Ex. 1 — [3 point] A set of M elastic flows shares a single link with capacity C.

1. Given a set of weights $w_i > 0$ for $i \in \{1, \dots M\}$, consider the NETWORK problem:

$$\begin{aligned} \underset{\mathbf{x} \in \mathbb{R}^M}{\text{maximize}} \quad & \sum_{i=1}^{M} w_i \log x_i \\ \text{subject to} \quad & \sum_{i=1}^{M} x_i \leq C. \end{aligned}$$

   Determine the optimal rate allocation $\mathbf{x}^*$ and the corresponding Lagrange multiplier λ^* as a function of the weights and of the capacity.

2. Consider now the SYSTEM problem:

$$\begin{aligned} \underset{\mathbf{x} \in \mathbb{R}^M}{\text{maximize}} \quad & \sum_{i=1}^{M} U_i(x_i) \\ \text{subject to} \quad & \sum_{i=1}^{M} x_i \leq C, \end{aligned}$$

   and assume it is solved through decomposition into the NETWORK problem and M USER problems. What is the optimal rate allocation if all the utility functions are equal, i.e. $U_i = U_j$ for each $i, j \in \{1, \dots M\}$?

3. Consider $M = 2$, $C = 1$ and $U_1(x) = U_2(x) = \log(x)$. What is the final payoff (utility minus bid) of the users if the SYSTEM problem is solved through the decomposition approach?[1]

4. Imagine that user 1 understands the decomposition approach and then knows the function $\lambda(w_1, w_2, C)$ (what is this function?). User 1 could solve the problem

$$\underset{w_1 \geq 0}{\text{maximize}} \quad \log\left(\frac{w_1}{\lambda(w_1, w_2, 1)}\right) - w_1,$$

   while user 2 solves the usual USER problem. Calculate the final rate allocation and the final bids and show that user 1 can achieve a higher payoff than user 2.

Ex. 2 — [2 point] Consider a wireless radio that can transmit simultaneously on M independent channels. The channels have the same bandwidth but different background noise power N_i ($i \in \{1, \dots M\}$). The maximum rate achievable on channel i is given by Shannon formula:

$$R_i = B \log_2\left(1 + \frac{S_i}{N_i}\right),$$

[1] The decomposition theorem holds also for this type of utility functions.

where S_i is the signal power over channel i. The transmitter has a maximum transmission power P_T (then $\sum_{i=1}^{M} S_i \leq P_T$) and wants to allocate it through the different channels in order to maximize the total transmission rate.

1. Formulate the corresponding optimization problem.
2. Characterize the optimal power allocation.
3. Assume that $M = 2$, $B = 1$, $N_1 = 1$, $N_2 = 2$. What is the signal power on each channel if $P_T = 1$? and if $P_T = 3$?