Distributed Optimization and Games

Introduction to Game Theory

Giovanni Neglia
INRIA – EPI Maestro
18 January 2017
What is Game Theory About?

- Mathematical/Logical analysis of situations of conflict and cooperation

Goal: to prescribe how rational players should act

Invest on scribe notes or on individual homework?

```
- Rose
  - S
    - Colin
      - S
        - $S_C = S_R = 20$
        - $H_C = 10, H_R = 10$
      - H
        - $S_C = S_R = 16$
        - $H_C = 16, H_R = 10$

- Rose
  - H
    - Colin
      - S
        - $S_C = S_R = 12$
        - $H_C = 16, H_R = 16$
      - H
        - $S_C = S_R = 20$
        - $H_C = 10, H_R = 10$
```
What is a Game?

- A Game consists of
  - at least two players
  - a set of strategies for each player
  - a preference relation over possible outcomes

- Player is general entity
  - individual, company, nation, protocol, animal, etc

- Strategies
  - actions which a player chooses to follow

- Outcome
  - determined by mutual choice of strategies

- Preference relation
  - modeled as utility (payoff) over set of outcomes
Short history of GT

- Forerunners:
  - Waldegrave’s first minimax mixed strategy solution to a 2-person game (1713), Cournot’s duopoly (1838), Zermelo’s theorem on chess (1913), Borel’s minimax solution for 2-person games with 3 or 5 strategies (20s)
  - 1928: von Neumann’s theorem on two-person zero-sum games
  - 1944: von Neumann and Morgenstern, Theory of Games and Economic Behaviour
  - 1950-53: Nash’s contributions (Nash equilibrium, bargaining theory)
  - 1952-53: Shapley and Gillies’ core (basic concept in cooperative GT)
  - 60s: Aumann’s extends cooperative GT to non-transferable utility games
  - 1967-68: Harsanyi’s theory of games of incomplete information
  - 1972: Maynard Smith’s concept of an Evolutionarily Stable Strategy

- Nobel prizes in economics
  - 1994 to Nash, Harsanyi and Selten for “their pioneering analysis of equilibria in the theory of non-cooperative games”
  - 2005 to Aumann and Schelling “for having enhanced our understanding of conflict and cooperation through game-theory analysis”
  - 2012 to Roth and Shapley “for the theory of stable allocations and the practice of market design”

- Movies:
  - 2001 “A beautiful mind” on John Nash’s life

- See also:
  - www.econ.canterbury.ac.nz/personal_pages/paul_walker/gt/hist.htm
Applications of Game Theory

- Economy
- Politics (vote, coalitions)
- Biology (Darwin’s principle, evolutionary GT)
- Anthropology
- War
- Management-labor arbitration
- Philosophy (morality and free will)
- National Football league draft

- “Recently” applied to computer networks
  - Nagle, RFC 970, 1985: “datagram networks as a multi-player game”
  - wider interest starting around 2000
## Matrix Game (Normal Form)

### Simultaneous Play
- Players analyze the game and then write their strategy on a piece of paper.

### Strategy Set
- **Player 1**: Rose
- **Player 2**: Colin

### Payoff Matrix

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<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
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<tbody>
<tr>
<td><strong>A</strong></td>
<td>(2, 2)</td>
<td>(0, 0)</td>
<td>(-2, -1)</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>(-5, 1)</td>
<td>(3, 4)</td>
<td>(3, -1)</td>
</tr>
</tbody>
</table>

### Notes
- **Payoff to Player 1**
- **Payoff to Player 2**
Students' game

<table>
<thead>
<tr>
<th></th>
<th>Colin</th>
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<tbody>
<tr>
<td></td>
<td>S</td>
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<tr>
<td>Rose</td>
<td></td>
</tr>
<tr>
<td>S</td>
<td>15, 15</td>
</tr>
<tr>
<td>H</td>
<td>16, 13</td>
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</table>
More Formal Game Definition

- Normal form (strategic) game
  - a finite set $N$ of players
  - a set strategies $S_i$ for each player $i \in N$
  - payoff function $u_i(s)$ for each player $i \in N$
    - where $s \in S = \times_{j \in N} S_j$ is an outcome
    - sometimes also $u_i(A, B, \ldots)$ $A \in S_1, B \in S_2, \ldots$
    - $u_i : S \to \mathbb{R}$
Two-person Zero-sum Games

- One of the first games studied
  - most well understood type of game
- Players interest are strictly opposed
  - what one player gains the other loses
  - game matrix has single entry (gain to player 1)
- A “strong” solution concept
Dominance

Strategy $S$ (weakly) dominates a strategy $T$ if every possible outcome when $S$ is chosen is at least as good as corresponding outcome in $T$, and one is strictly better.

- $S$ strictly dominates $T$ if every possible outcome when $S$ is chosen is strictly better than corresponding outcome in $T$.

Dominance Principle
- Rational players never choose dominated strategies.

Higher Order Dominance Principle
- Iteratively remove dominated strategies.
Higher order dominance may be enough

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<tbody>
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</tr>
<tr>
<td>H</td>
<td>16, 13</td>
<td>14, 14</td>
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</tbody>
</table>

GT prescribes: Rose H - Colin H
Higher order dominance may be enough

GT prescribes:
Rose C - Colin B

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<tr>
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<th>A</th>
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<tr>
<td></td>
<td>-16</td>
<td>0</td>
<td>5</td>
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A priori D is not dominated by C

Strictly dominated by B

(Weakly) Dominated by C
… but not in general

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<td>7</td>
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<tr>
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<td>3</td>
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<tr>
<td>D</td>
<td>-16</td>
<td>0</td>
<td>0</td>
<td>16</td>
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*dominated strategy (dominated by B)*
Analyzing the Reduced Game: Movement Diagram

Outcome \((C, B)\) is “stable”
- Pure strategy Nash Equilibrium
- mutual best responses

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<tr>
<td><strong>D</strong></td>
<td>-16</td>
<td>0</td>
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If Rose plays D, A is Colin’s best response.
**Students' game**

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Games without pure strategy NE

- An example?

<table>
<thead>
<tr>
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<th>R</th>
<th>P</th>
<th>S</th>
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<td>R</td>
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<td>-1</td>
<td>1</td>
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<td>P</td>
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<td>-1</td>
</tr>
<tr>
<td>S</td>
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Games without pure strategy NE

An example? An even simpler one

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<td>B</td>
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Some practice: find all the pure strategy NE

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<td>A</td>
<td>4</td>
<td>3</td>
<td>8</td>
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<tr>
<td>B</td>
<td>9</td>
<td>5</td>
<td>1</td>
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<tr>
<td>C</td>
<td>2</td>
<td>7</td>
<td>6</td>
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Games with no pure strategy NE

What should players do?
- resort to randomness to select strategies
Games with no pure strategy NE

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Rose

- but we can find mixed strategies equilibria
Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy (equalizing strategy), that equalizes the opponent payoffs
  - how to calculate it?

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Mixed strategies equilibria

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Rose considers Colin’s game

Rose

- 4
- 1

Colin

1/5
4/5
Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy, that equalizes the opponent payoffs
  - how to calculate it?

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Colin considers Rose’s game

- 3/5
- 2/5
Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy, that equalizes the opponent payoffs
  - how to calculate it?

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<td>B</td>
</tr>
<tr>
<td>A</td>
<td>5, 0</td>
<td>-1, 4</td>
</tr>
<tr>
<td>B</td>
<td>3, 2</td>
<td>2, 1</td>
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Rose playing (1/5, 4/5)
Colin playing (3/5, 2/5) is an equilibrium

Rose gains 13/5
Colin gains 8/5
Good news: Nash’s theorem [1950]

- Every two-person games has at least one equilibrium either in pure strategies or in mixed strategies
  - Proved using fixed point theorem
  - Generalized to N person game
- This equilibrium concept called Nash equilibrium in his honor
  - A vector of strategies (a profile) is a Nash Equilibrium (NE) if no player can unilaterally change its strategy and increase its payoff
A useful property

- Given a finite game, a profile is a mixed NE of the game if and only if for every player i, every pure strategy used by i with non-null probability is a best response to other players mixed strategies in the profile.
  - see Osborne and Rubinstein, A course in game theory, Lemma 33.2
Game of Chicken (aka. Hawk-Dove Game)

- Driver who swerves loses

Driver 2

<table>
<thead>
<tr>
<th></th>
<th>Driver 1</th>
<th></th>
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<tbody>
<tr>
<td>swerve</td>
<td>0, 0</td>
<td>-1, 5</td>
</tr>
<tr>
<td>stay</td>
<td>5, -1</td>
<td>-10, -10</td>
</tr>
</tbody>
</table>

Drivers want to do opposite of one another

Two equilibria: not equivalent, not interchangeable!

- playing an equilibrium strategy does not lead to equilibrium
### Students' game

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<td>13, 16</td>
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<tr>
<td></td>
<td>16, 13</td>
<td></td>
<td>14, 14</td>
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</tbody>
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- **Better outcome**: 15, 15
- **Single NE**: 14, 14
Students' game

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</tr>
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<td>H</td>
<td>16,13</td>
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Def: outcome $o^*$ is Pareto Optimal if no other outcome would give to all the players a payoff not smaller and a payoff higher to at least one of them.

Conflict between group rationality (Pareto principle) and individual rationality (dominance principle)
Students’ game = Prisoner’s Dilemma

- One of the most studied and used games
  - proposed in 1950
- Two suspects arrested for joint crime
  - each suspect when interrogated separately, has option to confess

|       | Suspect 2
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<tbody>
<tr>
<td>NC</td>
<td>2, 2</td>
</tr>
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</table>
| C     | 10, 1  

payoff is years in jail (smaller is better)

better outcome

single NE
Distributed Optimization and Games

Auctions
Giovanni Neglia
INRIA – EPI Maestro
18 January 2017
Our starting problem

- We want to give an object to the person who values it the most, i.e.

  \[
  \begin{align*}
  &\text{maximize} & \sum_{i=1}^{N} x_i v_i \\
  &\text{subject to} & \sum_{i=1}^{N} x_i = 1 \\
  &\text{over} & x_i \in \{0, 1\}
  \end{align*}
  \]

- Difficulty: we do not know values \(v_i\) ...
- and we cannot ask to people (they would lie)
- Solution: auctions, but we need to introduce money
Types of auctions

- 1\textsuperscript{st} price & descending bids (Dutch auctions)
- 2\textsuperscript{nd} price & ascending bids (English auctions)
A class of games for which there is a function $P(s_1, s_2, \ldots, s_N)$ such that

- For each $i \in U(s_1, s_2, \ldots, x_i, \ldots, s_N) > U(s_1, s_2, \ldots, y_i, \ldots, s_N)$ if and only if $P(s_1, s_2, \ldots, x_i, \ldots, s_N) > P(s_1, s_2, \ldots, y_i, \ldots, s_N)$

Properties of potential games: Existence of a pure-strategy NE and convergence to it of best-response dynamics.

The routing games we considered are particular potential games.
How it works

- Companies bid for keywords
- On the basis of the bids Google puts their link on a given position (first ads get more clicks)
- Companies are charged a given cost for each click (the cost depends on all the bids)
- Why Google adopted this solution:
  - It has no idea about the value of a click...
  - It lets the company reveal it
Some numbers (2014)

- $\approx 90\%$ of Google revenues (66 billions$) from ads
  - investor.google.com/financial/tables.html

- Costs
  - "calligraphy pens" $1.70
  - "Loan consolidation" $50
  - "mesothelioma" $50 per click

- Click fraud problem
Outline

❖ Preliminaries
  ❖ Auctions
  ❖ Matching markets

❖ Possible approaches to ads pricing

❖ Google mechanism

❖ References
  ❖ Easley, Kleinberg, "Networks, Crowds and Markets", ch.9,10,15
Game Theoretic Model

- **N** players (the bidders)
- Strategies/actions: \( b_i \) is player \( i \)'s bid
- For player \( i \) the good has value \( v_i \)
- \( p_i \) is player \( i \)'s payment if he gets the good
- Utility:
  - \( v_i - p_i \) if player \( i \) gets the good
  - 0 otherwise
- Assumption here: values \( v_i \) are independent and private
  - i.e. very particular goods for which there is not a reference price
Game Theoretic Model

- N players (the bidders)
- Strategies: $b_i$ is player i’s bid
- Utility:
  - $v_i - b_i$ if player i gets the good
  - 0 otherwise
- Difficulties:
  - Utilities of other players are unknown!
  - Better to model the strategy space as continuous (differently from the games we looked at)
2\textsuperscript{nd} price auction

- Player with the highest bid gets the good and pays a price equal to the 2\textsuperscript{nd} highest bid
- There is a dominant strategies
  - I.e. a strategy that is more convenient independently from what the other players do
  - Be truthful, i.e. bid how much you evaluate the good ($b_i=v_i$)
  - Social optimality: the bidder who value the good the most gets it!
$b_i = v_i$ is the highest bid

Bidding more than $v_i$ is not convenient
$b_i = v_i$ is the highest bid

Bidding less than $v_i$ is not convenient (may be unconvenient)
$b_i = v_i$ is not the highest bid

Bidding more than $v_i$ is not convenient (may be inconvenient)
$b_i = v_i$ is not the highest bid

Bidding less than $v_i$ is not convenient