Introduction to Game Theory

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INRIA – EPI Maestro
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One of the first games studied
- most well understood type of game

Players interest are strictly opposed
- what one player gains the other loses
- game matrix has single entry (gain to player 1)

A “strong” solution concept
Dominance

- Strategy $S$ (weakly) dominates a strategy $T$ if every possible outcome when $S$ is chosen is at least as good as corresponding outcome in $T$, and one is strictly better
  - $S$ strictly dominates $T$ if every possible outcome when $S$ is chosen is strictly better than corresponding outcome in $T$

- Dominance Principle
  - rational players never choose dominated strategies

- Higher Order Dominance Principle
  - iteratively remove dominated strategies
Higher order dominance may be enough

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Rose’s S strategy dominated by H

GT prescribes:
Rose H - Colin H
Higher order dominance may be enough

GT prescribes:
Rose C - Colin B

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(Weakly) Dominated by C

A priori D is not dominated by C

Strictly dominated by B
... but not in general

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*dominated strategy (dominated by B)*
Analyzing the Reduced Game: Movement Diagram

Outcome (C, B) is “stable”
- Pure strategy Nash Equilibrium
- mutual best responses

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If Rose plays D, A is Colin’s best response
# Students' game

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Games without pure strategy NE

An example?

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Games without pure strategy NE

- An example? An even simpler one

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Some practice: find all the pure strategy NE

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Games with no pure strategy NE

What should players do?
- resort to randomness to select strategies

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Games with no pure strategy NE

...but we can find mixed strategies equilibria

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Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy (equalizing strategy), that equalizes the opponent payoffs
  - how to calculate it?

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Mixed strategies equilibria

- Same idea of equilibrium
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Rose considers Colin’s game

1

4

1/5

4/5
Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy, that equalizes the opponent payoffs
  - how to calculate it?

Colin considers Rose’s game

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3/5  2/5
Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy, that equalizes the opponent payoffs
  - how to calculate it?

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Rose playing (1/5, 4/5)
Colin playing (3/5, 2/5)
is an equilibrium

Rose gains 13/5
Colin gains 8/5
Good news:
Nash’s theorem [1950]

- Every two-person games has at least one equilibrium either in pure strategies or in mixed strategies
  - Proved using fixed point theorem
  - Generalized to N person game
- This equilibrium concept called Nash equilibrium in his honor
  - A vector of strategies (a profile) is a Nash Equilibrium (NE) if no player can unilaterally change its strategy and increase its payoff
A useful property

Given a finite game, a profile is a mixed NE of the game if and only if for every player i, every pure strategy used by i with non-null probability is a best response to other players mixed strategies in the profile.

- see Osborne and Rubinstein, *A course in game theory*, Lemma 33.2
Game of Chicken (aka. Hawk-Dove Game)

- driver who swerves loses

Drivers want to do opposite of one another

Two equilibria: not equivalent, not interchangeable!

- playing an equilibrium strategy does not lead to equilibrium
# Students' game

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Better outcome: S,S

Single NE: H,H

**Notes:**
- Rose prefers S
- Colin prefers H
- (15, 15) is a Nash equilibrium
### Students' game

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- **Def:** outcome \( o^* \) is Pareto Optimal if no other outcome would give to all the players a payoff not smaller and a payoff higher to at least one of them.

- **Conflict between group rationality (Pareto principle) and individual rationality (dominance principle)**
Students’ game = Prisoner’s Dilemma

- One of the most studied and used games
  - Proposed in 1950

- Two suspects arrested for joint crime
  - Each suspect when interrogated separately, has option to confess

|             | Suspect 2
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<tr>
<td>Suspect 1</td>
<td>NC</td>
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<td>NC</td>
<td>2, 2</td>
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Payoff is years in jail (smaller is better)

Better outcome

Single NE
Distributed Optimization and Games

Auctions
Giovanni Neglia
INRIA – EPI Maestro
20 January 2016
Our starting problem

- We want to give an object to the person who values it the most, i.e.

\[
\text{maximize } \sum_{i=1}^{N} x_i v_i \\
\text{subject to } \sum_{i=1}^{N} x_i = 1 \\
\text{over } x_i \in \{0, 1\}
\]

- Difficulty: we do not know values \( v_i \) ...
- and we cannot ask to people (they would lie)
- Solution: auctions, but we need to introduce money
Types of auctions

- 1\textsuperscript{st} price & descending bids (Dutch auctions)
- 2\textsuperscript{nd} price & ascending bids (English auctions)
A class of games for which there is a function $P(s_1, s_2, \ldots, s_N)$ such that

- For each $i \in U(s_1, s_2, \ldots, x_i, \ldots, s_N) > U(s_1, s_2, \ldots, y_i, \ldots, s_N)$ if and only if $P(s_1, s_2, \ldots, x_i, \ldots, s_N) > P(s_1, s_2, \ldots, y_i, \ldots, s_N)$

Properties of potential games:
- Existence of a pure-strategy NE and convergence to it of best-response dynamics
- The routing games we considered are particular potential games.
How it works

- Companies bid for keywords
- On the basis of the bids Google puts their link on a given position (first ads get more clicks)
- Companies are charged a given cost for each click (the cost depends on all the bids)
- Why Google adopted this solution:
  - It has no idea about the value of a click...
  - It lets the company reveal it
Some numbers (2014)

- ≈ 90% of Google revenues (66 billions$) from ads
  - investor.google.com/financial/tables.html

- Costs
  - "calligraphy pens" $1.70
  - "Loan consolidation" $50
  - "mesothelioma" $50 per click

- Click fraud problem
Outline

- Preliminaries
  - Auctions
    - Matching markets
- Possible approaches to ads pricing
- Google mechanism

References
- Easley, Kleinberg, "Networks, Crowds and Markets", ch.9,10,15
Game Theoretic Model

- N players (the bidders)
- Strategies/actions: $b_i$ is player i’s bid
- For player i the good has value $v_i$
- $p_i$ is player i’s payment if he gets the good
- Utility:
  - $v_i - p_i$ if player i gets the good
  - 0 otherwise
- Assumption here: values $v_i$ are independent and private
  - i.e. very particular goods for which there is not a reference price
Game Theoretic Model

- N players (the bidders)
- Strategies: \( b_i \) is player i’s bid
- Utility:
  - \( v_i - b_i \) if player i gets the good
  - 0 otherwise
- Difficulties:
  - Utilities of other players are unknown!
  - Better to model the strategy space as continuous (differently from the games we looked at)
2nd price auction

- Player with the highest bid gets the good and pays a price equal to the 2nd highest bid
- There is a dominant strategies
  - I.e. a strategy that is more convenient independently from what the other players do
  - Be truthful, i.e. bid how much you evaluate the good \((b_i = v_i)\)
  - Social optimality: the bidder who value the good the most gets it!
\( b_i = v_i \) is the highest bid

Bidding more than \( v_i \) is not convenient
$b_i = v_i$ is the highest bid

$U_i = v_i - b_k > v_i - b_i = 0$

Bidding less than $v_i$ is not convenient (may be inconvenient)
\[ b_i = v_i \text{ is not the highest bid} \]

Bidding more than \( v_i \) is not convenient (may be inconvenient)
$b_i = v_i$ is not the highest bid

Bidding less than $v_i$ is not convenient
Seller revenue

- \( N \) bidders
- Values are independent random values between 0 and 1
- Expected \( i^{\text{th}} \) largest utility is \( \frac{N+1-i}{N+1} \)
- Expected seller revenue is \( \frac{N-1}{N+1} \)
1st price auction

- Player with the highest bid gets the good and pays a price equal to her/his bid
- Being truthful is not a dominant strategy anymore!
  - Consider for example if I knew other players’ utilities
- How to study it?
1\textsuperscript{st} price auction

- Assumption: for each player the other values are i.i.d. random variables between 0 and 1
  - to overcome the fact that utilities are unknown
- Player \( i \)'s strategy is a function \( s() \) mapping value \( v_i \) to a bid \( b_i \)
  - \( s() \) strictly increasing, differentiable function
  - \( 0 \leq s(v) \leq v \quad \Rightarrow \quad s(0)=0 \)
- We investigate if there is a strategy \( s() \) common to all the players that leads to a Nash equilibrium
1st price auction

- Assumption: for each player the other values are i.i.d. random variables between 0 and 1.
- Player i’s strategy is a function s() mapping value $v_i$ to a bid $b_i$.
- Expected payoff of player i if all the players play s():
  \[ U_i(s(v_1), \ldots, s(v_i), \ldots, s(v_N)) = v_i^{N-1} (v_i - s(v_i)) \]

prob. i wins  i’s payoff if he/she wins
1st price auction

- Expected payoff of player $i$ if all the players play $s()$:
  - $U_i(s(v_1),...,s(v_i),...,s(v_N)) = v_i^{N-1} (v_i - s(v_i))$

- What if $i$ plays a different strategy $t()$?
  - If all players playing $s()$ is a NE, then:
    - $U_i(s(v_1),...,s(v_i),...,s(v_N)) = v_i^{N-1} (v_i - s(v_i))$
    - $\geq v_i^{N-1} (v_i - t(v_i)) = U_i(s(v_1),...,t(v_i),...,s(v_N))$

- Difficult to check for all the possible functions $t()$ different from $s()$

- Help from the revelation principle
The Revelation Principle

- All the strategies are equivalent to bidder $i$ supplying to $s()$ a different value of $v_i$. 
1st price auction

- Expected payoff of player $i$ if all the players play $s()$:
  - $U_i(s(v_1),...,s(v_i),...,s(v_N)) = v_i^{N-1} (v_i-s(v_i))$

- What if $i$ plays a different strategy $t()$?

- By the revelation principle:
  - $U_i(s(v_1),...,t(v_i),...,s(v_N)) = \text{eq } U_i(s(v_1),...,s(v),...,s(v_N)) = v^{N-1} (v_i-s(v))$

- If $v_i^{N-1} (v_i-s(v_i)) \geq v^{N-1} (v_i-s(v))$ for each $v$ (and for each $v_i$)
  - Then all players playing $s()$ is a NE
1st price auction

- If $v_i^{N-1} (v_i-s(v_i)) \geq v^{N-1} (v_i-s(v))$ for each $v$ (and for each $v_i$)
  - Then all players playing $s()$ is a NE

- $f(v)=v_i^{N-1} (v_i-s(v_i)) - v^{N-1} (v_i-s(v))$ is minimized for $v=v_i$

- $f'(v)=0$ for $v=v_i$,
  - i.e. $(N-1) v_i^{N-2} (v_i-s(v_i)) + v_i^{N-1} s'(v_i) = 0$ for each $v_i$
  - $s'(v_i) = (N-1)(1 - s(v_i)/v_i)$, $s(0)=0$
  - Solution: $s(v_i)=(N-1)/N v_i$
1\textsuperscript{st} price auction

- All players bidding according to 
  \( s(v) = \frac{(N-1)}{N} v \) is a NE

- Remarks
  - They are not truthful
  - The more they are, the higher they should bid

- Expected seller revenue
  - \( \left( \frac{(N-1)}{N} \right) \mathbb{E}[v_{\text{max}}] = \left( \frac{(N-1)}{N} \right) \frac{N}{(N+1)} = \frac{(N-1)}{(N+1)} \)
  - Identical to 2\textsuperscript{nd} price auction!
  - A general revenue equivalence principle
Outline

- Preliminaries
  - Auctions
  - Matching markets
- Possible approaches to ads pricing
- Google mechanism

References

- Easley, Kleinberg, "Networks, Crowds and Markets", ch.9,10,15
How to match a set of different goods to a set of buyers with different evaluations.

$v_{ij}$: value that buyer $j$ gives to good $i$.
Matching Markets

How to match a set of different goods to a set of buyers with different evaluations

\[ \text{maximize} \quad \sum_{i,j=1}^{N} x_{ij} v_{ij} \]

subject to \[ \sum_{j=1}^{N} x_{ij} = 1, \quad \sum_{i=1}^{N} x_{ij} = 1, \]

over \[ x_{ij} \in \{0,1\} \]

\( v_{ij} \): value that buyer \( j \) gives to good \( i \)
Matching Markets

Which goods buyers like most? Preferred seller graph

How to match a set of different goods to a set of buyers with different evaluations
Matching Markets

Which goods buyers like most? Preferred seller graph

- Given the prices, look for a perfect matching on the preferred seller graph
- There is no such matching for this graph
Which goods buyers like most? Preferred seller graph

- But with different prices, there is
Which goods buyers like most? Preferred seller graph

- But with different prices, there is
- Such prices are market clearing prices
Market Clearing Prices

- They always exist
  - And can be easily calculated if valuations are known

- They are socially optimal in the sense that
  - they achieve the maximum total valuation of any assignment of sellers to buyers
  - Or, equivalently, they maximize the sum of all the payoffs in the network (both sellers and buyers)
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Ads pricing

How to rank ads from different companies

$r_i$: click rate for an ad in position $i$
(assumed to be independent from the ad and known a priori)

$v_i$: value that company $i$ gives to a click

1. $r_1$
2. $r_2$
3. $r_3$

1. $v_1$
2. $v_2$
3. $v_3$
Ads pricing as a matching market

Ads positions

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>r₁</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>r₂</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>r₃</td>
<td></td>
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</tbody>
</table>

companies

<p>| | | |</p>
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<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>v₁r₁, v₁r₂, v₁r₃</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>v₂r₁, v₂r₂, v₂r₃</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>v₃r₁, v₃r₂, v₃r₃</td>
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</tbody>
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rᵢ: click rate for an ad in position i (assumed to be independent from the ad and known a priori)

vᵢ: value that company i gives to a click

Problem: Valuations are not known!

... but we could look for something as 2nd price auctions
The VCG mechanism

- The correct way to generalize 2nd price auctions to multiple goods
- Vickrey-Clarke-Groves
- Every buyers should pay a price equal to the social value loss for the others buyers
  - Example: consider a 2nd price auction with $v_1 > v_2 > ... v_N$
    - With 1 present the others buyers get 0
    - Without 1, 2 would have got the good with a value $v_2$
    - then the social value loss for the others is $v_2$
The VCG mechanism

- The correct way to generalize 2nd price auctions to multiple goods
- Vickrey-Clarke-Groves
- Every buyers should pay a price equal to the social value loss for the others buyers
  - If $V_B^S$ is the maximum total valuation over all the possible perfect matchings of the set of sellers $S$ and the set of buyers $B$,
  - If buyer $j$ gets good $i$, he/she should be charged $V_{B-j}^S - V_{B-j}^{S-i}$
VCG example

<table>
<thead>
<tr>
<th>Ads positions</th>
<th>companies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
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<tr>
<td>2</td>
<td>2</td>
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$r_i$: click rate for an ad in position $i$ (assumed to be independent from the ad and known a priori)

$v_i$: value that company $i$ gives to a click
VCG example

Ads positions

1
2
3

companies

1 30, 15, 6
2 20, 10, 4
3 10, 5, 2
VCG example

This is the maximum weight matching

- 1 gets 30, 2 gets 10 and 3 gets 2
VCG example

If 1 weren’t there, 2 and 3 would get 25 instead of 12,

Then 1 should pay 13
If 2 weren’t there, 1 and 3 would get 35 instead of 32,
Then 2 should pay 3
If 3 weren’t there, nothing would change for 1 and 2,
Then 3 should pay 0
The VCG mechanism

- Every buyers should pay a price equal to the social value loss for the others buyers
  - If $V_B^S$ is the maximum total valuation over all the possible perfect matchings of the set of sellers $S$ and the set of buyers $B$,
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- Under this price mechanism, truth-telling is a dominant strategy