exp. loss when playing B = 0*q+3*(1-q) = 3-3q

- How should Rose choose its prob. distribution?
  - Colin cannot take advantage of q=8/10
  - For q=8/10 Rose guarantees a gain of?
Rose playing the mixed strategy $(8/10, 2/10)$ and Colin playing the mixed strategy $(3/10, 7/10)$ is the equilibrium of the game.

- No player has any incentives to change, because any other choice would allow the opponent to gain more.
- Rose gains $3/5$ and Colin loses $3/5$.
mx2 game

- By playing $p=3/10$, Colin guarantees max exp. loss = $3/5$
  - it loses $3/5$ if Rose plays A or B, it wins $13/5$ if Rose plays C
- Rose should not play strategy C
mx2 game

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<tr>
<td><strong>A</strong></td>
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<tr>
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Then Rose should play mixed strategy (8/10, 2/10, 3/5) guaranteeing a gain not less than 3/5.
Minimax Theorem

- Every two-person zero-sum game has a solution, i.e., there is a unique value $v$ (value of the game) and there are optimal (pure or mixed) strategies such that
  - Rose’s optimal strategy guarantees to her a payoff $\geq v$ (no matter what Colin does)
  - Colin’s optimal strategies guarantees to him a payoff $\leq v$ (no matter what Rose does)

- This solution can always be found as the solution of a $k \times k$ subgame

- Proved by John von Neumann in 1928!
  - birth of game theory...
How to solve mxm games

- if all the strategies are used at the equilibrium, the probability vector is such to make equivalent for the opponent all its strategies
  - a linear system with m-1 equations and m-1 variables
  - if it has no solution, then we need to look for smaller subgames

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Example:
- $2x - 5y + 3(1-x-y) = 0x + 3y - 5(1-x-y)$
- $2x - 5y + 3(1-x-y) = 1x - 2y + 3(1-x-y)$
How to solve 2x2 games

- If the game has no saddle point
  - calculate the absolute difference of the payoffs achievable with a strategy
  - invert them
  - normalize the values so that they become probabilities

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<tbody>
<tr>
<td>p</td>
<td>A 2 0</td>
</tr>
<tr>
<td>1-p</td>
<td>B -5 3</td>
</tr>
</tbody>
</table>

A | \( |2-0| = 2 \) | \( 8 \rightarrow \frac{8}{10} \)
B | \( |-5-3| = 8 \) | \( 2 \rightarrow \frac{2}{10} \)
How to solve mxn matrix games

1. Eliminate dominated strategies
2. Look for saddle points (solution of 1x1 games), if found stop
3. Look for a solution of all the hxh games, with h=min{m,n}, if found stop
4. Look for a solution of all the (h-1)x(h-1) games, if found stop
5. ...
h+1. Look for a solution of all the 2x2 games, if found stop

Remark: when a potential solution for a specific kxk game is found, it should be checked that Rose’s m-k strategies not considered do not provide her a better outcome given Colin’s mixed strategy, and that Colin’s n-k strategies not considered do not provide him a better outcome given Rose’s mixed strategy.
Two-person non zero-sum games

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Slides are based on a previous course with D. Figueiredo (UFRJ) and H. Zhang (Suffolk University)
Outline

- **Two-person zero-sum games**
  - **Matrix games**
    - Pure strategy equilibria (dominance and saddle points), ch 2
    - Mixed strategy equilibria, ch 3
  - Game trees, ch 7

- **Two-person non-zero-sum games**
  - Nash equilibria...
    - ...And its limits (equivalence, interchangeability, Prisoner’s dilemma), ch. 11 and 12
  - Strategic games, ch. 14
  - Subgame Perfect Nash Equilibria (not in the book)
  - Repeated Games, partially in ch. 12
  - Evolutionary games, ch. 15

- **N-persons games**
Two-person Non-zero Sum Games

- Players are not strictly opposed
  - payoff sum is non-zero

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<td>Player 1</td>
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<td>A</td>
<td>3, 4</td>
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<tr>
<td>B</td>
<td>5, 1</td>
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- Situations where interest is not directly opposed
  - players could cooperate
  - communication may play an important role
    - for the moment assume no communication is possible
What do we keep from zero-sum games?

- Dominance
- Movement diagram
  - Pay attention to which payoffs have to be considered to decide movements

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- Enough to determine pure strategies equilibria
  - But still there are some differences (see after)
What can we keep from zero-sum games?

- As in zero-sum games, pure strategies equilibria do not always exist...

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- ...but we can find mixed strategies equilibria