Game Theory: introduction and applications to computer networks

Introduction

Giovanni Neglia
INRIA – EPI Maestro
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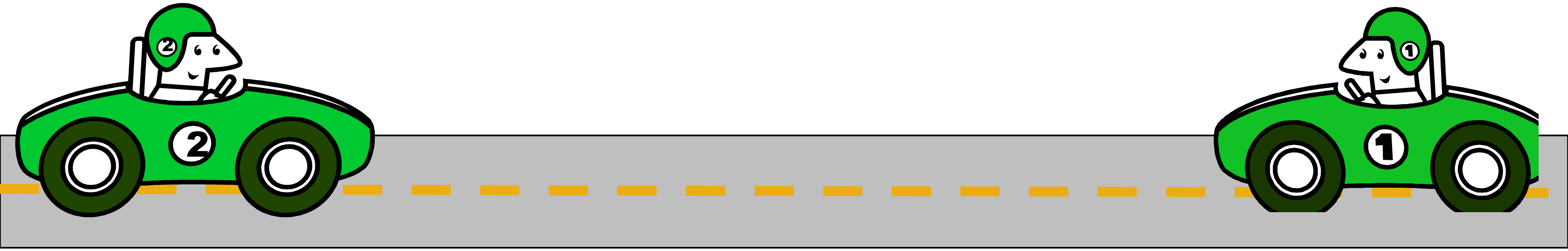
Part of the slides are based on a previous course with D. Figueiredo (UFRJ) and H. Zhang (Suffolk University)

What is Game Theory About?

- Mathematical/Logical analysis of situations of conflict and cooperation



- Game of Chicken
  - driver who steers away looses
  - what should drivers do?
- Goal: to prescribe how rational players should act

What is a Game?

- A Game consists of
  - at least two players
  - a set of strategies for each player
  - a preference relation over possible outcomes
- Player is general entity
  - individual, company, nation, protocol, animal, etc
- Strategies
  - actions which a player chooses to follow
- Outcome
  - determined by mutual choice of strategies
- Preference relation
  - modeled as utility (payoff) over set of outcomes

Short history of GT

- Forerunners:
  - Waldegrave's first minimax mixed strategy solution to a 2-person game (1713), Cournot's duopoly (1838), Zermelo's theorem on chess (1913), Borel's minimax solution for 2-person games with 3 or 5 strategies (20s)
- 1928: von Neumann's theorem on two-person zero-sum games
- 1944: von Neumann and Morgenstern, Theory of Games and Economic Behaviour
- 1950-53: Nash's contributions (Nash equilibrium, bargaining theory)
- 1952-53: Shapley and Gillies' core (basic concept in cooperative GT)
- 60s: Aumann's extends cooperative GT to non-transferable utility games
- 1967-68: Harsanyi's theory of games of incomplete information
- 1972: Maynard Smith's concept of an Evolutionarily Stable Strategy
- Nobel prizes in economics
  - 1994 to Nash, Harsanyi and Selten "for their pioneering analysis of equilibria in the theory of non-cooperative games"
  - 2005 to Aumann and Schelling "for having enhanced our understanding of conflict and cooperation through game-theory analysis"
  - 2012 to Roth and Shapley "for the theory of stable allocations and the practice of market design"
- Movies:
  - 2001 "A beautiful mind" on John Nash's life
- See also:
  - www.econ.canterbury.ac.nz/personal_pages/paul_walker/gt/hist.htm

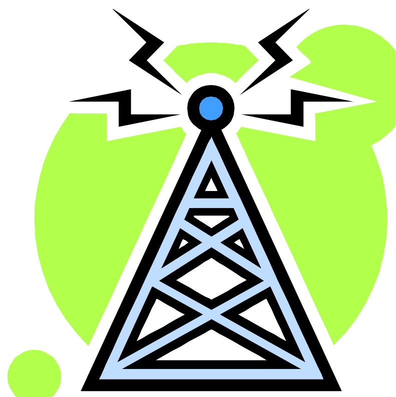
Applications of Game Theory

- Economy
- Politics (vote, coalitions)
- Biology (Darwin's principle, evolutionary GT)
- Anthropology
- War
- Management-labor arbitration
- Philosophy (morality and free will)
- National Football league draft

Applications of Game Theory

- "Recently" applied to computer networks
  - Nagle, RFC 970, 1985
    - "datagram networks as a multi-player game"
  - wider interest starting around 2000
- Which are the strategies available?
  - Network elements follow protocol!!!

Power games



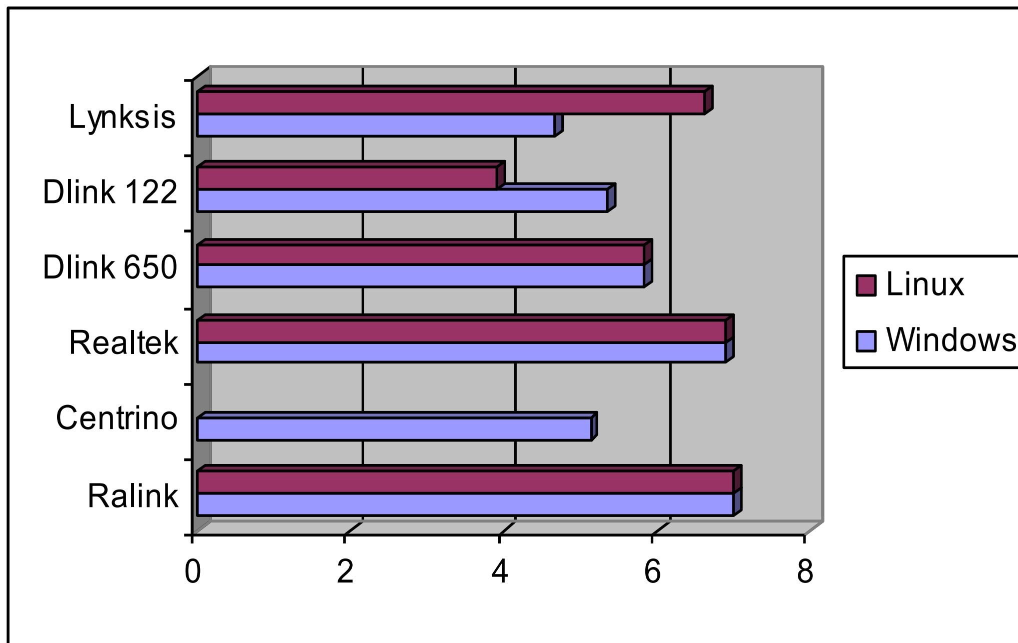
$$SNIR_1 = \frac{H_{1,BS} P_1}{N + H_{2,1} P_2}$$

Medium Access Control Games

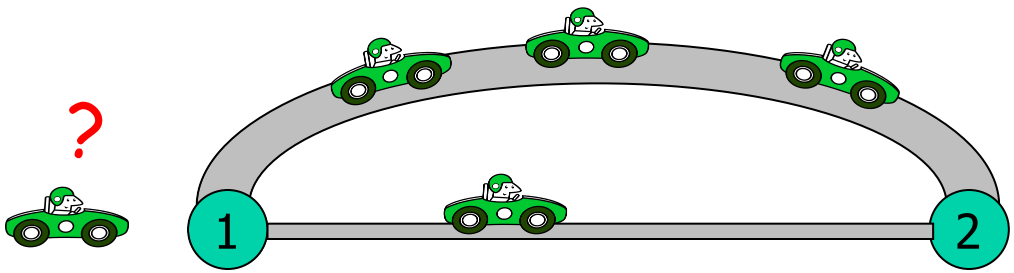
$$Thr_1 = \frac{p_1(1-p_2)P}{(1-p_1)(1-p_2)\sigma + [1-(1-p_1)(1-p_2)]T}$$

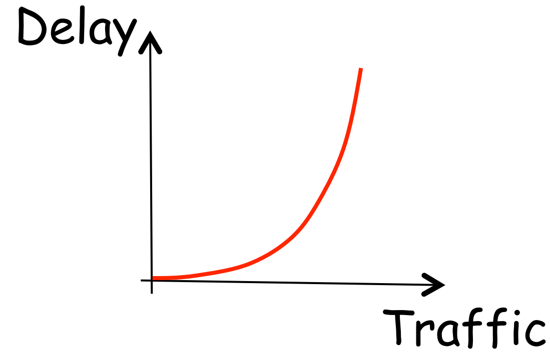
Medium Access Control Games

- Despite of the Wi-Fi certification, several cards exhibit very heterogeneous performance, due to arbitrary protocol implementations
  - "Experimental Assessment of the Backoff Behavior of Commercial IEEE 802.11b Network Cards," G Bianchi et al, INFOCOM 2007



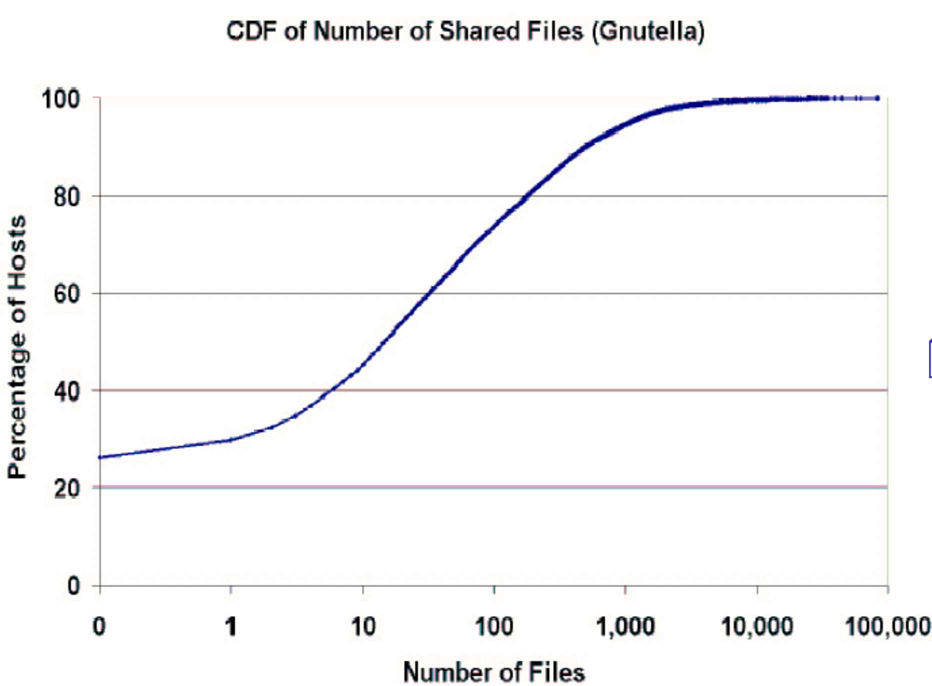
Routing games





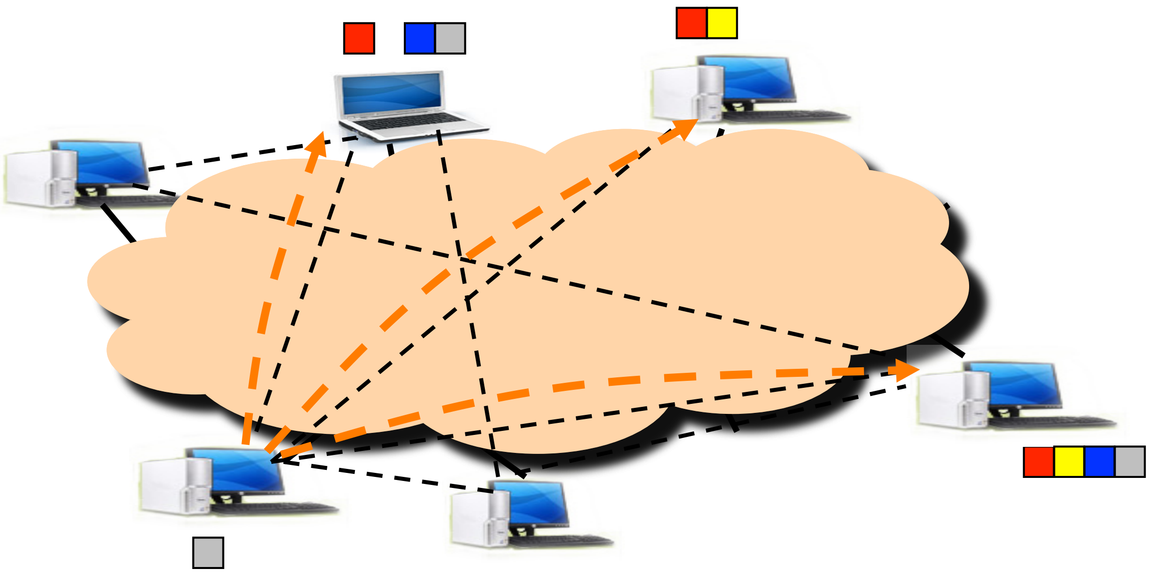
- Possible in the Internet (see later)

Free riders in P2P networks



- Individuals not willing to pay the cost of a public good, they hope that someone else will bear the cost instead
- Few servers become the hot spots: Anonymous?, Copyright?, Privacy? Scalability?, Is it P2P?

Connection games in P2P



- Each peer may open multiple TCP connections to increase its downloading rate

Diffusion of BitTorrent variants

- Try to exploit BitTorrent clients weaknesses

BitThief

- Are they really dangerous?
    - Evolutionary game theory says that

      Yes they can be

Space for GT in Networks

- User behaviors (to share or not to share)
  - Client variants
- Protocols do not specify everything...
  - power level to use
  - number of connections to open
- ...and/or are not easy to enforce
  - how control a P2P network
  - not-compliant WiFi implementation
- ...and software easy to modify

Limitations of Game Theory

- Real-world conflicts are complex
  - models can at best capture important aspects
- Players are considered rational
  - determine what is best for them given that others are doing the same
  - Men are not, but computers are more
- No unique prescription
  - not clear what players should do

- But it can provide intuitions, suggestions and partial prescriptions
  - the best mathematical tool we have

Syllabus

- References
  - [S] Straffin, Game Theory and Strategy (main one, chapters indicated)
  - [EK] Easley and Kleinberg, Network Crowds and Markets
  - [OR] Osborne and Rubinstein, A course in game theory, MIT Press
- Two-person zero-sum games
  - Matrix games
    - Pure strategy equilibria (dominance and saddle points), [S2]
    - Mixed strategy equilibria, [S3]
  - Game trees (?), [S7]
- Two-person non-zero-sum games
  - Nash equilibria…
    - …And its limits (equivalence, interchangeability, Prisoner's dilemma), [S11-12]
  - Subgame Perfect Nash Equilibria (?)
  - Routing games [EK8]
- Auction theory

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Two-person zero-sum games

Giovanni Neglia
INRIA – EPI Maestro

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Matrix Game (Normal form)

Strategy set for Player 1

Player 2, Colin

Strategy set for Player 2

		A	B	C
Player 1, Rose	A	(2, 2)	(0, 0)	(-2, -1)
	B	(-5, 1)	(3, 4)	(3, -1)

Payoff to Player 1

Payoff to Player 2

- Simultaneous play
  - players analyze the game and then write their strategy on a piece of paper

More Formal Game Definition

- Normal form (strategic) game
  - a finite set N of players
  - a set strategies S_i for each player $i \in N$
  - payoff function $u_i(s)$ for each player $i \in N$
    - where $s \in S = \times_{j \in N} S_j$ is an outcome
    - sometimes also $u_i(A, B, \ldots) \quad A \in S_1, B \in S_2, \ldots$
    - $u_i : S \to \Re$

Two-person Zero-sum Games

- One of the first games studied
  - most well understood type of game
- Players interest are strictly opposed
  - what one player gains the other loses
  - game matrix has single entry (gain to player 1)
- A “strong” solution concept

Let' s play!

		Colin			
		A	B	C	D
	A	12	-1	1	0
Rose	B	5	1	7	-20
	C	3	2	4	3
	D	-16	0	0	16

- Divide in pairs, assign roles (Rose/Colin) and play 20 times
- Log how many times you have played each strategy and how much you have won

Analyzing the Game

		Colin			
		A	B	C	D
Rose	A	12	-1	1	0
	B	5	1	7	-20
	C	3	2	4	3
	D	-16	0	0	16

dominated
strategy
(dominated by B)

Dominance

- Strategy S *(weakly) dominates* a strategy T if every possible outcome when S is chosen is at least as good as corresponding outcome in T, and one is strictly better
  - S *strictly dominates* T if every possible outcome when S is chosen is strictly better than corresponding outcome in T
- Dominance Principle
  - rational players never choose dominated strategies
- Higher Order Dominance Principle
  - iteratively remove dominated strategies

Higher order dominance may be enough

Colin

Rose		A	B	C	D
	A	12	-1	1	0
	B	3	1	4	-18
	C	5	2	4	3
	D	-16	0	5	-1

Higher order dominance may be enough

GT prescribes:

Rose C - Colin B

Colin

Rose	A	B	C	D
A	12	-1	1	0
B	3	1	4	-18
C	5	2	4	3
D	-16	0	5	-1

(Weakly) Dominated by C

A priori D is **not** dominated by C

Strictly dominated by B

… but not in the first game

		Colin			
		A	B	C	D
Rose	A	12	-1	1	0
	B	5	1	7	-20
	C	3	2	4	3
	D	-16	0	0	16

dominated strategy (dominated by B)

Analyzing the Reduced Game: Movement Diagram

Colin

Rose		A	B	D
	A	12	-1	0
	B	5	1	20
	C	3	2	3
	D	-16	0	16

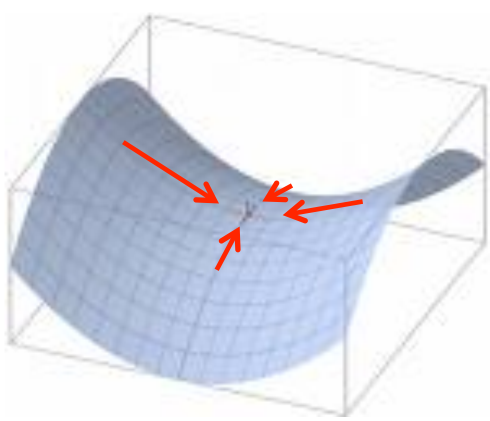
Outcome (C, B) is "stable"

- ***saddle point of game***
- mutual ***best responses***

Saddle Points

- An outcome (x,y) is a *saddle point* if the corresponding entry u(x,y) is both less than or equal to any value in its row and greater than or equal to any value in its column
  - u(x,y) <= u(x,w) for all w in $S_2=S_{Colin}$
  - u(x,y) >= u(v,y) for all v in $S_1=S_{Rose}$

	A	B	D
A	12	-1	0
B	5	1	-20
C	3	2	3
D	-16	0	16



Saddle Points Principle

- Players should choose outcomes that are saddle points of the game
  - Because it is an equilibrium…
  - … but not only

Performance Evaluation

Second Part
Lecture 5

Giovanni Neglia
INRIA – EPI Maestro
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Saddle Points main theorem

- The game has a saddle point iff
  $\max_v \min_w u(v,w) = \min_w \max_v u(v,w)$

Colin

Rose	A	B	D	$\min_w$
A	12	-1	0	-1
B	5	1	-20	-20
C	3	2	3	2
D	-16	0	16	-16
$\max_v$	12	2	16	

- Rose C ε argmax $\min_w u(v,w)$ most cautious strategy for Rose: it secures the maximum worst case gain independently from Colin's action (the game ***maximin value***)

- Colin B ε argmin $\max_v u(v,w)$ most cautious strategy for Colin: it secures the minimum worst case loss (the game ***minimax value***)

Saddle Points main theorem

- Another formulation:
  - The game has a saddle point iff

    maximin = minimax,
- This value is called the **value of the game**

Saddle Points main theorem

- The game has a saddle point iff
  $\max_v \min_w u(v,w) = \min_w \max_v u(v,w)$

N.C.

Two preliminary remarks

1. It holds (always)
   $\max_v \min_w u(v,w) <= \min_w \max_v u(v,w)$
   because $\min_w u(v,w) <= u(v,w) <= \max_v u(v,w)$ for all v and w
2. By definition, if (x,y) is a saddle point
   - $u(x,y) <= u(x,w)$ for all w in S_{Colin}
     - i.e. $u(x,y) = \min_w u(x,w)$
   - $u(x,y) >= u(v,y)$ for all v in S_{Rose}
     - i.e. $u(x,y) = \max_v u(v,y)$

Saddle Points main theorem

- The game has a saddle point iff
  $\max_v \min_w u(v,w) = \min_w \max_v u(v,w)$

1. $\max_v \min_w u(v,w) <= \min_w \max_v u(v,w)$
2. if (x,y) is a saddle point
   - $u(x,y)=\min_w u(x,w)$, $u(x,y)=\max_v u(v,y)$

N.C.

$u(x,y)=\min_w u(x,w)<=\max_v \min_w u(v,w)<=\min_w \max_v u(v,w)<=\max_v u(v,y)=u(x,y)$

Saddle Points main theorem

- The game has a saddle point iff
  $\max_v \min_w u(v,w) = \min_w \max_v u(v,w)$

S.C.

x in argmax $\min_w u(v,w)$

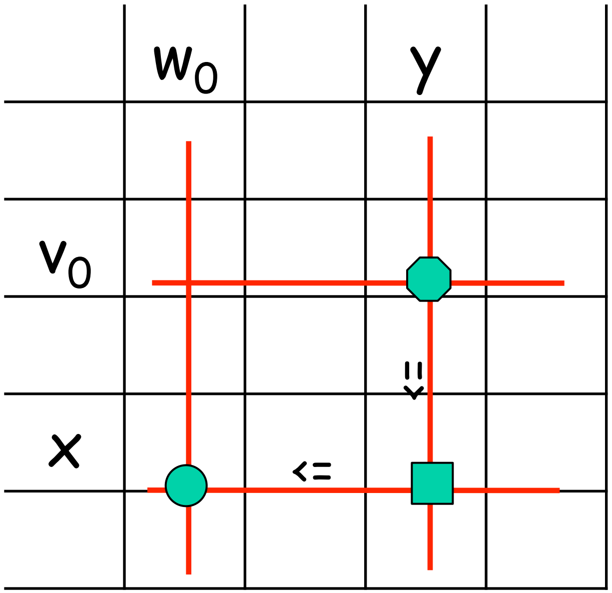
y in argmin $\max_v u(v,w)$

We prove that (x,y) is a saddle-point

w_0 in $\mathrm{argmin}_w\, u(x,w)$ $(\max_v \min_w u(v,w)=u(x,w_0))$

v_0 in $\mathrm{argmax}_v\, u(v,y)$ $(\min_w \max_v u(v,w)=u(v_0,y))$

$u(x,w_0)=\min_w u(x,w) \le u(x,y) \le \max_v u(v,y)=u(v_0,y)$



Note that $u(x,y) = \max_v \min_w u(v,w)$

Saddle Points main theorem

- The game has a saddle point iff
  $\max_v \min_w u(v,w) = \min_w \max_v u(v,w)$

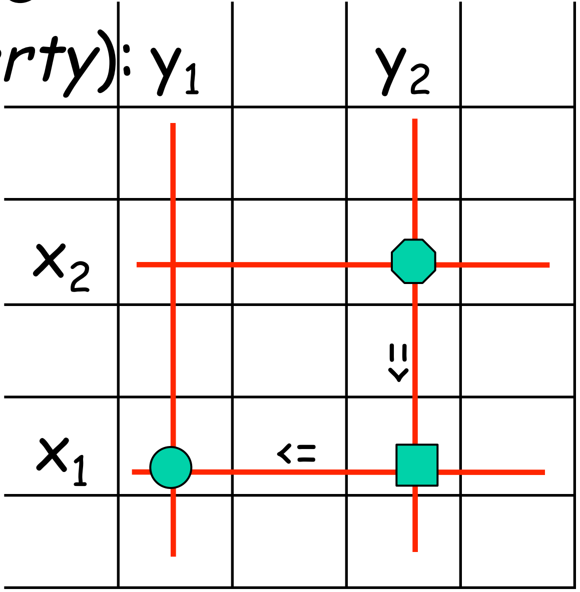
Rose \ Colin	A	B	D	$\min_w$
A	12	-1	0	-1
B	5	1	-20	-20
C	3	2	3	2
D	-16	0	16	-16
$\max_v$	12	2	16	

This result provides also another way to find saddle points

Properties

- Given two saddle points (x_1,y_1) and (x_2,y_2),
  - they have the same payoff (*equivalence property*):
    - it follows from previous proof:

      $u(x_1,y_1) = \max_v \min_w u(v,w) = u(x_2,y_2)$
  - (x_1,y_2) and (x_2,y_1) are also saddle points(*interchangeability property*):
    - as in previous proof



They make saddle point a very nice solution!

What is left?

- There are games with no saddle-point!
- An example?

	R	P	S	min
R	0	-1	1	-1
P	1	0	-1	-1
S	-1	1	0	-1
max	1	1	1	

maximin

minimax

maximin <> minimax

What is left?

- There are games with no saddle-point!
- An example? An even simpler one

	A	B	min
A	2	0	0
B	-5	3	-5
max	2	3	

maximin

minimax

Some practice: find all the saddle points

	A	B	C	D
A	3	2	4	2
B	2	1	3	0
C	2	2	2	2

	A	B	C
A	-2	0	4
B	2	1	3
C	3	-1	-2

	A	B	C
A	4	3	8
B	9	5	1
C	2	7	6

Games with no saddle points

	Colin	
Rose	**A**	**B**
A	2	0
B	-5	3

- What should players do?
  - resort to randomness to select strategies

Mixed Strategies

- Each player associates a probability distribution over its set of strategies
- Expected value principle: maximize the expected payoff

		Colin	1/3	2/3
			A	B
Rose		A	2	0
		B	-5	3

Rose's expected payoff when playing A = 1/3*2+2/3*0=2/3

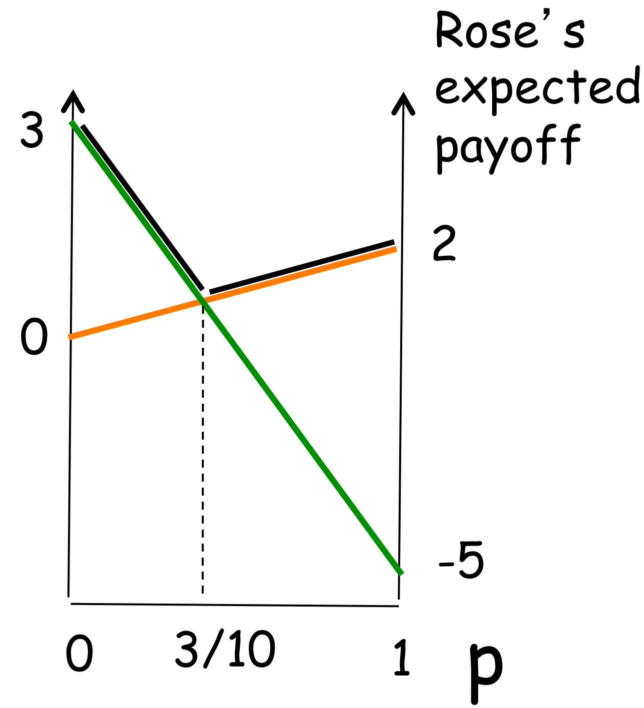
Rose's expected payoff when playing B = 1/3*-5+2/3*3=1/3

- How should Colin choose its prob. distribution?

2x2 game

Colin

	p	1-p
	A	B
Rose A	2	0
Rose B	-5	3



Rose's exp. gain when playing A = 2p + (1-p)*0 = 2p

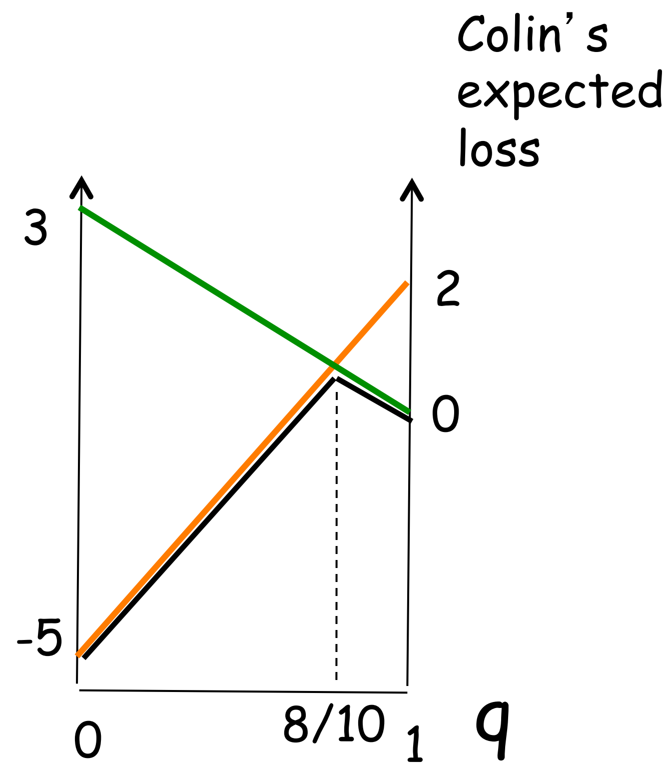
Rose's exp. gain when playing B = -5*p + (1-p)*3 = 3-8p

- How should Colin choose its prob. distribution?
  - Rose cannot take advantage of p=3/10
  - for p=3/10 Colin guarantees a loss of 3/5, what about Rose's?

2x2 game

Colin

Rose			A	B
	q	A	2	0
	1-q	B	-5	3



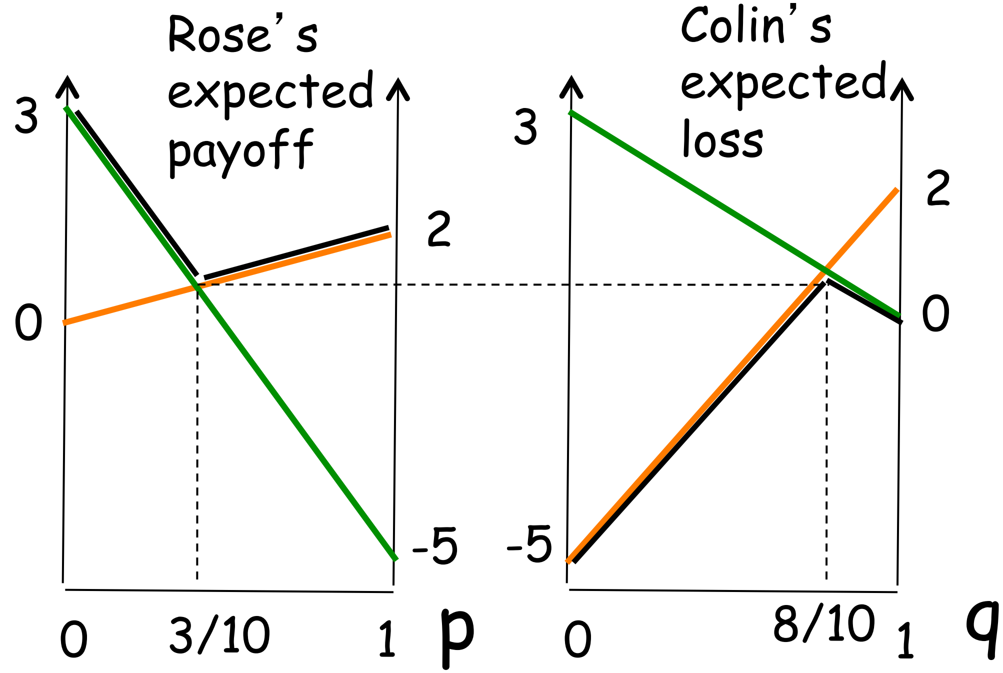
Colin's exp. loss when playing A = 2q -5*(1-q) = 7q-5

Colin's exp. loss when playing B = 0*q+3*(1-q) = 3-3q

- How should Rose choose its prob. distribution?
  - Colin cannot take advantage of q=8/10
  - for q=8/10 Rose guarantees a gain of?

2x2 game

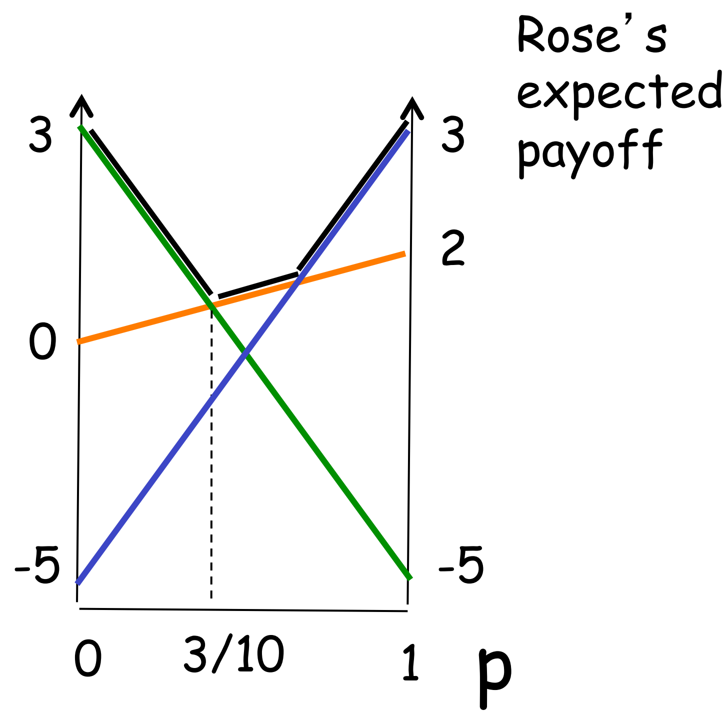
		Colin: p	1-p
Rose		A	B
q	A	2	0
1-q	B	-5	3



- Rose playing the mixed strategy (8/10,2/10) and Colin playing the mixed strategy (3/10,7/10) is the equilibrium of the game
  - No player has any incentives to change, because any other choice would allow the opponent to gain more
  - Rose gain 3/5 and Colin loses 3/5

mx2 game

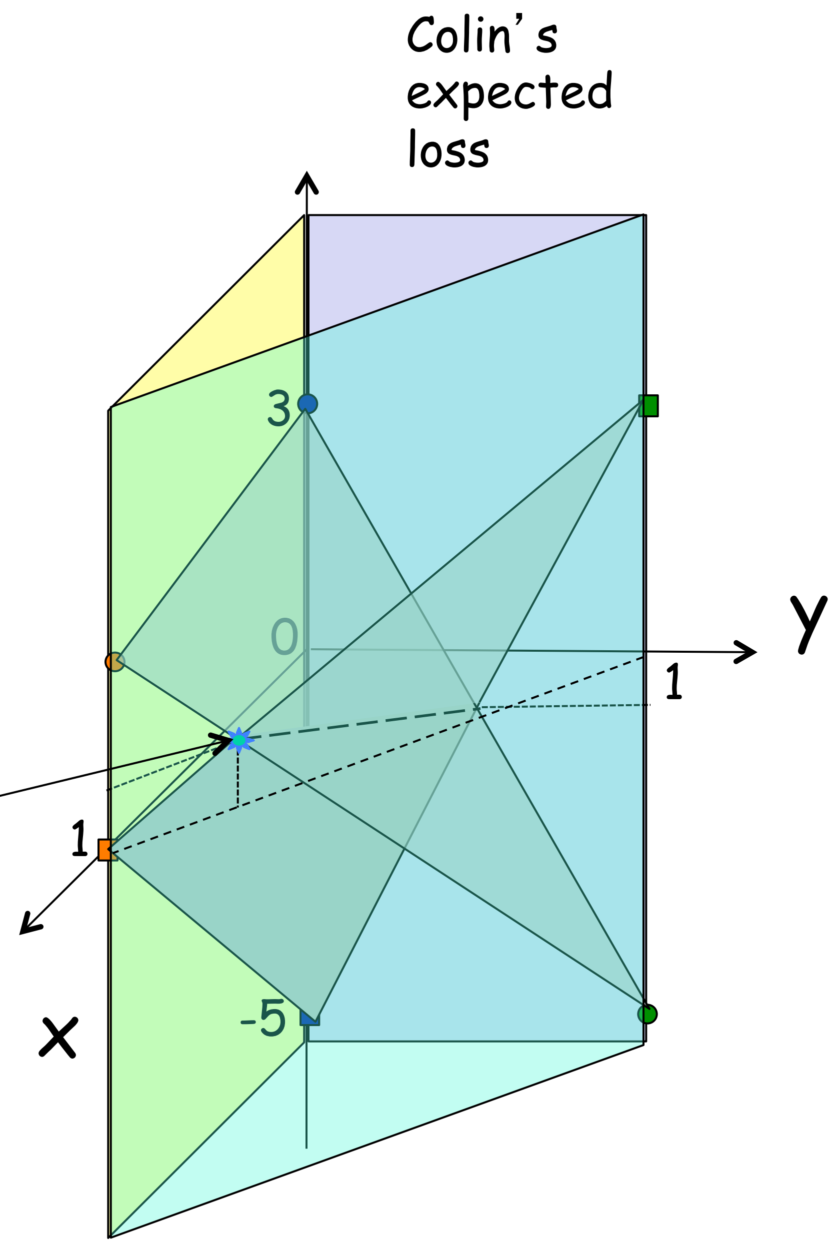
	Rose \ Colin	A (p)	B ($1-p$)
x	A	2	0
y	B	-5	3
$1-x-y$	C	3	-5



- By playing p=3/10, Colin guarantees max exp. loss = 3/5
  - it loses 3/5 if Rose plays A or B, it wins 13/5 if Rose plays C
- Rose should not play strategy C

mx2 game

		Colin	
		p	1-p
		A	$\boxed{B}$
Rose	x	A: 2	0
	y	B: -5	3
	1-x-y	C: 3	-5



- Then Rose should play mixed strategy(8/10,2/10,0)
- guaranteeing a gain not less than 3/5

Minimax Theorem

- Every two-person zero-sum game has a solution, i.e, there is a unique value v (*value of the game*) and there are optimal (pure or mixed) strategies such that
  - Rose's optimal strategy guarantees to her a payoff >= v (no matter what Colin does)
  - Colin's optimal strategies guarantees to him a payoff <= v (no matter what Rose does)
- This solution can always be found as the solution of a kxk subgame
- Proved by John von Neumann in 1928!
  - birth of game theory…

How to solve mxm games

- if all the strategies are used at the equilibrium, the probability vector is such to make equivalent for the opponent all its strategies
  - a linear system with m-1 equations and m-1 variables
  - if it has no solution, then we need to look for smaller subgames

Colin

Rose			A	B	C
	x	A	2	0	1
	y	B	-5	3	-2
	1-x-y	C	3	-5	3

Example:

- $2x-5y+3(1-x-y)=0x+3y-5(1-x-y)$
- $2x-5y+3(1-x-y)=1x-2y+3(1-x-y)$

How to solve 2x2 games

- If the game has no saddle point
  - calculate the absolute difference of the payoffs achievable with a strategy
  - invert them
  - normalize the values so that they become probabilities

Colin

			p	1-p
			A	B
Rose	q	A	2	0
	1-q	B	-5	3

$\rightarrow |2-0|=2 \quad\quad 8 \rightarrow 8/10$

$\rightarrow |-5-3|=8 \quad\quad 2 \rightarrow 2/10$

How to solve mxn matrix games

1. Eliminate dominated strategies
2. Look for saddle points (solution of 1x1 games), if found stop
3. Look for a solution of all the hxh games, with h=min{m,n}, if found stop
4. Look for a solution of all the (h-1)x(h-1) games, if found stop
5. …

h+1. Look for a solution of all the 2x2 games, if found stop

Remark: when a potential solution for a specific kxk game is found, it should be checked that Rose's m-k strategies not considered do not provide her a better outcome given Colin's mixed strategy, and that Colin's n-k strategies not considered do not provide him a better outcome given Rose's mixed strategy.