Game Theory: introduction and applications to computer networks

Two-person non zero-sum games

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28 January 2013

Slides are based on a previous course
with D. Figueiredo (UFRJ) and H. Zhang (Suffolk University)

Outline

- Two-person zero-sum games
  - Matrix games
    - Pure strategy equilibria (dominance and saddle points), ch 2
    - Mixed strategy equilibria, ch 3
  - Game trees, ch 7
- Two-person non-zero-sum games
  - Nash equilibria…
    - …And its limits (equivalence, interchangeability, Prisoner's dilemma), ch. 11 and 12
  - Strategic games, ch. 14
  - Subgame Perfect Nash Equilibria (not in the book)
  - Repeated Games, partially in ch. 12
  - Evolutionary games, ch. 15
- N-persons games

Two-person Non-zero Sum Games

- Players are not strictly opposed
  - payoff sum is non-zero

Player 2

Player 1		A	B
	A	3, 4	2, 0
	B	5, 1	-1, 2

- Situations where interest is not directly opposed
  - players could cooperate
  - communication may play an important role
    - for the moment assume no communication is possible

What do we keep from zero-sum games?

- Dominance
- Movement diagram
  - pay attention to which payoffs have to be considered to decide movements

		Player 2 A	Player 2 B
Player 1	A	5, 4	2, 0
	B	3, 1	-1, 2

- Enough to determine pure strategies equilibria
  - but still there are some differences (see after)

What can we keep from zero-sum games?

- As in zero-sum games, pure strategies equilibria do not always exist…

		Player 2	
		A	B
Player 1	A	5, 0	-1, 4
	B	3, 2	2, 1

- …but we can find mixed strategies equilibria

Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy (*equalizing strategy*), that equalizes the opponent payoffs
  - how to calculate it?

		Colin A	Colin B
Rose	A	5, 0	-1, 4
	B	3, 2	2, 1

Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy, that equalizes the opponent payoffs
  - how to calculate it?

		Colin A	Colin B
Rose	A	-0	-4
	B	-2	-1

Rose considers *Colin's game*

$\rightarrow$ 4 $\quad$ 1/5

$\rightarrow$ 1 $\quad$ 4/5

Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy, that equalizes the opponent payoffs
  - how to calculate it?

Colin considers *Rose's game*

		Colin A	Colin B
Rose	A	5	-1
	B	3	2
		3/5	2/5

Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy, that equalizes the opponent payoffs
  - how to calculate it?

		Colin A	Colin B
Rose	A	5, 0	-1, 4
	B	3, 2	2, 1

Rose playing (1/5,4/5)
Colin playing (3/5,2/5)
is an equilibrium

Rose gains 13/5
Colin gains 8/5

Good news: Nash's theorem [1950]

- Every two-person games has at least one equilibrium either in pure strategies or in mixed strategies
  - Proved using fixed point theorem
  - generalized to N person game
- This equilibrium concept called Nash equilibrium in his honor
  - A vector of strategies (a profile) is a Nash Equilibrium (NE) if no player can unilaterally change its strategy and increase its payoff

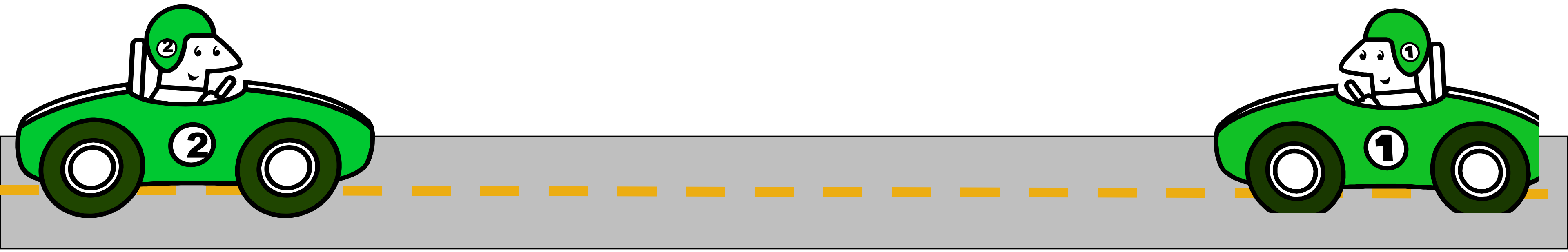
A useful property

- Given a finite game, a profile is a mixed NE of the game if and only if for every player i, every pure strategy used by i with non-null probability is a best response to other players mixed strategies in the profile
  - see Osborne and Rubinstein, *A course in game theory*, Lemma 33.2

Bad news: what do we lose?

- equivalence
- interchangeability
- identity of equalizing strategies with prudential strategies
- main cause
  - at equilibrium every player is considering the opponent’s payoffs ignoring its payoffs.
- New problematic aspect
  - group rationality versus individual rationality (cooperation versus competition)
  - absent in zero-sum games
- we lose the idea of **the** solution

Game of Chicken



- Game of Chicken (aka. Hawk-Dove Game)
  - driver who swerves looses

Driver 2

Driver 1	swerve	stay
swerve	0, 0	-1, 5
stay	5, -1	-10, -10

Drivers want to do opposite of one another

Two equilibria:
not equivalent
not interchangeable!

- playing an equilibrium strategy does not lead to equilibrium

The Prisoner's Dilemma

- One of the most studied and used games
  - proposed in 1950
- Two suspects arrested for joint crime
  - each suspect when interrogated separately, has option to confess

		Suspect 2	
		NC	C
Suspect 1	NC	2, 2	10, 1
	C	1, 10	5, 5

payoff is years in jail (**smaller is better**)

better outcome

single NE

Pareto Optimal

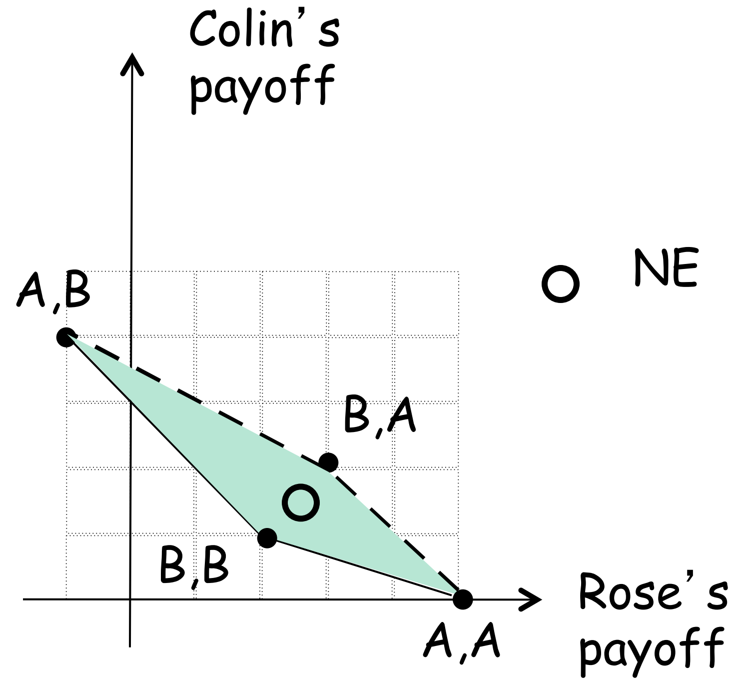
		Suspect 2	
		NC	C
Suspect 1	NC	2, 2	10, 1
	C	1, 10	5, 5

Pareto Optimal

- Def: outcome o* is Pareto Optimal if no other outcome would give to all the players a payoff not smaller and a payoff higher to at least one of them
- Pareto Principle: to be acceptable as a solution of a game, an outcome should be Pareto Optimal
  - the NE of the Prisoner's dilemma is not!
- Conflict between group rationality (Pareto principle) and individual rationality (dominance principle)

Payoff polygon

		Colin	
		A	B
Rose	A	5, 0	-1, 4
	B	3, 2	2, 1



- All the points in the convex hull of the pure strategy payoffs correspond to payoffs obtainable by mixed strategies
- The north-east boundary contains the Pareto optimal points

Another possible approach to equilibria

- NE ⇔ equalizing strategies
- What about prudential strategies?

Prudential strategies

- Each player tries to minimize its maximum loss (then it plays in its own game)

		Colin A	Colin B
Rose	A	5, 0	-1, 4
	B	3, 2	2, 1

Prudential strategies

- Rose assumes that Colin would like to minimize her gain
- Rose plays in Rose's game
- Saddle point in BB
- B is Rose's prudential strategy and guarantees to Rose at least 2 (Rose's *security level*)

		Colin	
		A	B
Rose	A	5	-1
	B	3	2

Prudential strategies

- Colin assumes that Rose would like to minimize his gain (maximize his loss)
- Colin plays in Colin's game
- mixed strategy equilibrium,
- (3/5,2/5) is Colin's prudential strategy and guarantees Colin a gain not smaller than 8/5

		Colin A	Colin B
Rose	A	0	-4
	B	-2	-1

Prudential strategies

- Prudential strategies
  - Rose plays B, Colin plays *A* w. prob. 3/5, B w. 2/5
  - Rose gains 13/5 (>2), Colin gains 8/5
- Is it stable?
  - No, if Colin thinks that Rose plays B, he would be better off by playing *A* (Colin's *counter-prudential strategy*)

		Colin	
		A	B
Rose	*A*	5, 0	-1, 4
	B	3, 2	2, 1

Prudential strategies

- are not the solution neither:
  - do not lead to equilibria
  - do not solve the group rationality versus individual rationality conflict
- dual basic problem:
  - look at your payoff, ignoring the payoffs of the opponents

Exercises

- Find NE and Pareto optimal outcomes:

	NC	C
NC	2, 2	10, 1
C	1, 10	5, 5

	A	B
A	2, 3	3, 2
B	1, 0	0, 1

	swerve	stay
swerve	0, 0	-1, 5
stay	5, -1	-10, -10

	A	B
A	2, 4	1, 0
B	3, 1	0, 4

Performance Evaluation

Routing as a Potential game

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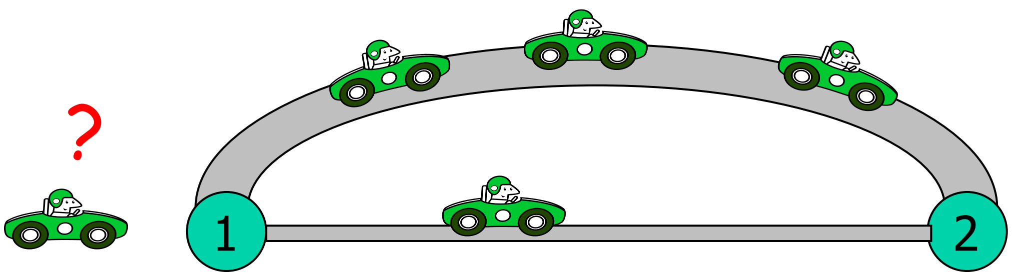
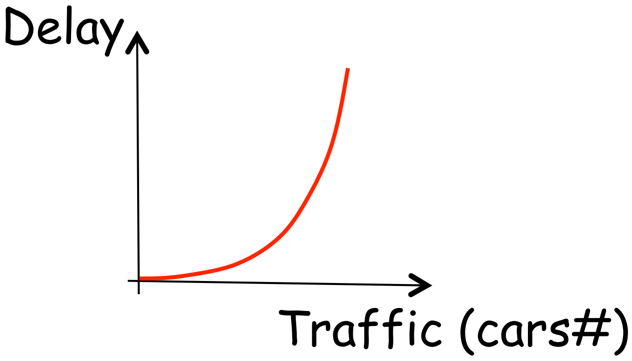
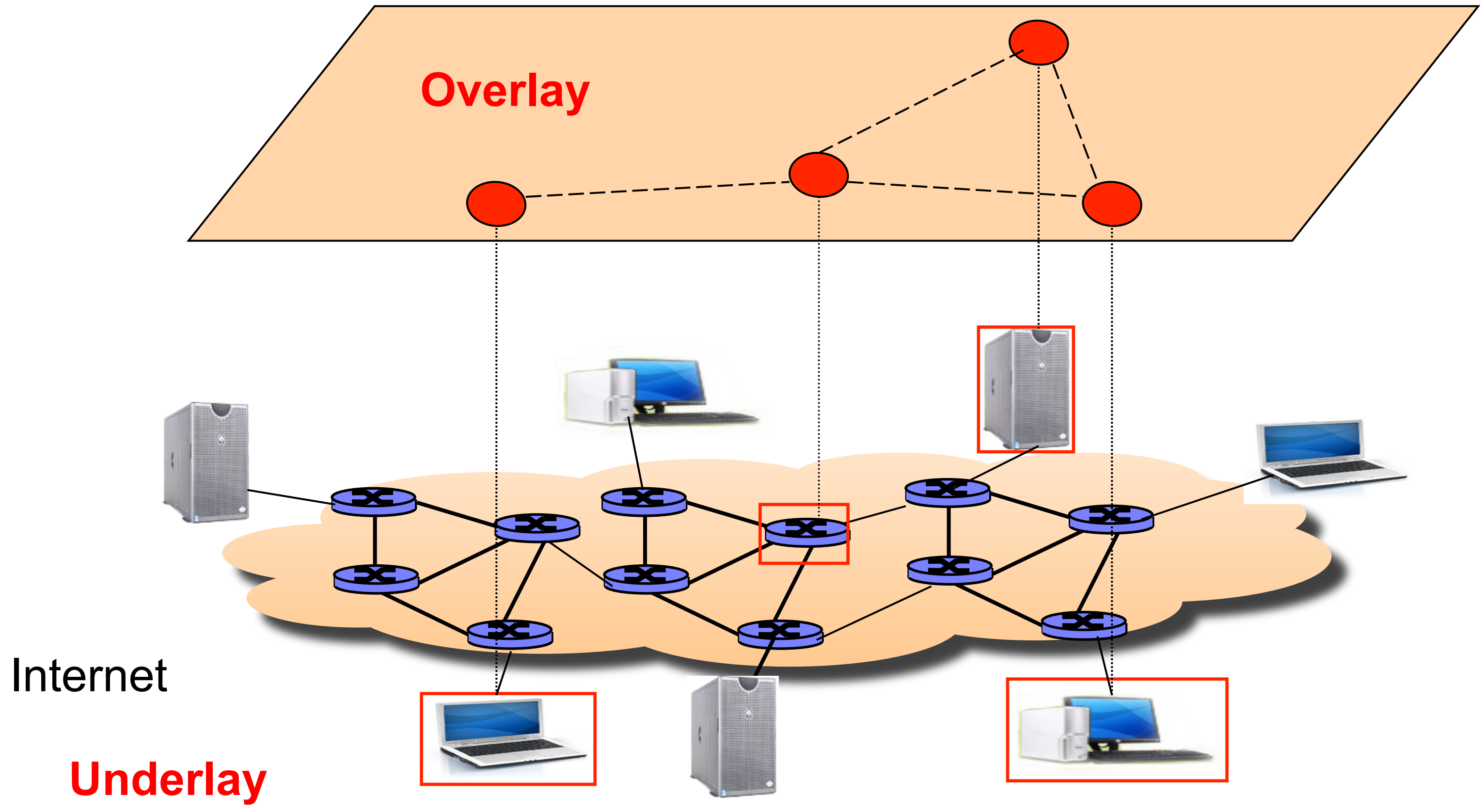
Routing games



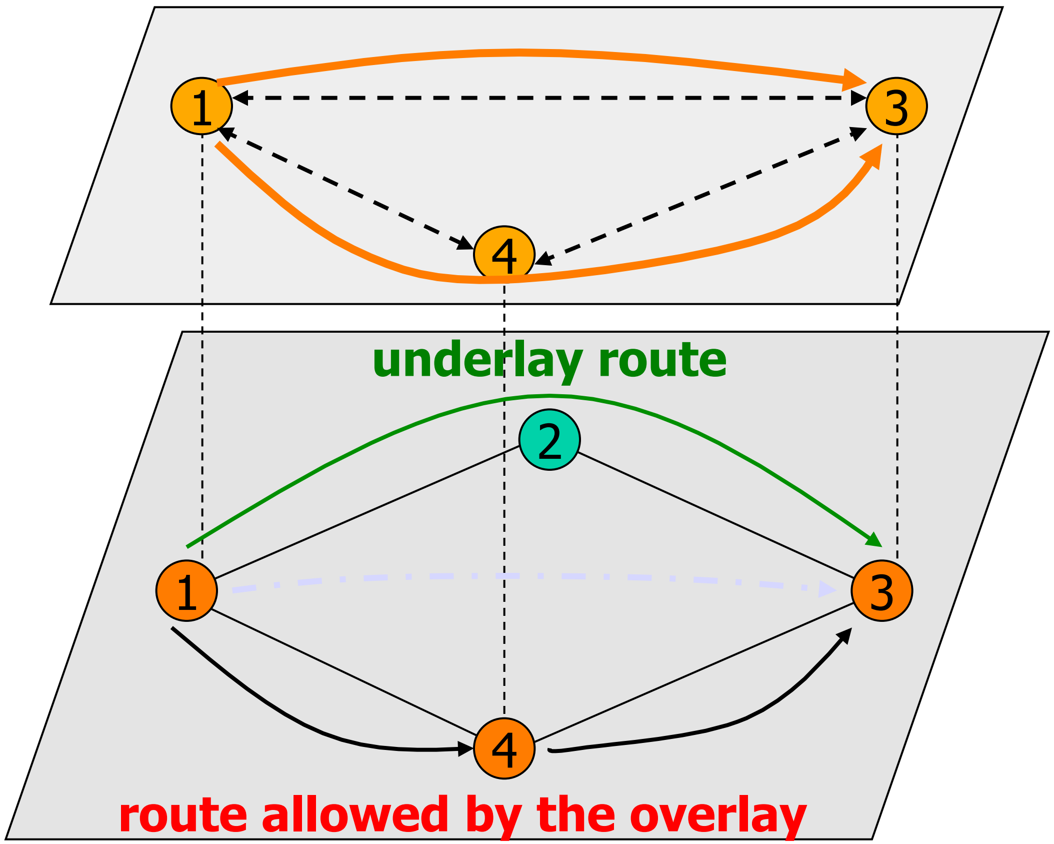


- Possible in the Internet?

Overlay networks



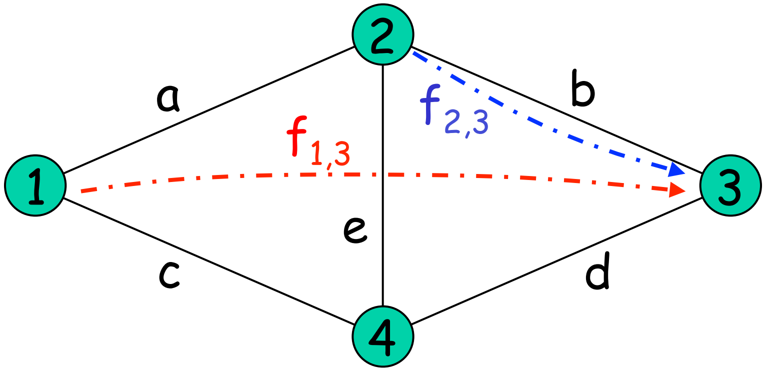
Routing games



An Overlay for routing:
Resilient Overlay Routing

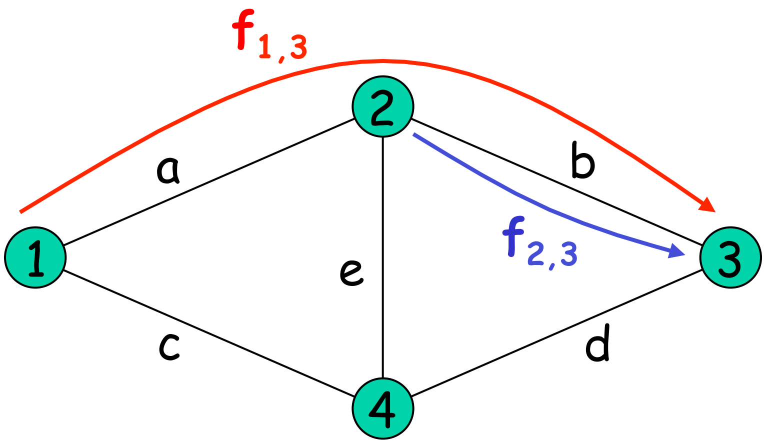
- Users can ignore ISP choices

Traffic demand



- unit traffic demands between pair of nodes

Delay costs



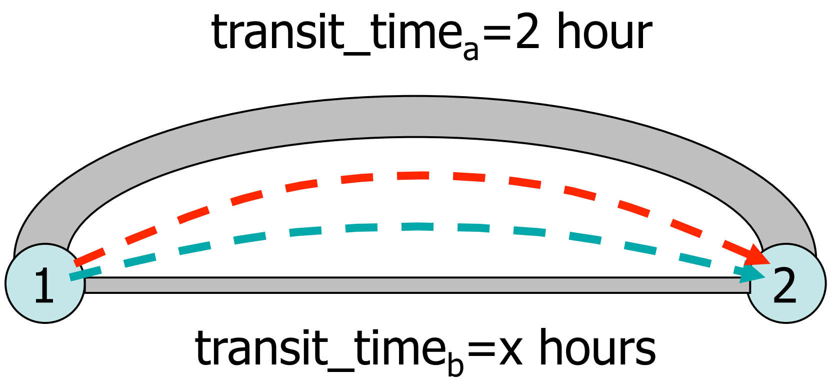
$R_{1,3} = \{a,b\}$, $R_{2,3}=\{b\}$

$f_a=f_{1,3}$, $f_b= f_{1,3} + f_{2,3}$, $f_c=f_d=0$

$c_\alpha(f_\alpha)$, $\alpha \in E=\{a,b,c,d,e\}$,
Non-negative,
non decreasing functions

- Social cost: $C_S = \Sigma_{\alpha \in E}\, f_\alpha * c_\alpha(f_\alpha)$
- User cost:
  - $C_{1,3}(f)= \Sigma_{\alpha \in R_{1,3}}\, c_\alpha(f_\alpha)$

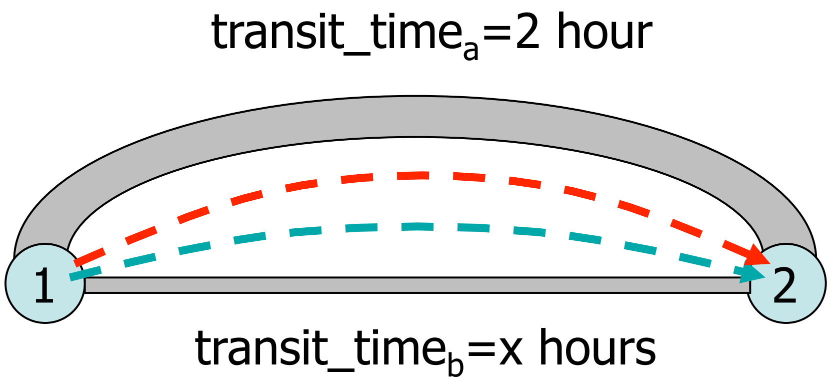
Pigou's example

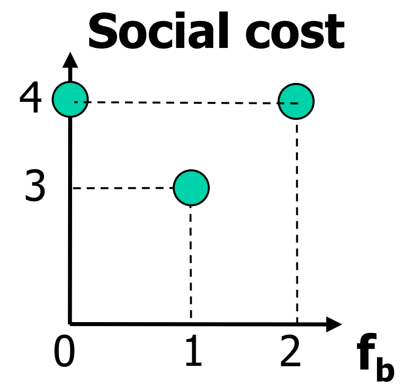


- Two possible roads between 1 and 2
  - a) a longer highway (almost constant transit time)
  - b) shorter but traffic sensitive city road
- 2 Selfish users (choose the road in order to minimize their delay)

		Colin	
		a	b
Rose	a	-2, -2	-2, -1
	b	-1, -2	-2, -2

Pigou's example



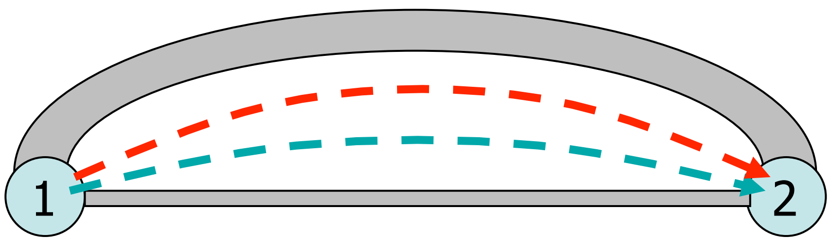


- Two possible roads between 1 and 2
  - a) a longer highway (almost constant transit time)
  - b) shorter but traffic sensitive city road
- 2 Selfish users (choose the road in order to minimize their delay)
  - There is 1 (pure-strategy) NE where they all choose the city road...
  - even if the optimal allocation is not worse for the single user!
- What if transit_time$_a$=2+ε?
- In what follows we only consider pure strategy NE

What is the cost of user selfishness for the community?

- Loss of Efficiency (LoE)
  - given a NE with social cost $C_S(f_{NE})$
  - and the traffic allocation with minimum social cost $C_S(f_{Opt})$
  - LoE = $C_S(f_{NE}) / C_S(f_{Opt})$

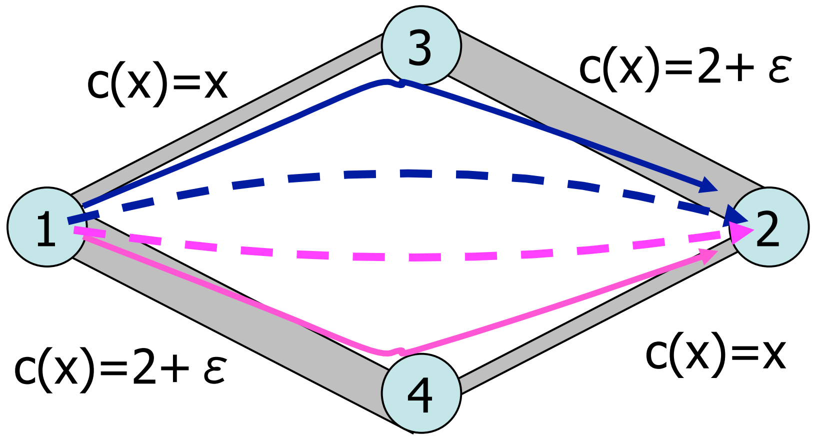
Pigou's example



- The LoE of (b,b) is 4/3
- The LoE of (b,a) and (a,b) is 1

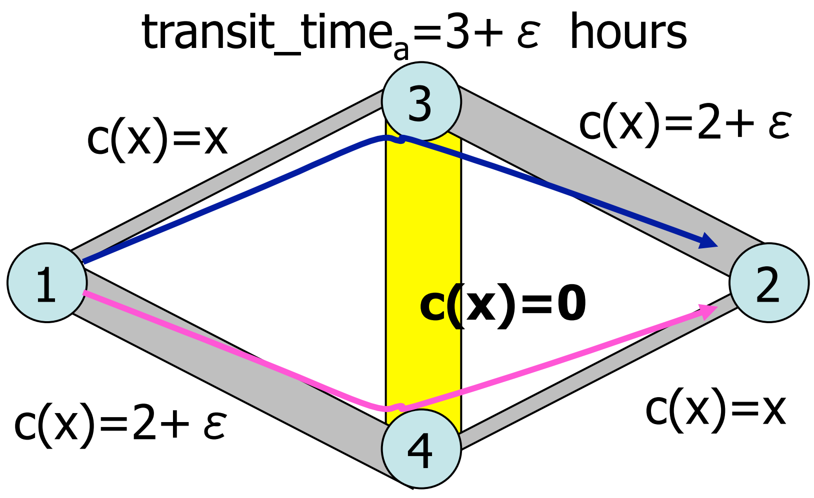
		Colin	
		a	b
Rose	a	-2, -2	-2, -1
	b	-1, -2	-2, -2

Braess's paradox

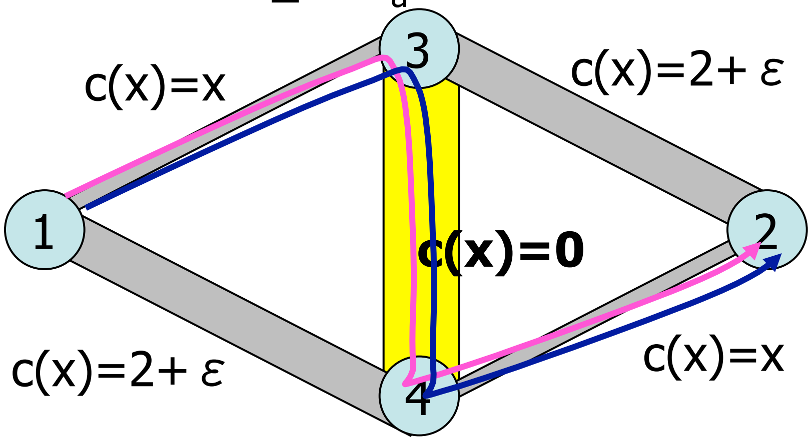


- User cost: $3+\varepsilon$
- Social cost: $C_{NE} = 6+2\varepsilon$ ($=C_{Opt}$)

Braess's paradox



Braess's paradox



- User cost: 4
- Social cost: $C_{NE} = 8 > 6+\varepsilon \quad (C_{Opt})$
- LoE = $8/(6+\varepsilon) \underset{\varepsilon \to 0}{\to} 4/3$

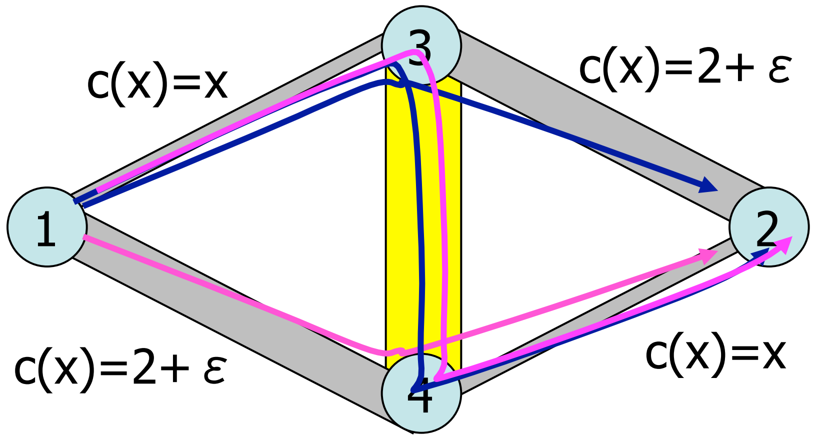
Routing games

1. Is there always a (pure strategy) NE?
2. Can we always find a NE with a "small" Loss of Efficiency (LoE)?

Always an equilibrium?

- Best Response dynamics
  - Start from a given routing and let each player play its Best Response strategy
  - What if after some time there is no change?

BR dynamics



1. Users costs: (3+ ε , 3+ ε)
2. Blue plays BR, costs: (3, 4+ ε)
3. Pink plays BR, costs: (4, 4)
4. Nothing changes….

Always an equilibrium?

- Best Response dynamics
  - Start from a given routing and let each player play its Best Response strategy
  - What if after some time there is no change?
  - Are we sure to stop?

Games with no saddle-point

- There are games with no saddle-point!
- An example?

	R	P	S	min
R	0	-1	1	-1
P	1	0	-1	-1
S	-1	1	0	-1
max	1	1	1	

maximin

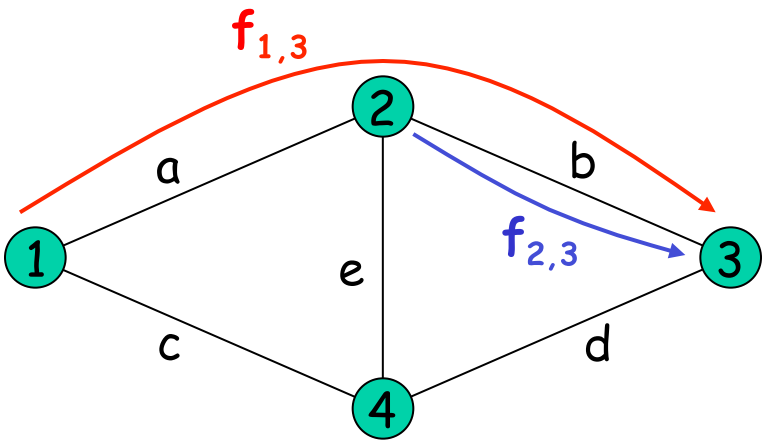
minimax

maximin <> minimax

Always an equilibrium?

- Best Response dynamics
  - Start from a given routing and let each player play its Best Response strategy
  - What if after some time there is no change?
  - Are we sure to stop?
    - In some cases we can define a potential function that keeps decreasing at each BR until a minimum is reached.
    - Is the social cost a good candidate?

Potential for routing games



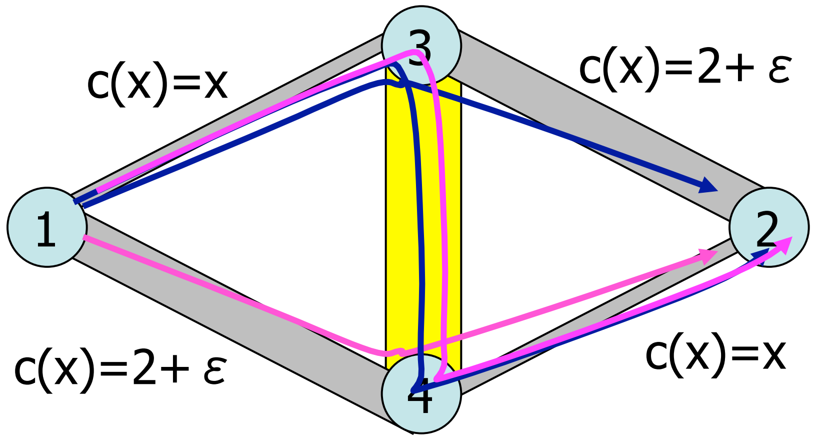
$P_{1,3} = \{a,b\}$, $P_{2,3}=\{b\}$

$f_a=f_{1,3}$, $f_b= f_{1,3} + f_{2,3}$, $f_c=f_d=0$

$c_\alpha(f_\alpha)$, $\alpha \in E=\{a,b,c,d,e\}$,
Non-negative,
non decreasing functions

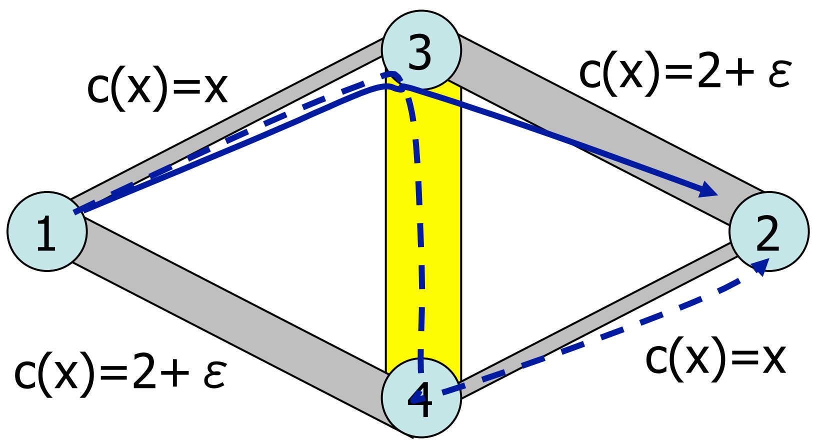
- Potential : $P = \sum_{\alpha \in E} P_\alpha(f_\alpha) = \sum_{\alpha \in E} \sum_{t=1,\ldots f\alpha} c_\alpha(t)$

Potential decreases at every BR



1. User costs: (3+ ε , 3+ ε), P=6+2 ε
2. Blue plays BR, costs: (3, 4+ ε), P=6+ ε
3. Pink plays BR, costs: (4, 4), P=6
4. Nothing changes….

Potential decreases at every BR



From route R
to route R'

- $f'_\alpha = f_\alpha + 1$ if α in R'-R, $f'_\alpha = f_\alpha - 1$ if α in R-R'
- $P_\alpha - P'_\alpha = -c(f_\alpha + 1)$ if α in R'-R,
- $P_\alpha - P'_\alpha = c(f_\alpha)$ if α in R-R'
- $P - P' = \Sigma_{\alpha\, \varepsilon\, R}\, c(f_\alpha) - \Sigma_{\alpha\, \varepsilon\, R'}\, c(f'_\alpha) =$

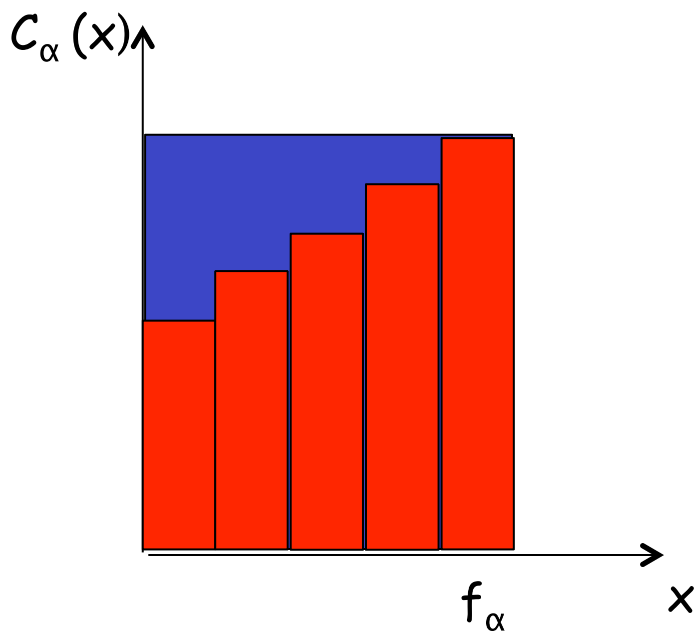
  =user difference cost between R and R'>0

BR dynamics converges to an equilibrium

- The potential decreases at every step
- There is a finite number of possible potential values
- After a finite number of steps a potential local minimum is reached
- The final routes identify a (pure strategy) NE

Always an equilibrium with small Loss of Efficiency?

- Consider only affine cost functions,
  i.e. $c_\alpha(x)=a_\alpha+b_\alpha x$
- We will use the potential to derive a bound on the social cost of a NE
  - $P(f) <= C_S(f) <= 2\ P(f)$



$P_\alpha(f_\alpha)$

$C_\alpha(f_\alpha)$

Always an equilibrium with small Loss of Efficiency?

- Consider only affine cost functions

  i.e. $c_\alpha(x)=a_\alpha+b_\alpha x$
- We will use the potential to derive a bound on the social cost of a NE
  - $P(f) <= C_S(f) <= 2\ P(f)$
- $P(f) = \Sigma_{\alpha \varepsilon E} P_\alpha = \Sigma_{\alpha \varepsilon E} \Sigma_{t=1,\ldots f\alpha}\ c(t) <=$

  $<= \Sigma_{\alpha \varepsilon E} \Sigma_{t=1,\ldots f\alpha}\ c(f_\alpha) = \Sigma_{\alpha \varepsilon E} f_\alpha c(f_\alpha) = C_S(f)$
- $P(f) = \Sigma_{\alpha \varepsilon E} P_\alpha = \Sigma_{\alpha \varepsilon E} \Sigma_{t=1,\ldots f\alpha}(a_a+b_a t) =$

  $= \Sigma_{\alpha \varepsilon E} f_a a_a + b_a f_a (f_a+1)/2 >= \Sigma_{\alpha \varepsilon E} f_a (a_a + b_a f_a)/2$

  $= C_S(f)/2$

Always an equilibrium with small Loss of Efficiency?

- Consider only affine cost functions

  i.e. $c_\alpha(x)=a_\alpha+b_\alpha x$
- $P(f) <= C_S(f) <= 2\ P(f)$
- Let's imagine to start from routing f_{Opt} with the optimal social cost $C_S(f_{Opt})$,
- Applying the BR dynamics we arrive to a NE with routing f_{NE} and social cost $C_S(f_{NE})$
- $C_S(f_{NE}) <= 2\ P(f_{NE}) <= 2\ P(f_{Opt}) <= 2\ C_S(f_{Opt})$
- The LoE of this equilibrium is at most 2

Same technique, different result

- Consider a network with a routing at the equilibrium
- Add some links
- Let the system converge to a new equilibrium
- The social cost of the new equilibrium can be at most 4/3 of the previous equilibrium social cost (as in the Braess Paradox)

Loss of Efficiency, Price of Anarchy, Price of Stability

- Loss of Efficiency (LoE)
  - given a NE with social cost $C_S(f_{NE})$
  - LoE = $C_S(f_{NE}) / C_S(f_{Opt})$
- Price of Anarchy (PoA) [Koutsoupias99]
  - Different settings G (a family of graph, of cost functions,...)
  - X_g =set of NEs for the setting g in G
  - PoA = $\sup_{g \in G} \sup_{NE \in X_g}\{C_S(f_{NE}) / C_S(f_{Opt})\}$ => "worst" loss of efficiency in G
- Price of Stability (PoS) [Anshelevish04]
  - PoS = $\sup_{g \in G} \inf_{NE \in X_g}\{C_S(f_{NE}) / C_S(f_{Opt})\}$ => guaranteed loss of efficiency in G

Stronger results for affine cost functions

- We have proven that for unit-traffic routing games the PoS is at most 2
- For unit-traffic routing games and single-source pairs the PoS is 4/3
- For non-atomic routing games the PoA is 4/3
  - non-atomic = infinite players each with infinitesimal traffic
- For other cost functions they can be much larger (even unbounded)

Potential games

- A class of games for which there is a function $P(s_1,s_2,\ldots s_N)$ such that
  - For each i $U_i(s_1,s_2,\ldots x_i,\ldots s_N) > U_i(s_1,s_2,\ldots y_i,\ldots s_N)$ if and only if $P(s_1,s_2,\ldots x_i,\ldots s_N) > P(s_1,s_2,\ldots y_i,\ldots s_N)$
- Properties of potential games: Existence of a pure-strategy NE and convergence to it of best-response dynamics
- The routing games we considered are particular potential games