Seller revenue

- $N$ bidders
- Values are independent random values between 0 and 1
- Expected $i^{th}$ largest utility is $(N+1-i)/(N+1)$
- Expected seller revenue is $(N-1)/(N+1)$
1\textsuperscript{st} price auction

- Player with the highest bid gets the good and pays a price equal to her/his bid.
- Being truthful is not a dominant strategy anymore!
  - Consider for example if I knew other players' utilities.
- How to study it?
$1^{st}$ price auction

- Assumption: for each player the other values are i.i.d. random variables between 0 and 1
  - to overcome the fact that utilities are unknown
- Player $i$'s strategy is a function $s()$ mapping value $v_i$ to a bid $b_i$
  - $s()$ strictly increasing, differentiable function
  - $0 \leq s(v) \leq v \implies s(0)=0$
- We investigate if there is a strategy $s()$ common to all the players that leads to a Nash equilibrium
1st price auction

- Assumption: for each player the other values are i.i.d. random variables between 0 and 1
- Player i’s strategy is a function s() mapping value $v_i$ to a bid $b_i$
- Expected payoff of player i if all the players plays s():
  \[ U_i(s(v_1),...s(v_i),...s(v_N)) = v_i^{N-1} (v_i - s(v_i)) \]
  prob. i wins, i’s payoff if he/she wins
1st price auction

- Expected payoff of player i if all the players play s():
  \[ U_i(s(v_1), \ldots s(v_i), \ldots s(v_N)) = v_i^{N-1} (v_i - s(v_i)) \]

- What if i plays a different strategy \( t() \)?
  - If all players playing s() is a NE, then:
    \[ U_i(s(v_1), \ldots s(v_i), \ldots s(v_N)) = v_i^{N-1} (v_i - s(v_i)) \]
    \[ \geq s^{-1}(t(v_i))^{N-1} (v_i - t(v_i)) = U_i(s(v_1), \ldots t(v_i), \ldots s(v_N)) \]

- Difficult to check for all the possible functions \( t() \) different from s()

- Help from the revelation principle
The Revelation Principle

- All the strategies are equivalent to bidder $i$ supplying to $s()$ a different value of $v_i$
1\textsuperscript{st} price auction

- Expected payoff of player i if all the players plays \( s(): \)
  - \( U_i(s(v_1),\ldots s(v_i),\ldots s(v_N)) = v_i^{N-1} (v_i-s(v_i)) \)

- What if \( i \) plays a different strategy \( t() \)?

- By the revelation principle:
  - \( U_i(s(v_1),\ldots t(v_i),\ldots s(v_N)) =_{\text{eq}} U_i(s(v_1),\ldots s(v),\ldots s(v_N)) = v_i^{N-1} (v_i-s(v)) \)

- If \( v_i^{N-1} (v_i-s(v_i)) \geq v_i^{N-1} (v_i-s(v)) \) for each \( v \) (and for each \( v_i \))
  - Then all players playing \( s() \) is a NE
1\textsuperscript{st} price auction

- If $v_i^{N-1} (v_i-s(v_i)) \geq v^{N-1} (v_i-s(v))$ for each $v$ (and for each $v_i$)
  - Then all players playing $s()$ is a NE
- $f(v)=v_i^{N-1} (v_i-s(v_i)) - v^{N-1} (v_i-s(v))$ is minimized for $v=v_i$
- $f'(v)=0$ for $v=v_i$,
  - i.e. $(N-1) v_i^{N-2} (v_i-s(v_i)) - v_i^{N-1} s'(v_i) = 0$ for each $v_i$
  - $s'(v_i) = (N-1)(1 - s(v_i)/v_i)$, $s(0)=0$
  - Solution: $s(v_i)=(N-1)/N v_i$
1st price auction

- All players bidding according to 
  \( s(v) = \frac{(N-1)}{N} v \) is a NE

Remarks
- They are not truthful
- The more they are, the higher they should bid

Expected seller revenue
- \( \left( \frac{(N-1)}{N} \right) E[v_{\text{max}}] = \left( \frac{(N-1)}{N} \right) \left( \frac{N}{N+1} \right) = \frac{(N-1)}{(N+1)} \)
- Identical to 2nd price auction!
- A general revenue equivalence principle
Outline

- Preliminaries
  - Auctions
  - Matching markets
- Possible approaches to ads pricing
- Google mechanism

References
- Easley, Kleinberg, "Networks, Crowds and Markets", ch.9,10,15
How to match a set of different goods to a set of buyers with different evaluations

$v_{ij}$: value that buyer $j$ gives to good $i$
Matching Markets

How to match a set of different goods to a set of buyers with different evaluations

$\mathbf{v}_{ij}$: value that buyer $j$ gives to good $i$

- **Goods**
  - 1
  - 2
  - 3

- **Buyers**
  - a: $v_{1a}, v_{2a}, v_{3a}$
  - b: $v_{1b}, v_{2b}, v_{3b}$
  - c: $v_{1c}, v_{2c}, v_{3c}$

Maximize

$$\sum_{i,j=1}^{N} x_{ij} v_{ij}$$

Subject to

$$\sum_{j=1}^{N} x_{ij} = 1, \quad \sum_{i=1}^{N} x_{ij} = 1,$$

Over

$$x_{ij} \in \{0,1\}$$
How to match a set of different goods to a set of buyers with different evaluations
Which goods buyers like most? Preferred seller graph

- Given the prices, look for a perfect matching on the preferred seller graph
- There is no such matching for this graph
Matching Markets

Which goods buyers like most? Preferred seller graph

- But with different prices, there is
Matching Markets

Which goods buyers like most? Preferred seller graph

- But with different prices, there is
- Such prices are market clearing prices
Market Clearing Prices

- They always exist
  - And can be easily calculated if valuations are known

- They are socially optimal in the sense that
  - they achieve the maximum total valuation of any assignment of sellers to buyers
  - Or, equivalently, they maximize the sum of all the payoffs in the network (both sellers and buyers)
Outline

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References
- Easley, Kleinberg, "Networks, Crowds and Markets", ch.9,10,15
Ads pricing

How to rank ads from different companies

$r_i$: click rate for an ad in position $i$ (assumed to be independent from the ad and known a priori)

$v_i$: value that company $i$ gives to a click
Ads pricing as a matching market

Ads positions

<table>
<thead>
<tr>
<th>1</th>
<th>$r_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$r_2$</td>
</tr>
<tr>
<td>3</td>
<td>$r_3$</td>
</tr>
</tbody>
</table>

Companies

<table>
<thead>
<tr>
<th>$a$</th>
<th>$v_ar_1, v_ar_2, v_ar_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>$v_br_1, v_br_2, v_br_3$</td>
</tr>
<tr>
<td>$c$</td>
<td>$v_cr_1, v_cr_2, v_cr_3$</td>
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</tbody>
</table>

$r_i$: click rate for an ad in position $i$ (assumed to be independent from the ad and known a priori)

$v_i$: value that company $i$ gives to a click

- Problem: Valuations are not known!
- ... but we could look for something as 2nd price auctions
The VCG mechanism

- The correct way to generalize 2nd price auctions to multiple goods
- Vickrey-Clarke-Groves
- Every buyers should pay a price equal to the social value loss for the others buyers
  - Example: consider a 2nd price auction with $v_a>v_b>\ldots>v_N$
    - With $a$ present the others buyers get 0
    - Without $a$, $b$ would have got the good with a value $v_b$
    - then the social value loss for the others is $v_b$
The VCG mechanism

- The correct way to generalize 2nd price auctions to multiple goods
- Vickrey-Clarke-Groves
- Every buyers should pay a price equal to the social value loss for the others buyers
  - If $V_{BS}$ is the maximum total valuation over all the possible perfect matchings of the set of sellers $S$ and the set of buyers $B$,
  - If buyer $\beta$ gets good $i$, he/she should be charged $V_{B-\beta}^S - V_{B-\beta}^{S-i}$
VCG example

Ads positions

1  \( r_1 = 10 \)

2  \( r_2 = 5 \)

3  \( r_3 = 2 \)

companies

a  \( v_a = 3 \)

b  \( v_b = 2 \)

c  \( v_c = 1 \)

\( r_i \): click rate for an ad in position \( i \) (assumed to be independent from the ad and known a priori)

\( v_i \): value that company \( i \) gives to a click
# VCG example

<table>
<thead>
<tr>
<th>Ads positions</th>
<th>companies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a</td>
</tr>
<tr>
<td>2</td>
<td>b</td>
</tr>
<tr>
<td>3</td>
<td>c</td>
</tr>
</tbody>
</table>

- **a**: 30, 15, 6
- **b**: 20, 10, 4
- **c**: 10, 5, 2
This is the maximum weight matching

- $a$ gets 30, $b$ gets 10, and $c$ gets 2
VCG example

If $a$ weren't there, $b$ and $c$ would get 25 instead of 12,

Then $a$ should pay 13
If $b$ weren’t there, $a$ and $c$ would get 35 instead of 32,

Then $b$ should pay 3
If $c$ weren’t there, nothing would change for $a$ and $b$,

Then $c$ should pay 0
The VCG mechanism

- Every buyer should pay a price equal to the social value loss for the others buyers
  - If $V^S_B$ is the maximum total valuation over all the possible perfect matchings of the set of sellers $S$ and the set of buyers $B$,
  - If buyer $j$ gets good $i$, he/she should be charged $V^S_{B-\beta} - V^S_{B-\beta-\{i\}}$
- Under this price mechanism, truth-telling is a dominant strategy
Outline

- Preliminaries
  - Auctions
  - Matching markets

- Possible approaches to ads pricing
  - Google mechanism

- References
  - Easley, Kleinberg, "Networks, Crowds and Markets", ch.9,10,15
Google’s GSP auction

- Generalized Second Price
- Once all the bids are collected $b_1 > b_2 > ... > b_N$
- Company $i$ pays $b_{i+1}$
- In the case of a single good (position), GSP is equivalent to a 2nd price auction, and also to VCG
- But why Google wanted to implement something different???
GSP properties

- Truth-telling may not be an equilibrium
GSP example

<table>
<thead>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a</td>
</tr>
<tr>
<td>r₁ = 10</td>
<td>vₐ = 7</td>
</tr>
<tr>
<td>2</td>
<td>b</td>
</tr>
<tr>
<td>r₂ = 4</td>
<td>v₉ = 6</td>
</tr>
<tr>
<td>3</td>
<td>c</td>
</tr>
<tr>
<td>r₃ = 0</td>
<td>v₉ = 1</td>
</tr>
</tbody>
</table>

rᵢ: click rate for an ad in position i (assumed to be independent from the ad and known a priori)

vᵢ: value that company i gives to a click

- If each player bids its true evaluation, a gets a payoff equal to 10
- If a bids 5, a gets a payoff equal to 24
GSP properties

- Truth-telling may not be an equilibrium
- There is always at least 1 socially optimal NE
GSP example

Ads positions | companies
---|---
1 | a | $v_a = 7$
2 | b | $v_b = 6$
3 | c | $v_c = 1$

$r_i$: click rate for an ad in position $i$ (assumed to be independent from the ad and known a priori)

$v_i$: value that company $i$ gives to a click

- **Multiple NE**
  - $a$ bids 5, $b$ bids 4 and $c$ bids 2
  - $a$ bids 3, $b$ bids 5 and $c$ bids 1
GSP properties

- Truth-telling may not be an equilibrium
- There is always at least 1 socially optimal NE
- Revenues can be higher or lower than VCG
  - Attention: the revenue equivalence principle does not hold for auctions with multiple goods!
  - Google was targeting higher revenues...
  - ...not clear if they did the right choice.
GSP example

<table>
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<tr>
<td>1</td>
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<tr>
<td>r₁=10</td>
<td>vₐ=7</td>
</tr>
<tr>
<td>2</td>
<td>b</td>
</tr>
<tr>
<td>r₂=4</td>
<td>vₐ=6</td>
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<tr>
<td>r₃=0</td>
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</table>

- **Multiple NE**
  - a bids 5, b bids 4, c bids 2  →  google’s revenue=48
  - a bids 3, b bids 5, c bids 1  →  google’s revenue=34
- **With VCG**, google’s revenue=44
Other issues

- Click rates are unknown and depend on the ad!
  - Concrete risk: low-quality advertiser bidding high may reduce the search engine’s revenue
  - Google’s solution: introduce and ad-quality factor taking into account actual click rate, relevance of the page and its ranking
    - Google is very secretive about how to calculate it => the market is more opaque

- Complex queries, nobody paid for
  - Usually engines extrapolate from simpler bids