Distributed Optimization and Games

Introduction to Game Theory

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Two-person Zero-sum Games

- One of the first games studied
    - most well understood type of game
- Players interest are strictly opposed
    - what one player gains the other loses
    - game matrix has single entry (gain to player 1)
- A "strong" solution concept

Dominance

- Strategy S *(weakly) dominates* a strategy T if every possible outcome when S is chosen is at least as good as corresponding outcome in T, and one is strictly better
  - S *strictly dominates* T if every possible outcome when S is chosen is strictly better than corresponding outcome in T
- Dominance Principle
  - rational players never choose dominated strategies
- Higher Order Dominance Principle
  - iteratively remove dominated strategies

Higher order dominance may be enough

		Colin	
		S	H
Rose	S	15, 15	13, 16
	H	16, 13	14, 14

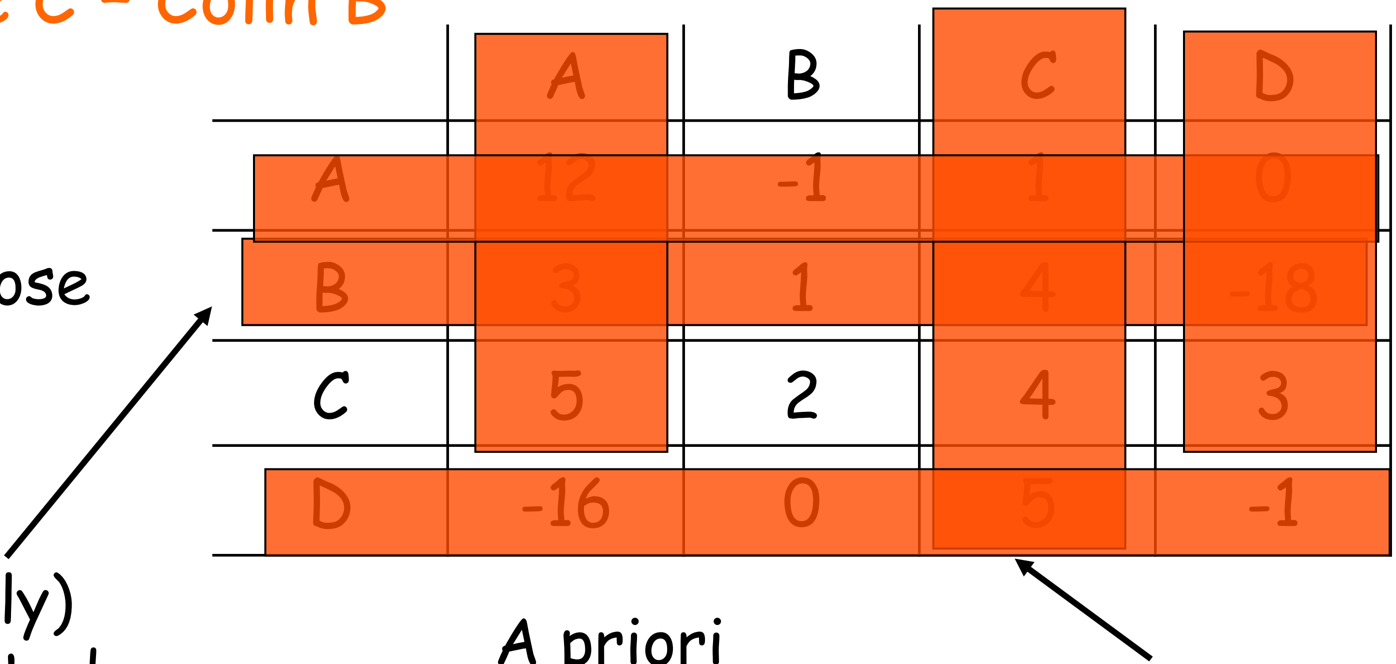
Rose's
S strategy
dominated
By H

GT prescribes:
Rose H - Colin H

Higher order dominance may be enough

GT prescribes:

Rose C – Colin B



Colin

Rose	A	B	C	D
A	12	-1	1	0
B	3	1	4	-18
C	5	2	4	3
D	-16	0	5	-1

(Weakly) Dominated by C

A priori D is **not** dominated by C

Strictly dominated by B

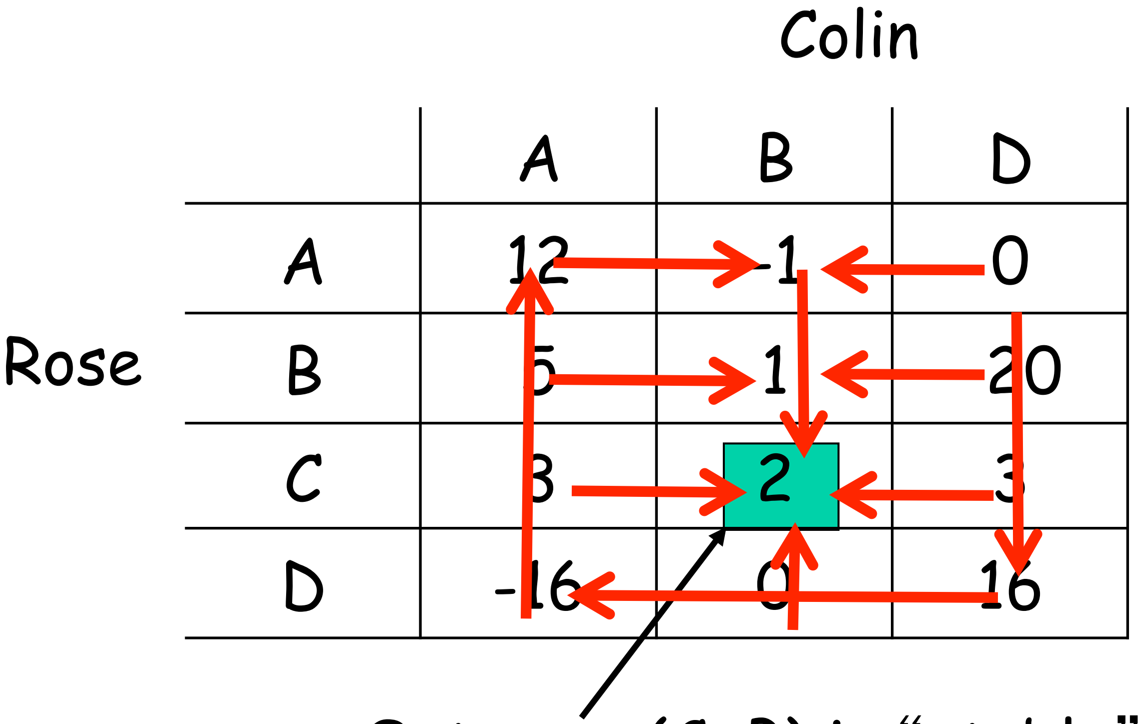
… but not in general

Colin

		A	B	C	D
Rose	A	12	-1	1	0
	B	5	1	7	-20
	C	3	2	4	3
	D	-16	0	0	16

dominated
strategy
(dominated by B)

Analyzing the Reduced Game: Movement Diagram



Colin

Rose

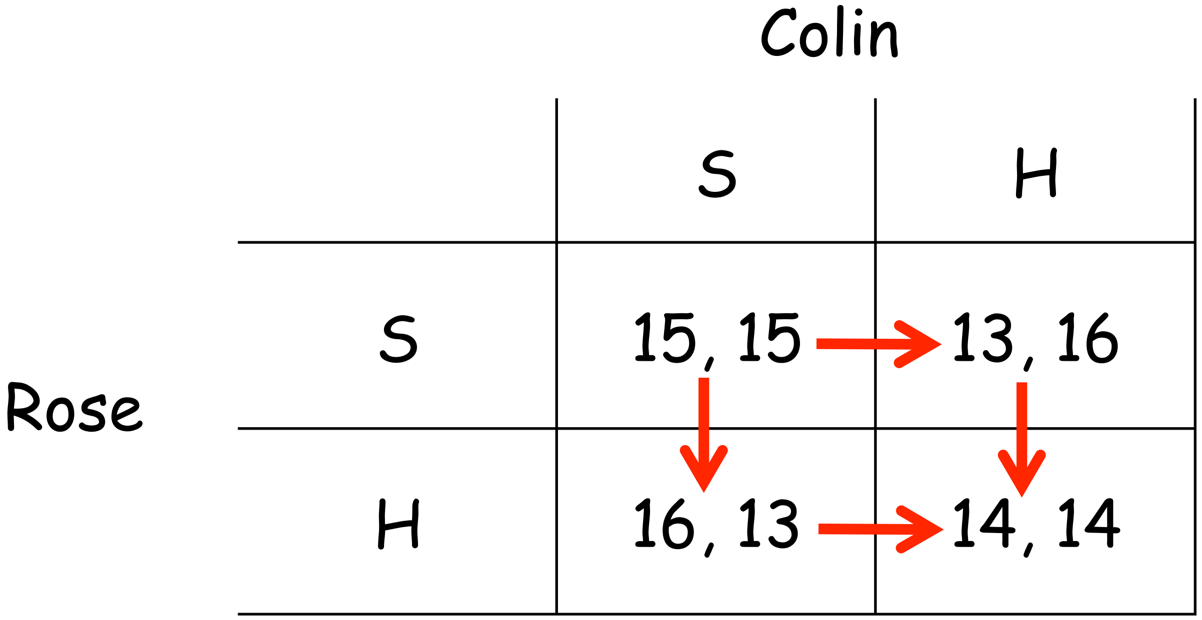
	A	B	D
A	12	-1	0
B	5	1	20
C	3	2	3
D	-16	0	16

If Rose plays D, A is Colin's **best response**

Outcome (C, B) is "stable"

- ***Pure strategy Nash Equilibrium***
- mutual best responses

Students' game



		Colin	
		S	H
Rose	S	15, 15	13, 16
	H	16, 13	14, 14

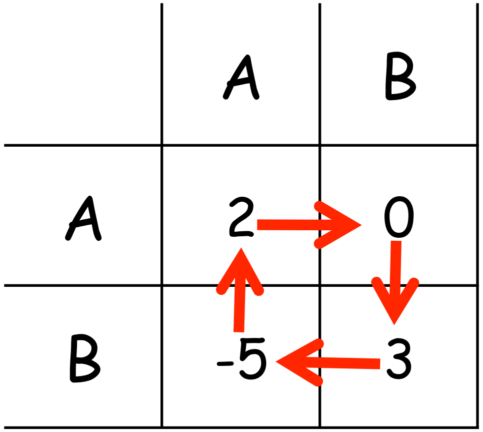
Games without pure strategy NE

- An example?

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Games without pure strategy NE

- An example? An even simpler one



	A	B
A	2	0
B	-5	3

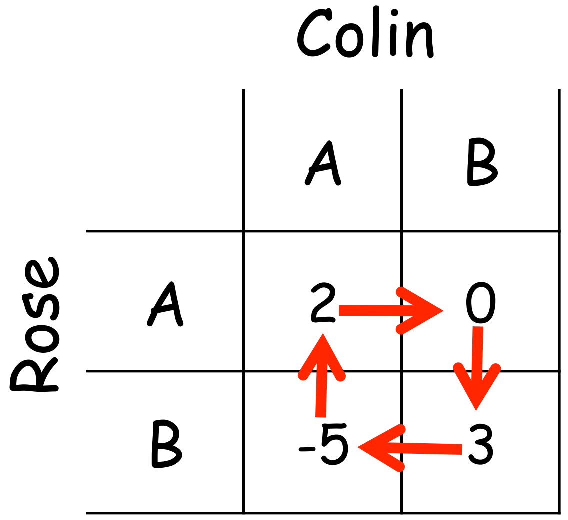
Some practice: find all the pure strategy NE

	A	B	C	D
A	3	2	4	2
B	2	1	3	0
C	2	2	2	2

	A	B	C
A	-2	0	4
B	2	1	3
C	3	-1	-2

	A	B	C
A	4	3	8
B	9	5	1
C	2	7	6

Games with no pure strategy NE

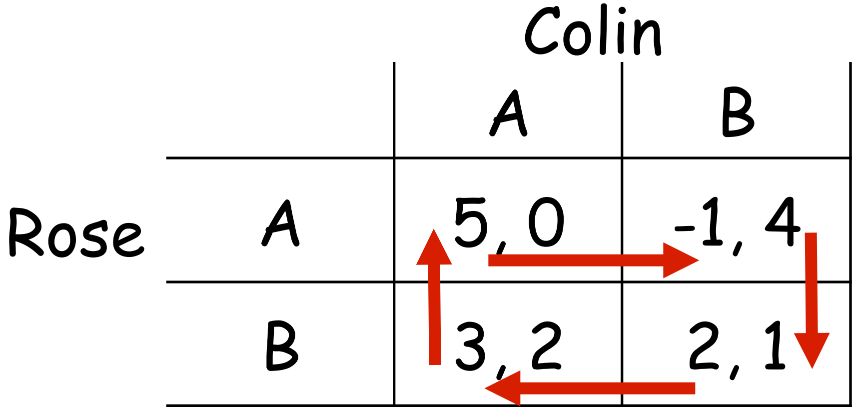


Colin

		A	B
Rose	A	2	0
	B	-5	3

- What should players do?
  - resort to randomness to select strategies

Games with no pure strategy NE



		Colin	
		A	B
Rose	A	5, 0	-1, 4
	B	3, 2	2, 1

- …but we can find mixed strategies equilibria

Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy (*equalizing strategy*), that equalizes the opponent payoffs
  - how to calculate it?

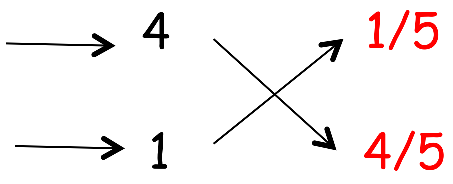
		Colin	
		A	B
Rose	A	5, 0	-1, 4
	B	3, 2	2, 1

Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy, that equalizes the opponent payoffs
  - how to calculate it?

		Colin	
		A	B
Rose	A	-0	-4
	B	-2	-1

Rose considers *Colin's game*



Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy, that equalizes the opponent payoffs
  - how to calculate it?

Colin considers *Rose's game*

		Colin	
		A	B
Rose	A	5	-1
	B	3	2
		3/5	2/5

Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy, that equalizes the opponent payoffs
  - how to calculate it?

		Colin	
		A	B
Rose	A	5, 0	-1, 4
	B	3, 2	2, 1

Rose playing (1/5,4/5)
Colin playing (3/5,2/5)
is an equilibrium

Rose gains 13/5
Colin gains 8/5

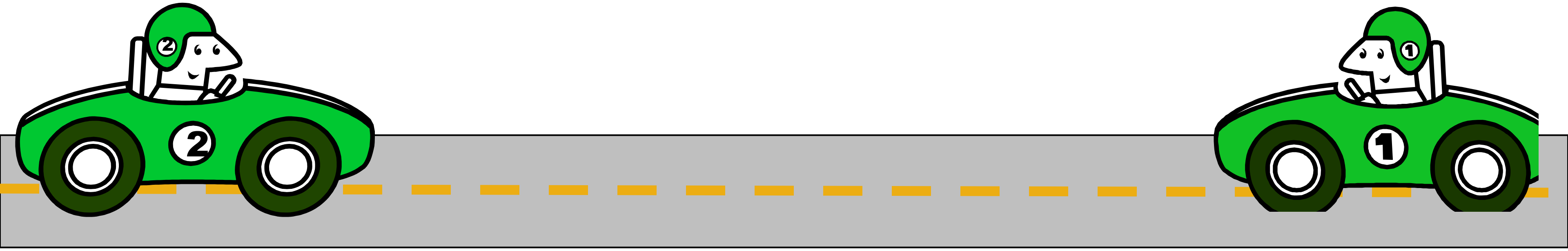
Good news:
Nash's theorem [1950]

- Every two-person games has at least one equilibrium either in pure strategies or in mixed strategies
  - Proved using fixed point theorem
  - generalized to N person game
- This equilibrium concept called Nash equilibrium in his honor
  - A vector of strategies (a profile) is a Nash Equilibrium (NE) if no player can unilaterally change its strategy and increase its payoff

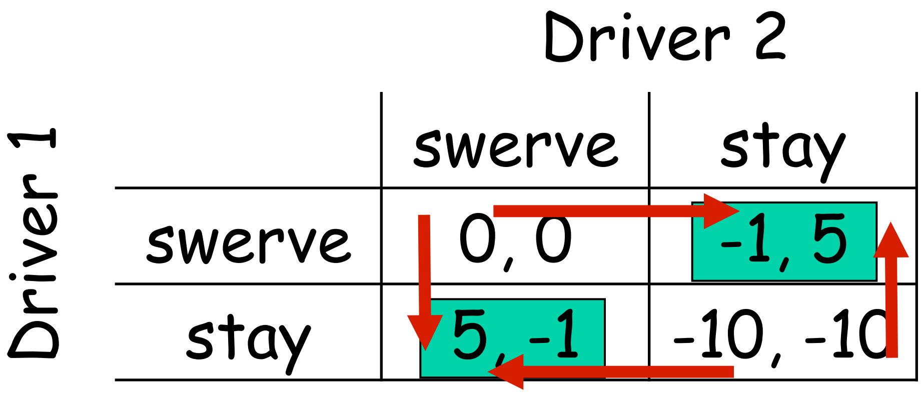
A useful property

- Given a finite game, a profile is a mixed NE of the game if and only if for every player i, every pure strategy used by i with non-null probability is a best response to other players mixed strategies in the profile
  - see Osborne and Rubinstein, *A course in game theory*, Lemma 33.2

Game of Chicken



- Game of Chicken (aka. Hawk-Dove Game)
  - driver who swerves looses



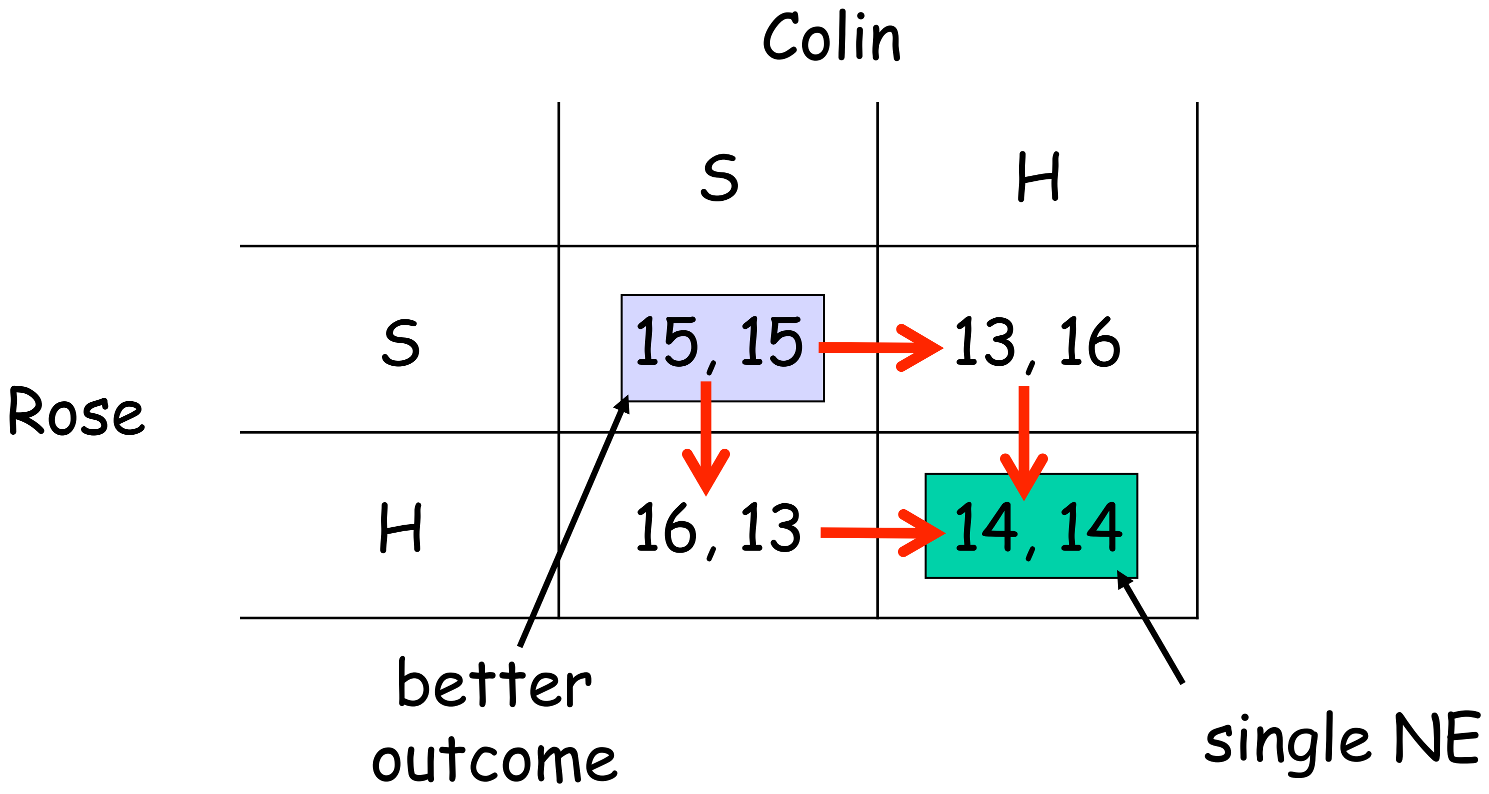
		Driver 2	
		swerve	stay
Driver 1	swerve	0, 0	-1, 5
	stay	5, -1	-10, -10

Drivers want to do opposite of one another

Two equilibria:
not equivalent
not interchangeable!

- playing an equilibrium strategy does not lead to equilibrium

Students' game

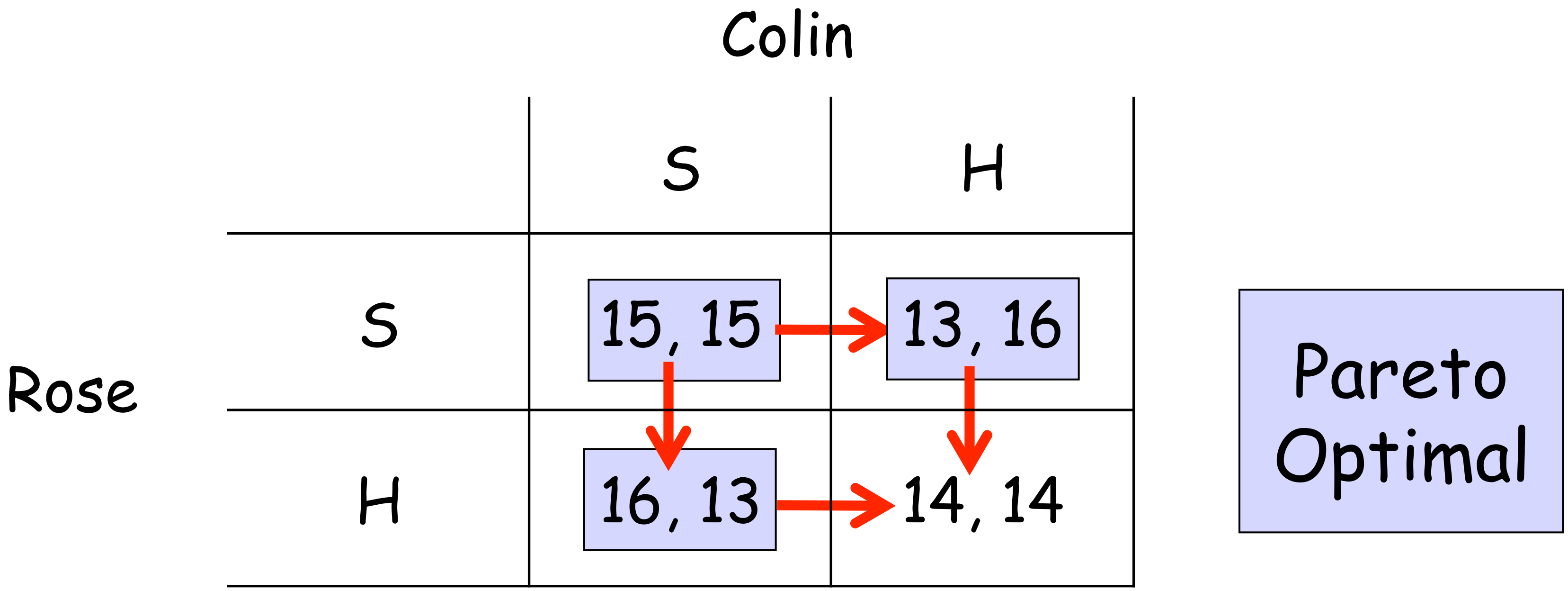


		Colin	
		S	H
Rose	S	15, 15	13, 16
	H	16, 13	14, 14

better outcome

single NE

Students' game



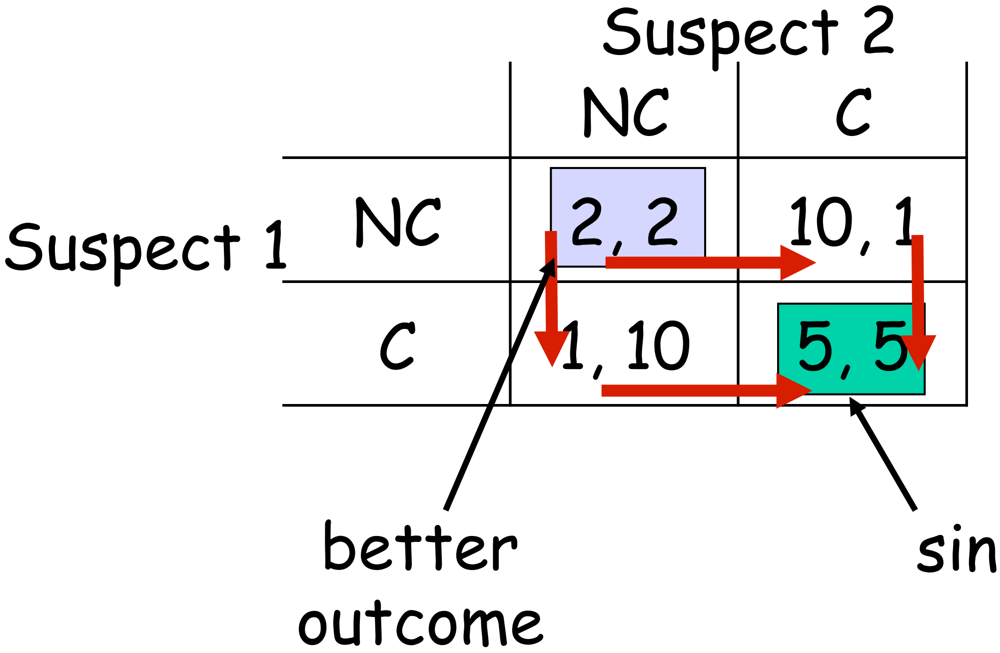
		Colin	
		S	H
Rose	S	15, 15	13, 16
	H	16, 13	14, 14

Pareto Optimal

- Def: outcome o* is Pareto Optimal if no other outcome would give to all the players a payoff not smaller and a payoff higher to at least one of them
- Conflict between group rationality (Pareto principle) and individual rationality (dominance principle)

Students' game = Prisoner's Dilemma

- One of the most studied and used games
  - proposed in 1950
- Two suspects arrested for joint crime
  - each suspect when interrogated separately, has option to confess



		Suspect 2	
		NC	C
Suspect 1	NC	2, 2	10, 1
	C	1, 10	5, 5

better outcome

single NE

payoff is years in jail (**smaller is better**)

Distributed Optimization and Games

Auctions

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Our starting problem

- We want to give an object to the person who values it the most, i.e.

maximize $$\sum_{i=1}^{N} x_i v_i$$

subject to $$\sum_{i=1}^{N} x_i = 1$$

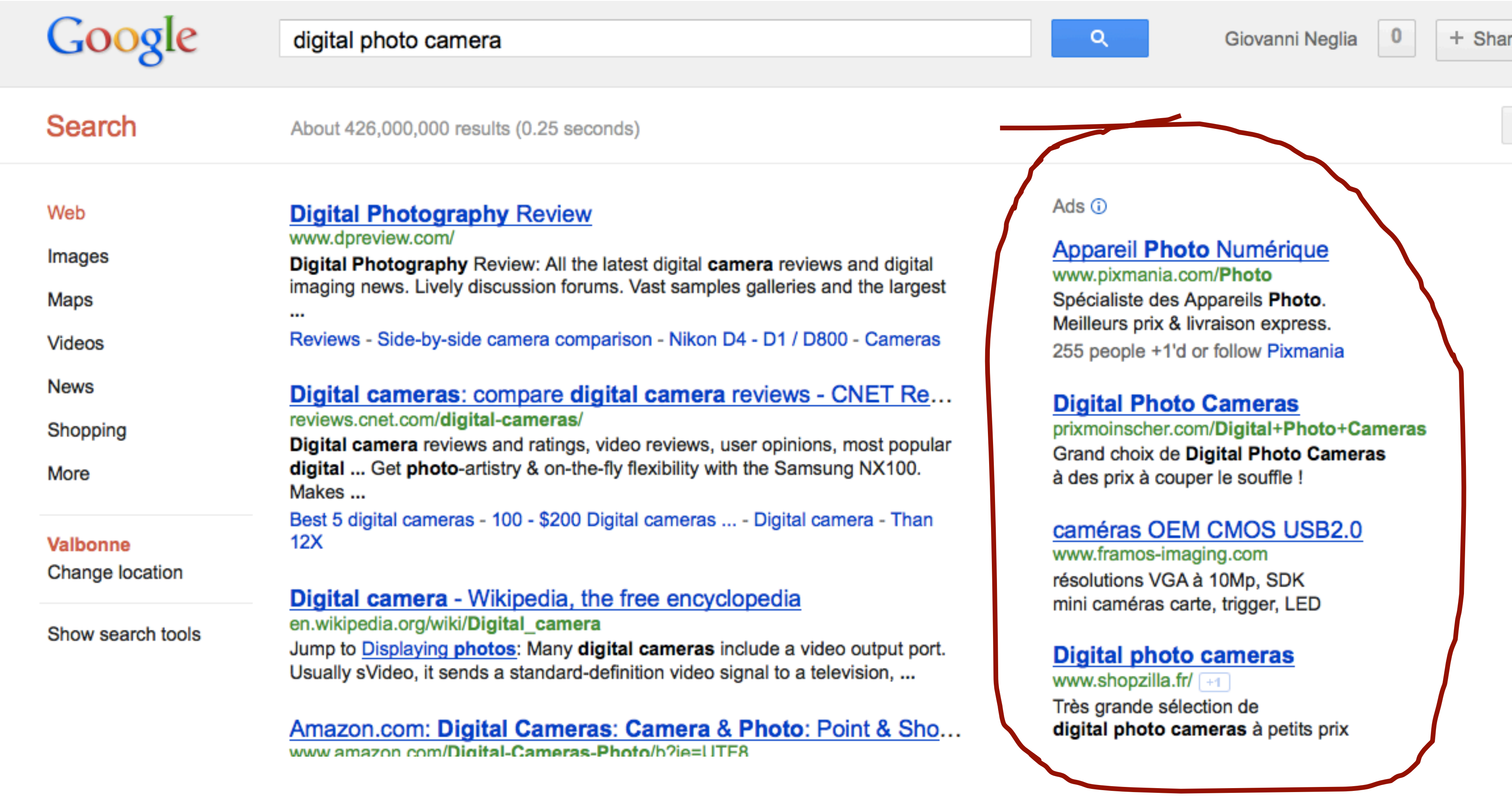
over $$x_i \in \{0,1\}$$

- Difficulty: we do not know values v_i ...
- and we cannot ask to people (they would lie)
- Solution: auctions, but we need to introduce money

Types of auctions

- 1st price & descending bids (Dutch auctions)
- 2nd price & ascending bids (English auctions)

Google



digital photo camera

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How it works

- Companies bid for keywords
- On the basis of the bids Google puts their link on a given position (first ads get more clicks)
- Companies are charged a given cost for each click (the cost depends on all the bids)
- Why Google adopted this solution:
    - It has no idea about the value of a click...
    - It lets the company reveal it

Some numbers (2014)

- ≈ 90% of Google revenues (66 billions$) from ads
  - investor.google.com/financial/tables.html
- Costs
  - "calligraphy pens" $1.70
  - "Loan consolidation" $50
  - "mesothelioma" $50 per click
- Click fraud problem

Outline

- Preliminaries
  - Auctions
  - Matching markets
- Possible approaches to ads pricing
- Google mechanism

- References
  - Easley, Kleinberg, "Networks, Crowds and Markets", ch.9,10,15

Game Theoretic Model

- N players (the bidders)
- Strategies/actions: b_i is player i's bid
- For player i the good has value v_i
- p_i is player i's payment if he gets the good
- Utility:
  - v_i-p_i if player i gets the good
  - 0 otherwise
- Assumption here: values v_i are *independent* and *private*
  - i.e. very particular goods for which there is not a reference price

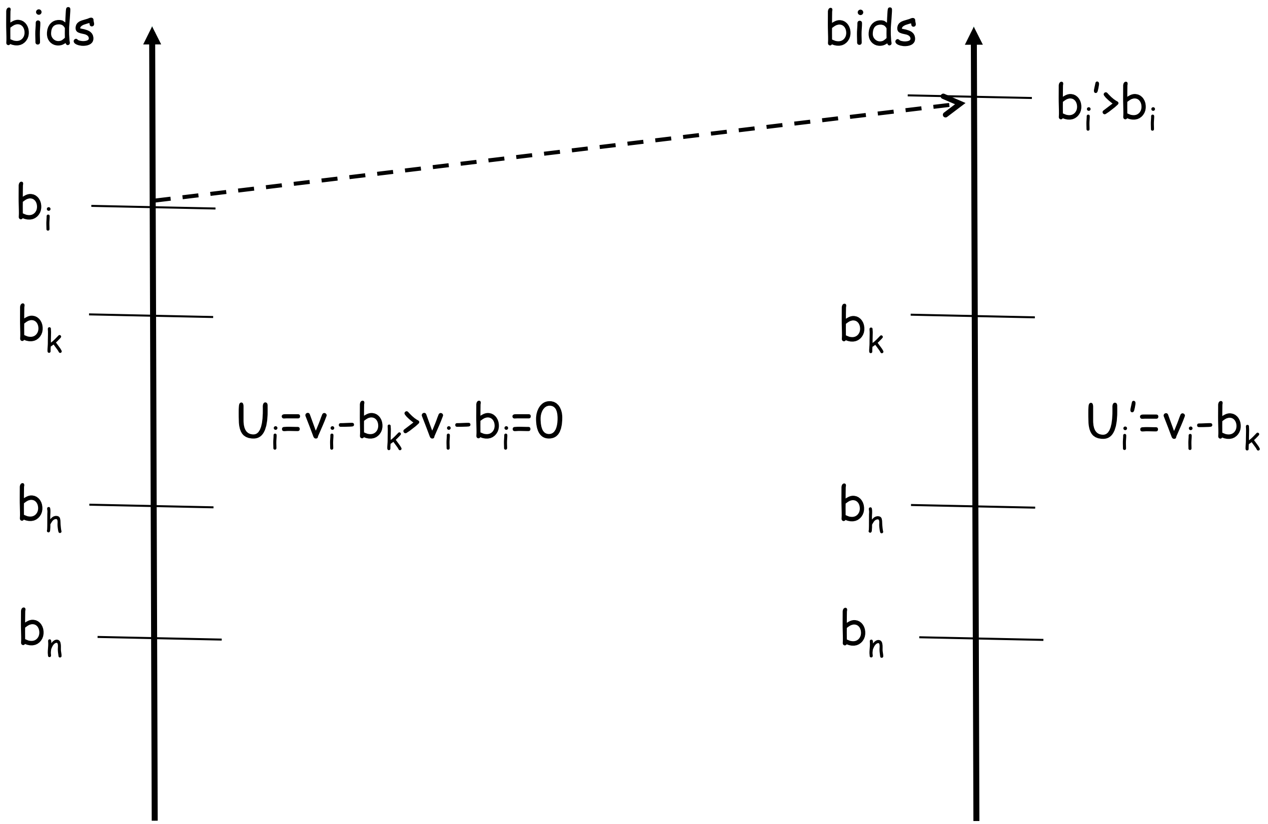
Game Theoretic Model

- N players (the bidders)
- Strategies: b_i is player i's bid
- Utility:
  - v_i-b_i if player i gets the good
  - 0 otherwise
- Difficulties:
  - Utilities of other players are unknown!
  - Better to model the strategy space as continuous (differently from the games we looked at)

2nd price auction

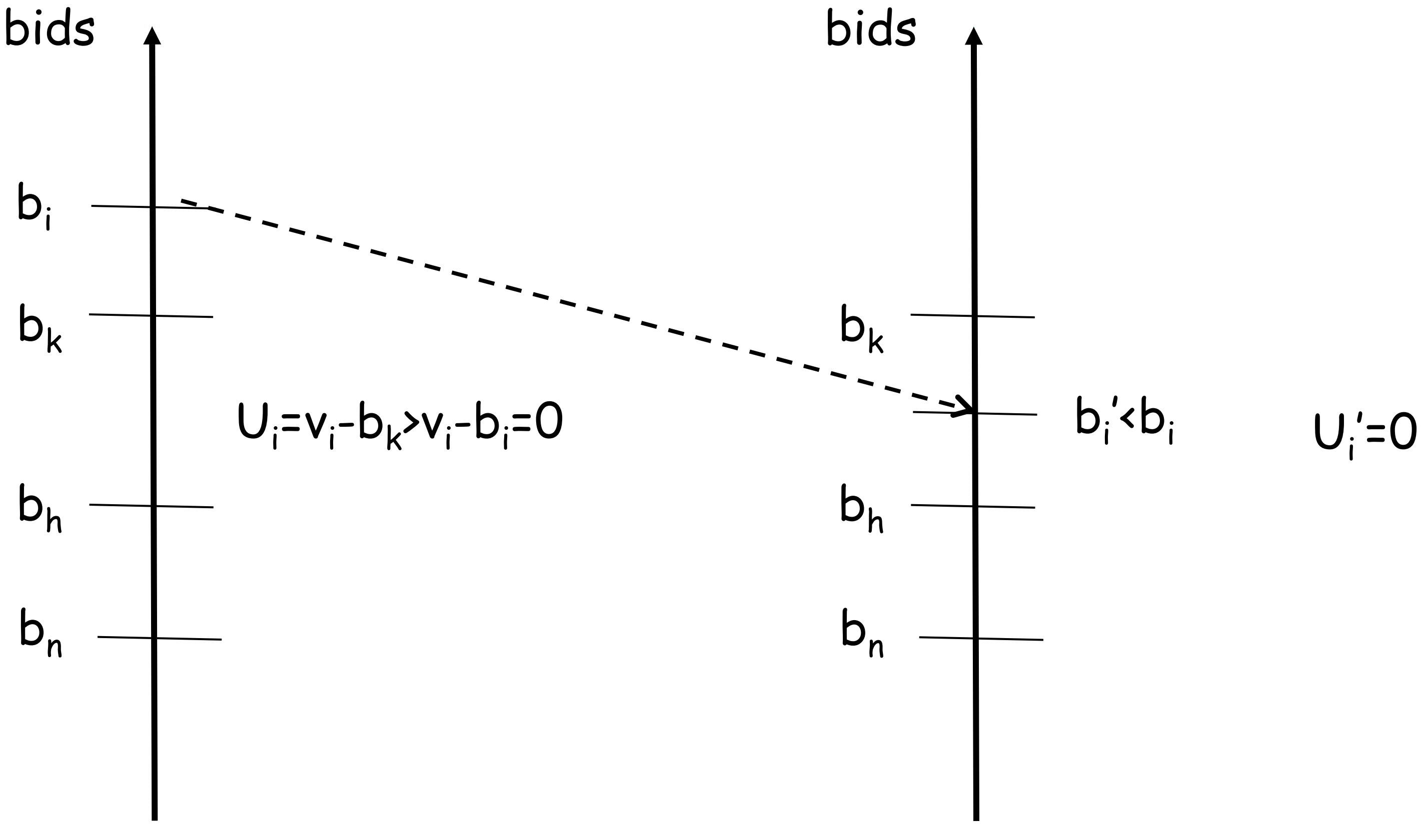
- Player with the highest bid gets the good and pays a price equal to the 2nd highest bid
- There is a dominant strategies
  - I.e. a strategy that is more convenient independently from what the other players do
  - **Be truthful**, i.e. bid how much you evaluate the good ($b_i = v_i$)
  - Social optimality: the bidder who value the good the most gets it!

$b_i=v_i$ is the highest bid



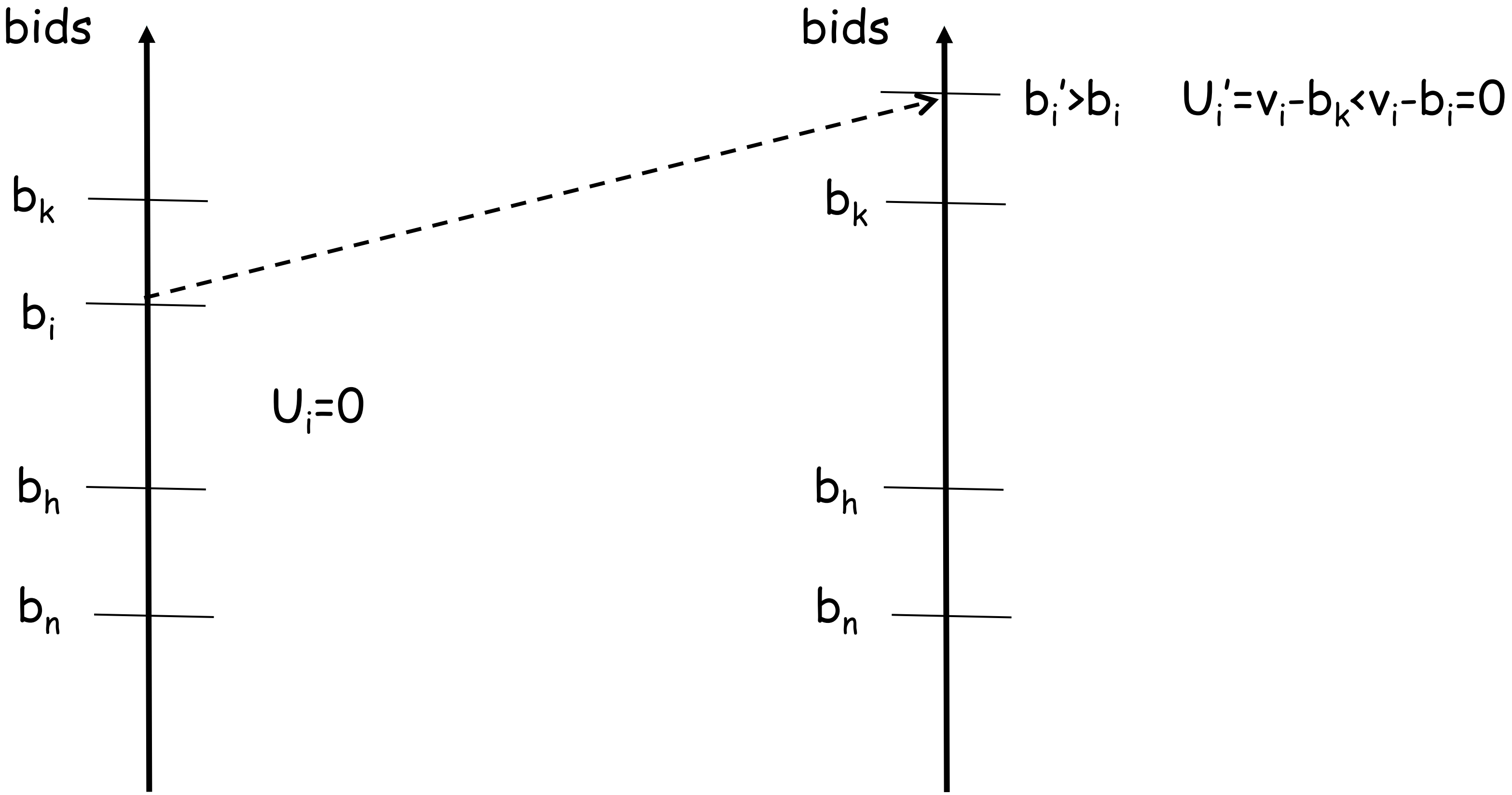
Bidding more than v_i is not convenient

$b_i=v_i$ is the highest bid



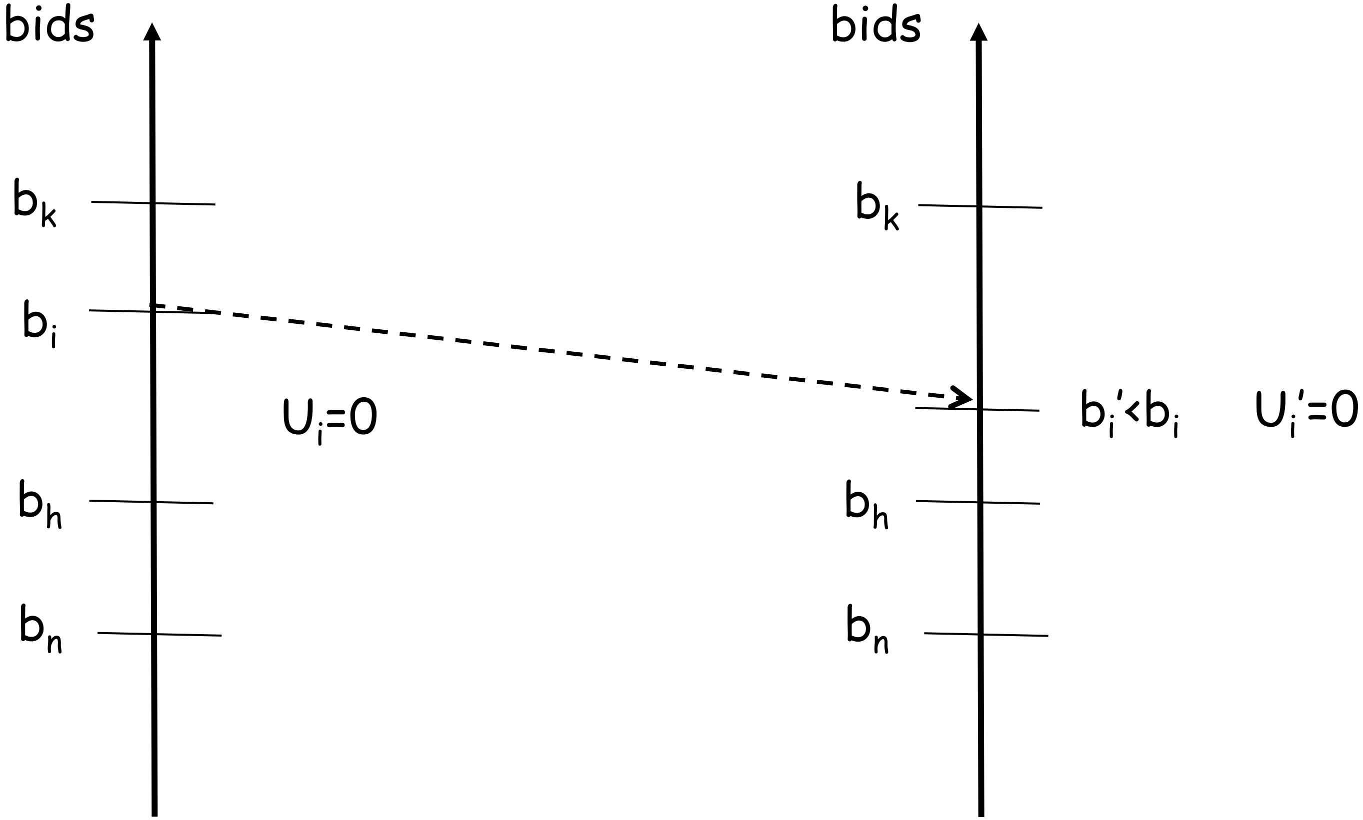
Bidding less than v_i is not convenient (may be unconvenient)

$b_i=v_i$ is not the highest bid



Bidding more than v_i is not convenient (may be unconvenient)

$b_i=v_i$ is not the highest bid



Bidding less than v_i is not convenient

Seller revenue

- N bidders
- Values are independent random values between 0 and 1
- Expected i^{th} largest utility is (N+1-i)/(N+1)
- Expected seller revenue is (N-1)/(N+1)

1^{st} price auction

- Player with the highest bid gets the good and pays a price equal to her/his bid
- Being truthful is not a dominant strategy anymore!
  - Consider for example if I knew other players' utilities
- How to study it?

1^{st} price auction

- Assumption: for each player the other values are i.i.d. random variables between 0 and 1
  - to overcome the fact that utilities are unknown
- Player i's strategy is a function s() mapping value v_i to a bid b_i
  - s() strictly increasing, differentiable function
  - $0 \leq s(v) \leq v \rightarrow s(0)=0$
- We investigate if there is a strategy s() common to all the players that leads to a Nash equilibrium

1st price auction

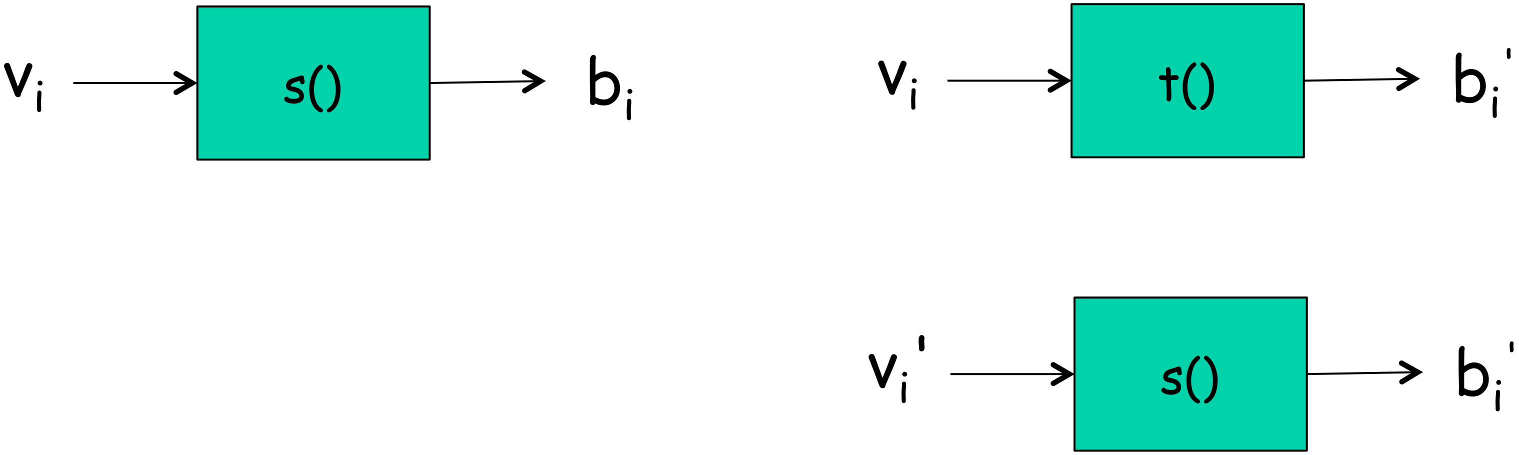
- Assumption: for each player the other values are i.i.d. random variables between 0 and 1
- Player i's strategy is a function s() mapping value v_i to a bid b_i
- Expected payoff of player i if all the players plays s():
  - $U_i(s(v_1),\ldots s(v_i),\ldots s(v_N)) = \underbrace{v_i^{N-1}}_{\text{prob. i wins}} \; \underbrace{(v_i - s(v_i))}_{\text{i's payoff if he/she wins}}$

1st price auction

- Expected payoff of player i if all the players play s():
  - $U_i(s(v_1),\ldots s(v_i),\ldots s(v_N)) = v_i^{N-1} (v_i - s(v_i))$
- What if i plays a different strategy t()?
  - If all players playing s() is a NE, then :
  - $U_i(s(v_1),\ldots s(v_i),\ldots s(v_N)) = v_i^{N-1} (v_i - s(v_i))$

    $\geq v_i^{N-1} (v_i - t(v_i)) = U_i(s(v_1),\ldots t(v_i),\ldots s(v_N))$
- Difficult to check for all the possible functions t() different from s()
- Help from the **revelation principle**

The Revelation Principle



- All the strategies are equivalent to bidder i supplying to s() a different value of v_i

1^{st} price auction

- Expected payoff of player i if all the players plays s():
  - $U_i(s(v_1),\ldots s(v_i),\ldots s(v_N)) = v_i^{N-1} (v_i - s(v_i))$
- What if i plays a different strategy t()?
- By the revelation principle:
  - $U_i(s(v_1),\ldots t(v_i),\ldots s(v_N)) =_{eq} U_i(s(v_1),\ldots s(v),\ldots s(v_N)) = v^{N-1} (v_i - s(v))$
- If $v_i^{N-1} (v_i - s(v_i)) \geq v^{N-1} (v_i - s(v))$ for each v (and for each v_i)
  - Then all players playing s() is a NE

1^{st} price auction

- If $v_i^{N-1}(v_i-s(v_i)) \geq v^{N-1}(v_i-s(v))$ for each v (and for each v_i)
  - Then all players playing s() is a NE
- $f(v)=v_i^{N-1}(v_i-s(v_i)) - v^{N-1}(v_i-s(v))$ is minimized for $v=v_i$
- $f'(v)=0$ for $v=v_i$,
  - i.e. $(N-1)\, v_i^{N-2}(v_i-s(v_i)) + v_i^{N-1} s'(v_i) = 0$ for each v_i
  - $s'(v_i) = (N-1)(1 - s(v_i)/v_i)$, $s(0)=0$
  - Solution: $s(v_i)=(N-1)/N\ v_i$

1^{st} price auction

- All players bidding according to
  $s(v) = (N-1)/N\ v$ is a NE
- Remarks
  - They are not truthful
  - The more they are, the higher they should bid
- Expected seller revenue
  - $((N-1)/N)\ E[v_{max}] = ((N-1)/N)\ (N/(N+1)) = (N-1)/(N+1)$
  - Identical to 2^{nd} price auction!
  - A general revenue equivalence principle

Outline

- Preliminaries
  - Auctions
  - Matching markets
- Possible approaches to ads pricing
- Google mechanism

- References
  - Easley, Kleinberg, "Networks, Crowds and Markets", ch.9,**10**,15

Matching Markets

v_{ij}: value that buyer j gives to good i

v_{11}, v_{21}, v_{31}

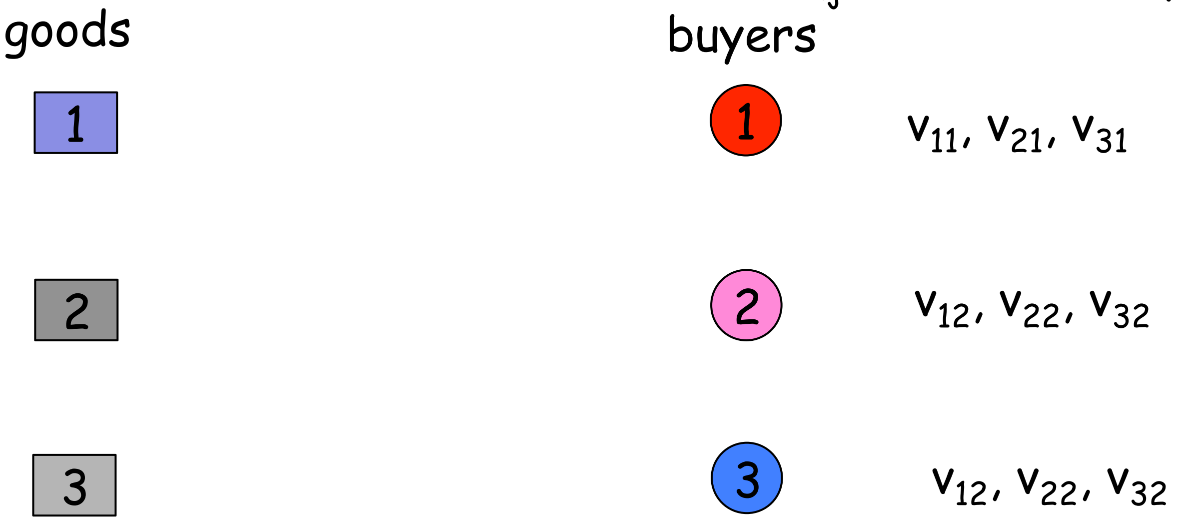
v_{12}, v_{22}, v_{32}

v_{12}, v_{22}, v_{32}

How to match a set of different goods to a set of buyers with different evaluations

Matching Markets

v_{ij}: value that buyer j gives to good i


goods

buyers

1

2

3

1 v_{11}, v_{21}, v_{31}

2 v_{12}, v_{22}, v_{32}

3 v_{12}, v_{22}, v_{32}

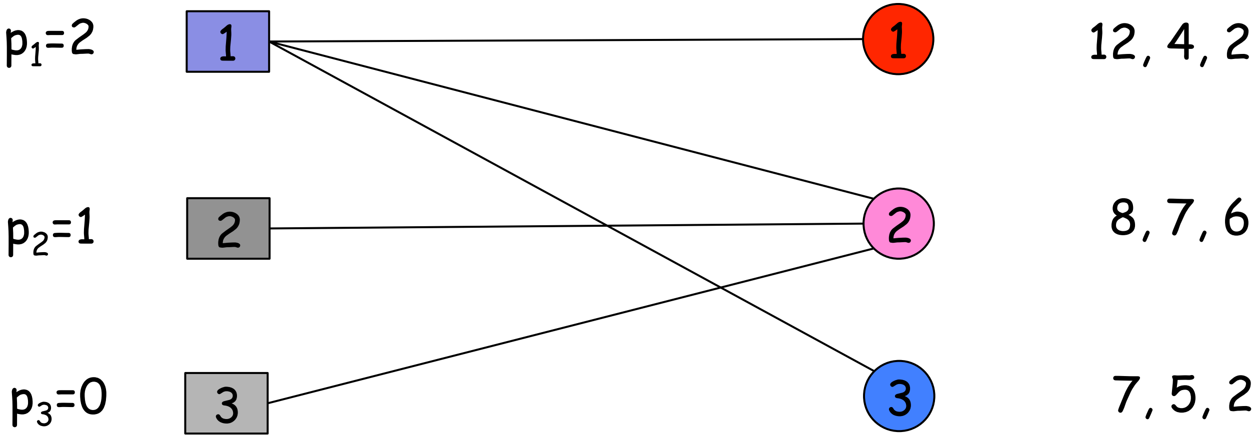
How to match a set of different goods to a set of buyers with different evaluations

maximize $$\sum_{i,j=1}^{N} x_{ij} v_{ij}$$

subject to $$\sum_{j=1}^{N} x_{ij} = 1, \quad \sum_{i=1}^{N} x_{ij} = 1,$$

over $$x_{ij} \in \{0,1\}$$

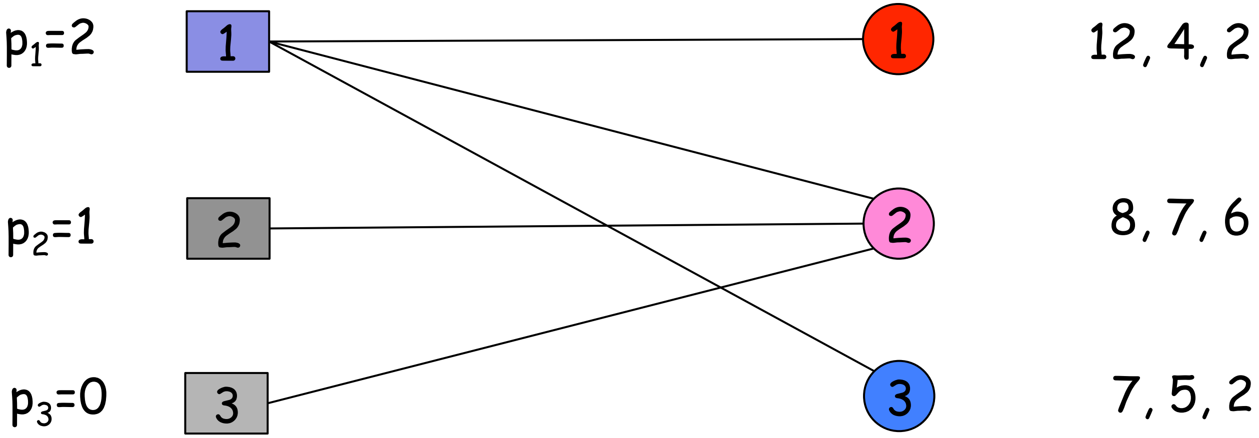
Matching Markets



Which goods buyers like most? Preferred seller graph

How to match a set of different goods to a set of buyers with different evaluations

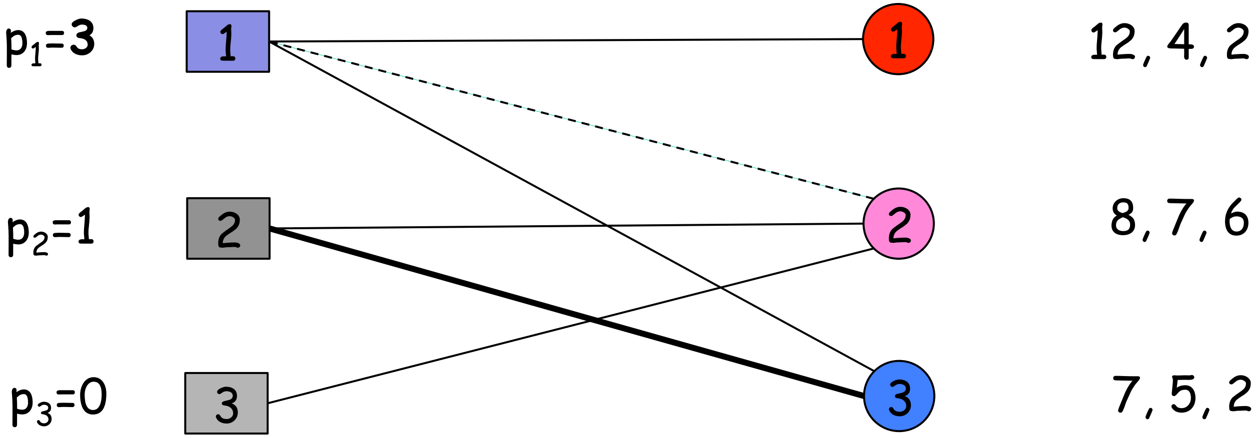
Matching Markets



Which goods buyers like most? Preferred seller graph

- Given the prices, look for a perfect matching on the preferred seller graph
- There is no such matching for this graph

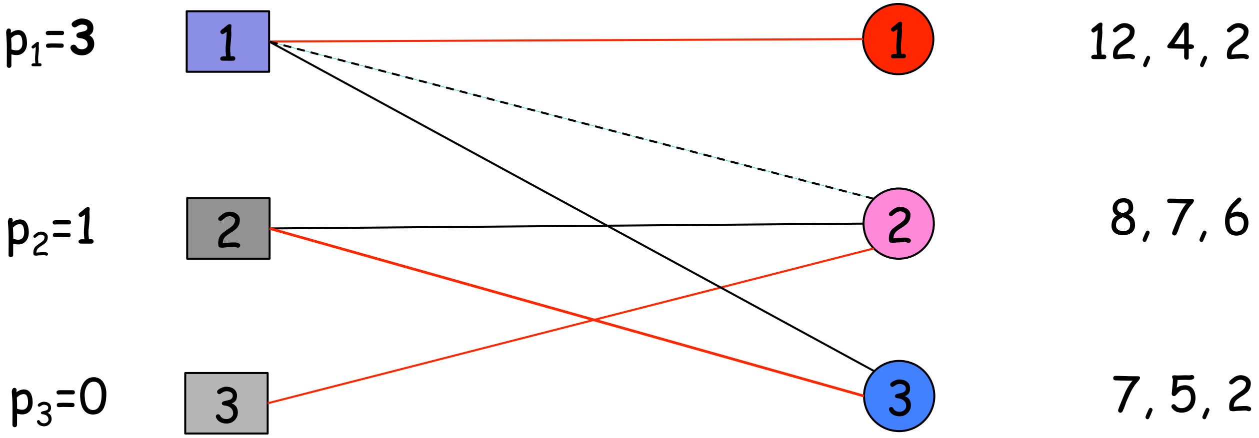
Matching Markets



Which goods buyers like most? Preferred seller graph

- But with different prices, there is

Matching Markets



Which goods buyers like most? Preferred seller graph

- But with different prices, there is
- Such prices are **market clearing prices**

Market Clearing Prices

- They always exist
  - And can be easily calculated if valuations are known
- They are socially optimal in the sense that
  - they achieve the maximum total valuation of any assignment of sellers to buyers
  - Or, equivalently, they maximize the sum of all the payoffs in the network (both sellers and buyers)

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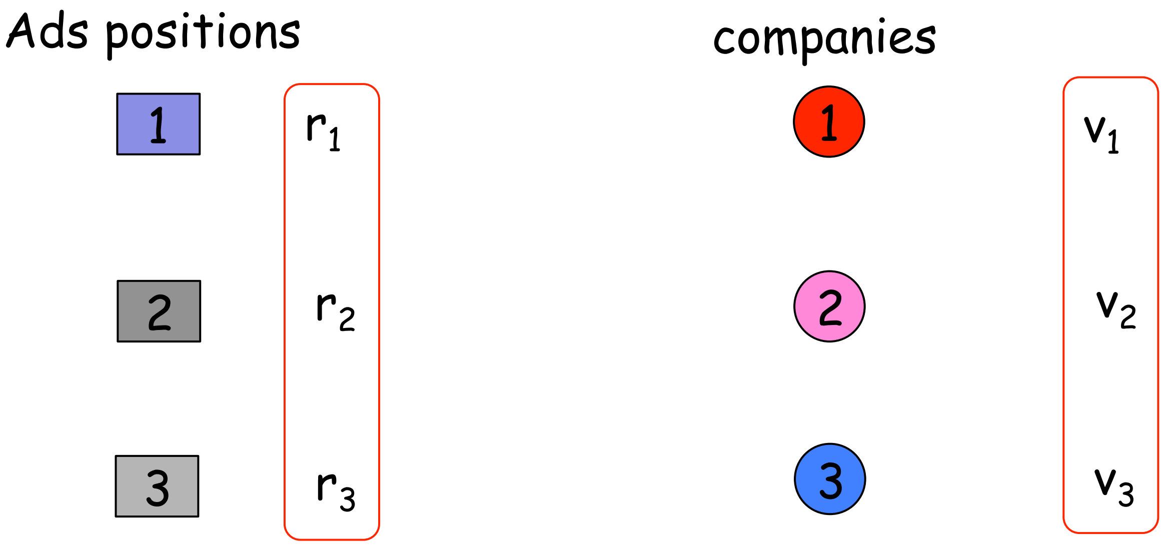
Ads pricing

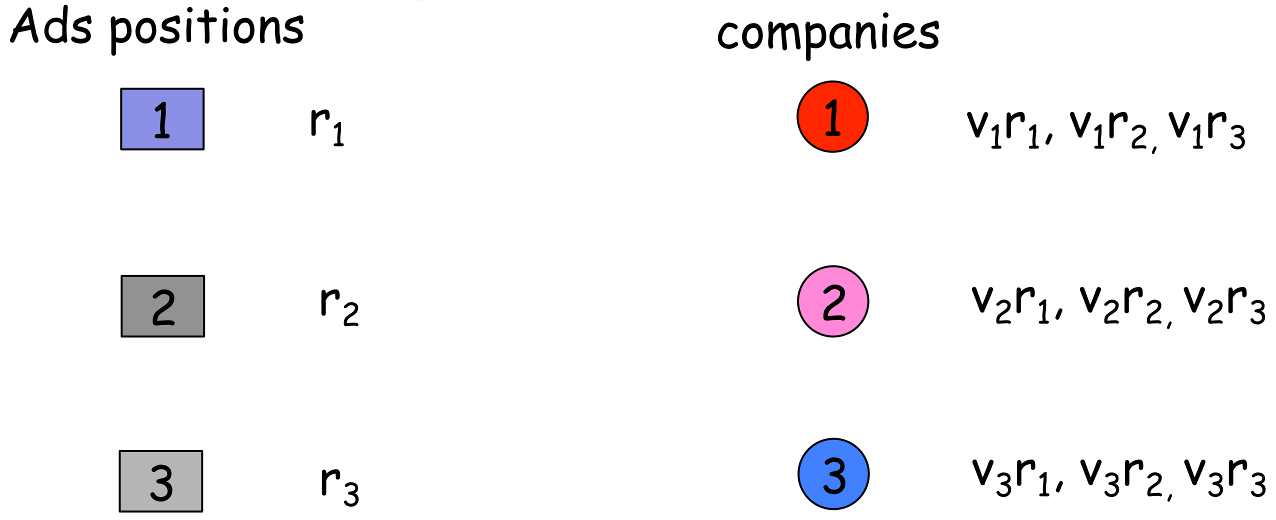


r_i: click rate for an ad in position i
(assumed to be independent
from the ad and known a priori)

v_i: value that company i
gives to a click

How to rank ads from different companies

Ads pricing as a matching market



Ads positions		companies	
1	r_1	1	v_1r_1, v_1r_2, v_1r_3
2	r_2	2	v_2r_1, v_2r_2, v_2r_3
3	r_3	3	v_3r_1, v_3r_2, v_3r_3

r_i: click rate for an ad in position i
(assumed to be independent
from the ad and known a priori)

v_i: value that company i
gives to a click

- Problem: Valuations are not known!
- … but we could look for something as 2nd price auctions

The VCG mechanism

- The correct way to generalize 2nd price auctions to multiple goods
- Vickrey-Clarke-Groves
- Every buyers should pay a price equal to the social value loss for the others buyers
  - Example: consider a 2nd price auction with $v_1 > v_2 > \ldots v_N$
    - With 1 present the others buyers get 0
    - Without 1, 2 would have got the good with a value v_2
    - then the social value loss for the others is v_2

The VCG mechanism

- The correct way to generalize 2nd price auctions to multiple goods
- Vickrey-Clarke-Groves
- Every buyers should pay a price equal to the social value loss for the others buyers
  - If V_B^S is the maximum total valuation over all the possible perfect matchings of the set of sellers S and the set of buyers B,
  - If buyer j gets good i, he/she should be charged $V_{B-j}^S - V_{B-j}^{S-i}$

VCG example

Ads positions

1 $r_1=10$

2 $r_2=5$

3 $r_3=2$

companies

1 $v_1=3$

2 $v_2=2$

3 $v_3=1$

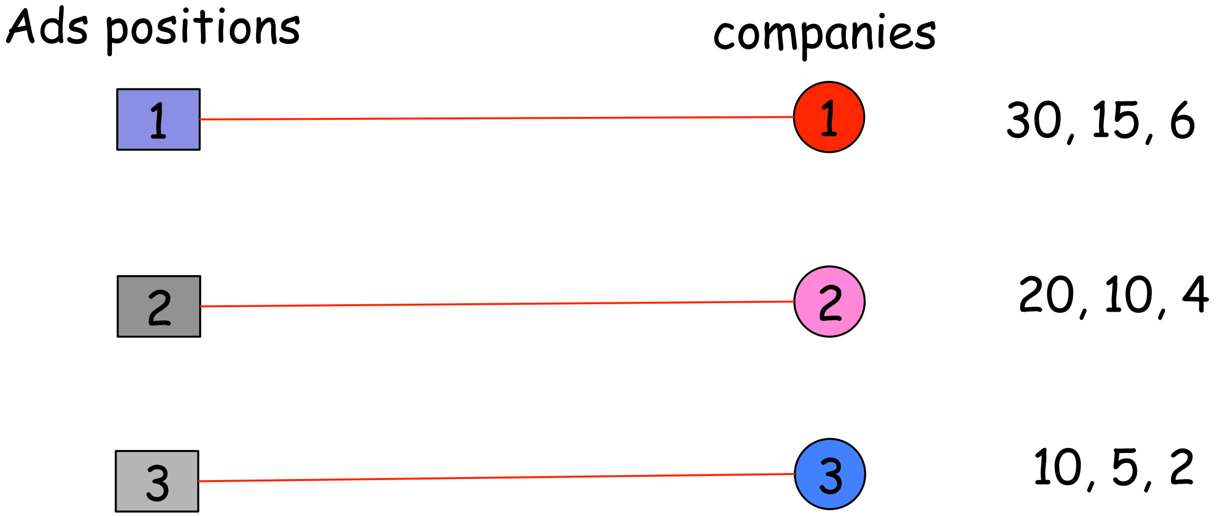
r_i: click rate for an ad in position i
(assumed to be independent
from the ad and known a priori)

v_i: value that company i
gives to a click

VCG example

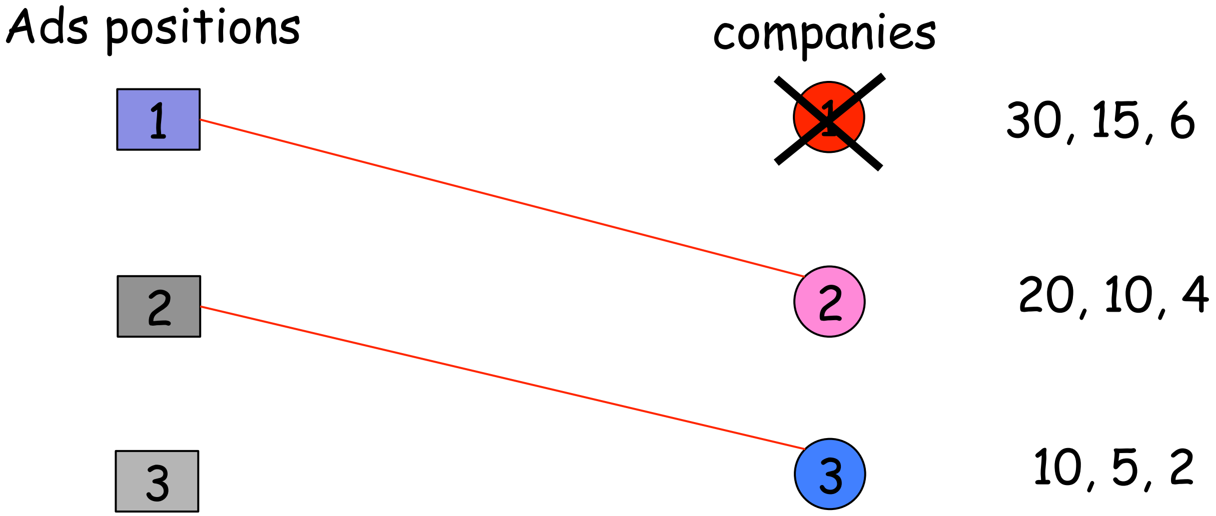
Ads positions	companies	
1	1	30, 15, 6
2	2	20, 10, 4
3	3	10, 5, 2

VCG example



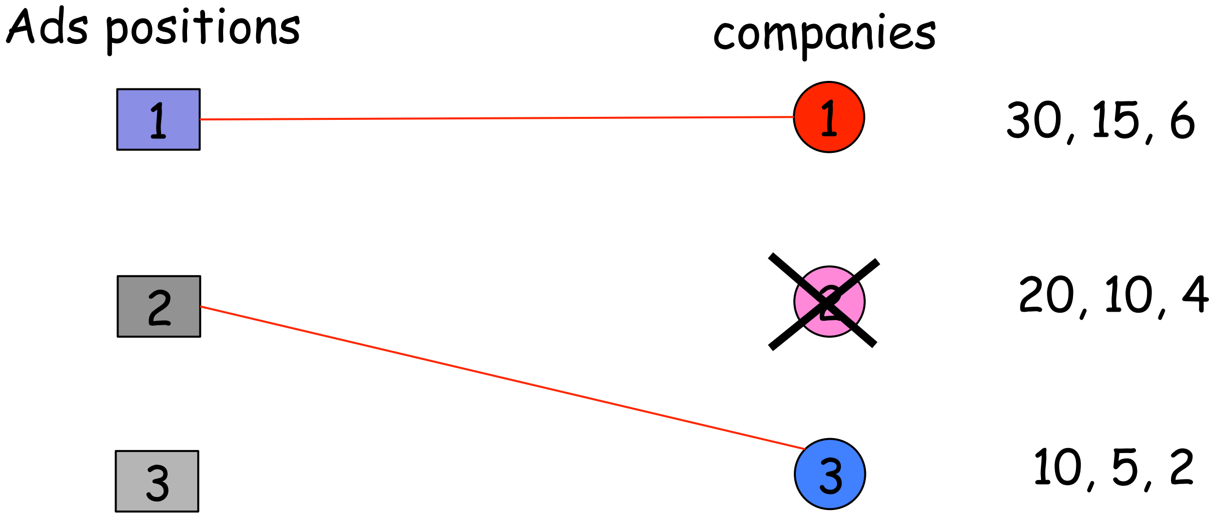
- This is the maximum weight matching
- 1 gets 30, 2 gets 10 and 3 gets 2

VCG example



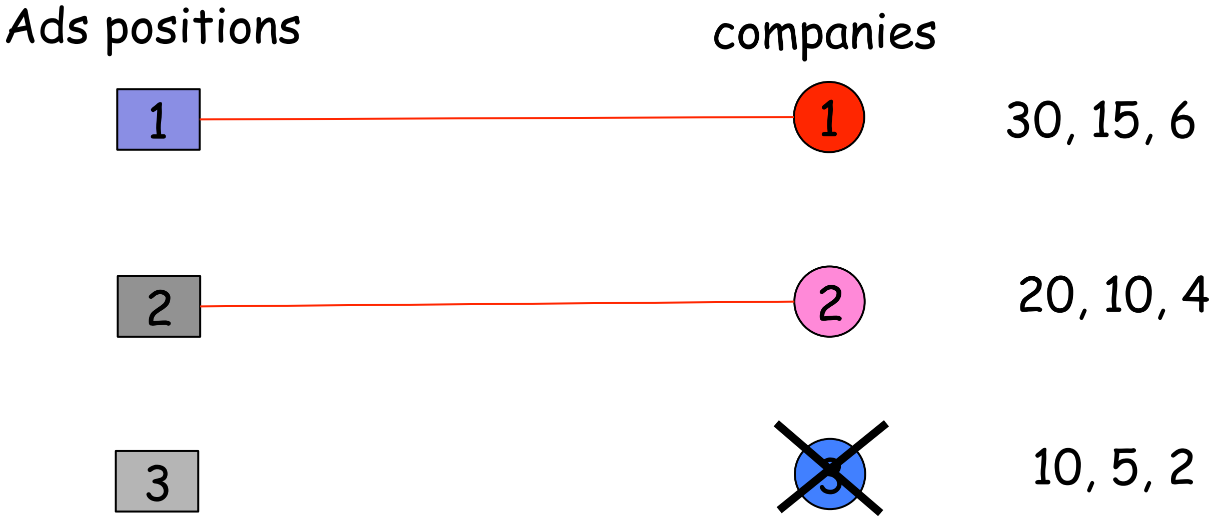
- If 1 weren't there, 2 and 3 would get 25 instead of 12,
- Then 1 should pay 13

VCG example



- If 2 weren't there, 1 and 3 would get 35 instead of 32,
- Then 2 should pay 3

VCG example



- If 3 weren't there, nothing would change for 1 and 2,
- Then 3 should pay 0

The VCG mechanism

- Every buyers should pay a price equal to the social value loss for the others buyers
  - If V_B^S is the maximum total valuation over all the possible perfect matchings of the set of sellers S and the set of buyers B,
  - If buyer j gets good i, he/she should be charged $V_{B-j}^{S} - V_{B-j}^{S-i}$
- Under this price mechanism, truth-telling is a dominant strategy