Adaptive multigrid methods

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September 11, 2013

Introduction

$$\mathbf{Au} = \mathbf{f}$$

We want to combine two ideas:
Multigrid methods and **Mesh adaptation**

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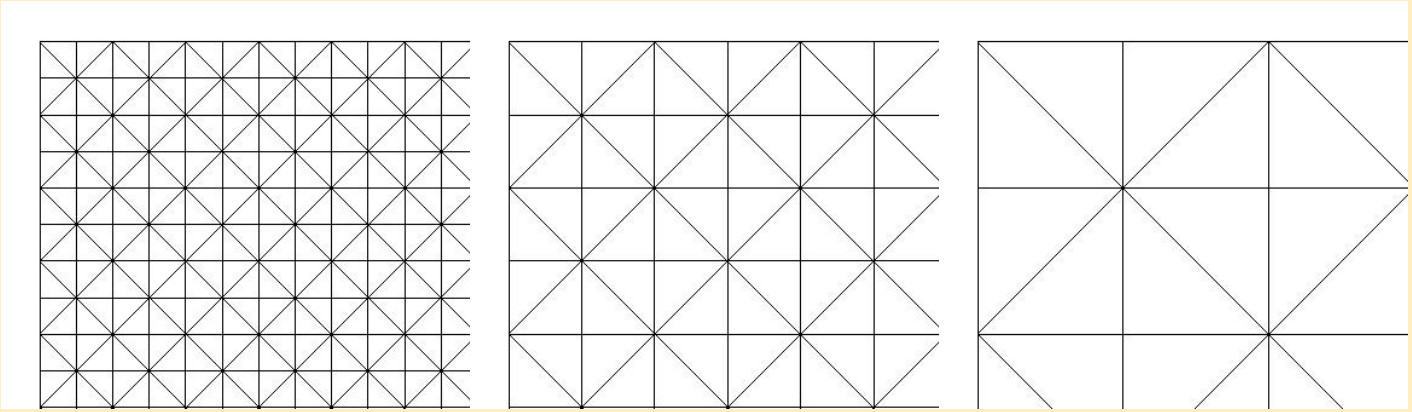
Multigrid methods

Historic

Dates back to sixties (Bakhvalov, Fedorenko).
Well established theory (Hackbusch, 1985)
Full-Multigrid Method (FMG):
Soon identifies as one of the first O(N) solution algorithm.

Theory

MG is iterative.



2G:

After advancing the equation by means of a smoother S (for exemple, Jacobi : $S = diag(A)^{-1}(diag(A) - A)$) on the considered grid, we transfer, with a **restriction** R, the residual on a coarser mesh and we compute there a correction which will be retransfered on the fine mesh with a **prolongation** P and applied on the fine grid iterate.

Extension to **MG** via recursion.

I. Multigrid methods

Properties of multigrids

MG need two main assumptions:

- **smoothing property:**

There are a constant C_S and a functional $\eta(\nu)$ independent of mesh size h (of the finest mesh) such that $||AS^\nu|| \leq C_S h^{-2m} \eta(\nu)$ and $\eta(\nu) \underset{\nu \to +\infty}{\longrightarrow} 0$

with $2m =$PDE equation order

- **approximation property:**

There is a constant C_A independent of h such that $||A^{-1} - P\bar{A}^{-1}R|| \leq C_A h^{2m}$ with P and R the transfer operators.

Then $||u_h^{k_{cycle}} - u_h|| < \rho_{cycle}^k$, ρ not depending on mesh size.

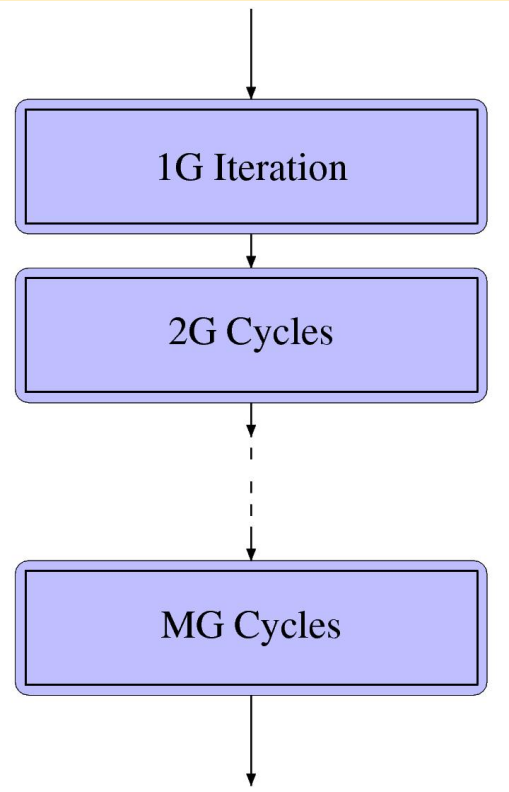
Failure of MG

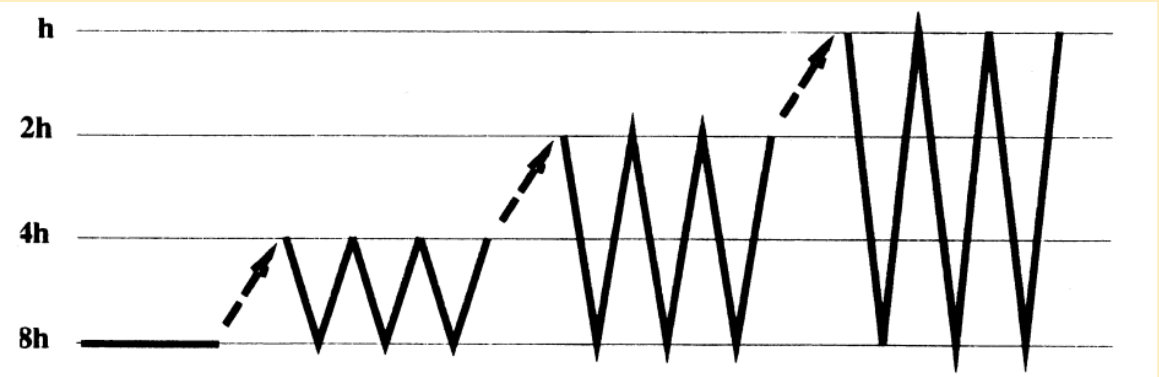
MG assumes a strong smoothness of equation and of solution. Otherwise, singular modes slow down the iteration convergence. Possible cure: use of MG as preconditionner of a quasi-Newton iteration (GMRES).

Full-multigrid method

The FMG algorithm applies successively MG to a sequence of meshes. We begin on a coarse mesh, then we transfer the computed solution on a finer mesh, etc...

Multigrid cycles are stopped when algorithmic error $||u_h^{k_{cycle}} - u||$ is less than approximation error $||u_h - u||$.





Full-multigrid method

FMG needs smoothing and approximation properties and an **approximation convergence property**:
$\forall i_{phase} > 0, ||u^{i_{phase}+1} - Pu^{i_{phase}}|| \leq C_1(h_{i_{phase}+1})^{\alpha}$, α accurary order.
Then $||u_h^{k_{cycle}} - u|| < (1+\epsilon)||u - u_h||$, k_{cycle} independent of h (*)
And then, FMG algorithm has a complexity of $O(N)$, with N the number of vertices of the finest grid.

Failure of FMG

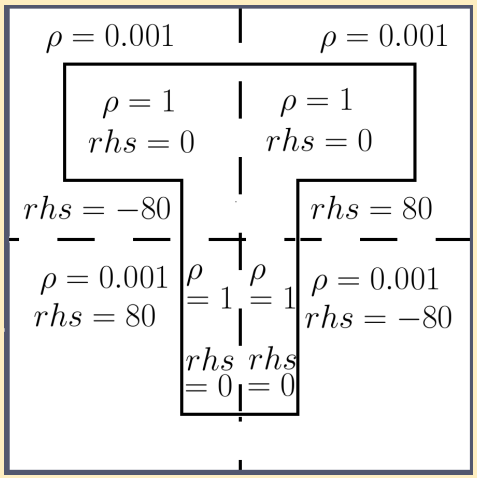
Approximation convergence property is not always true, so iterative convergence (*) should be controlled.
- Many works based on a priori CV (Arioli, 2004):$||u_h - u|| \leq Kh^2$.
- Another idea is use of a posteriori estimators.

Convergence property does not hold for coarse meshes.
Potential cure: mesh adaptation.

Experiment with FMG: Tatebe test case

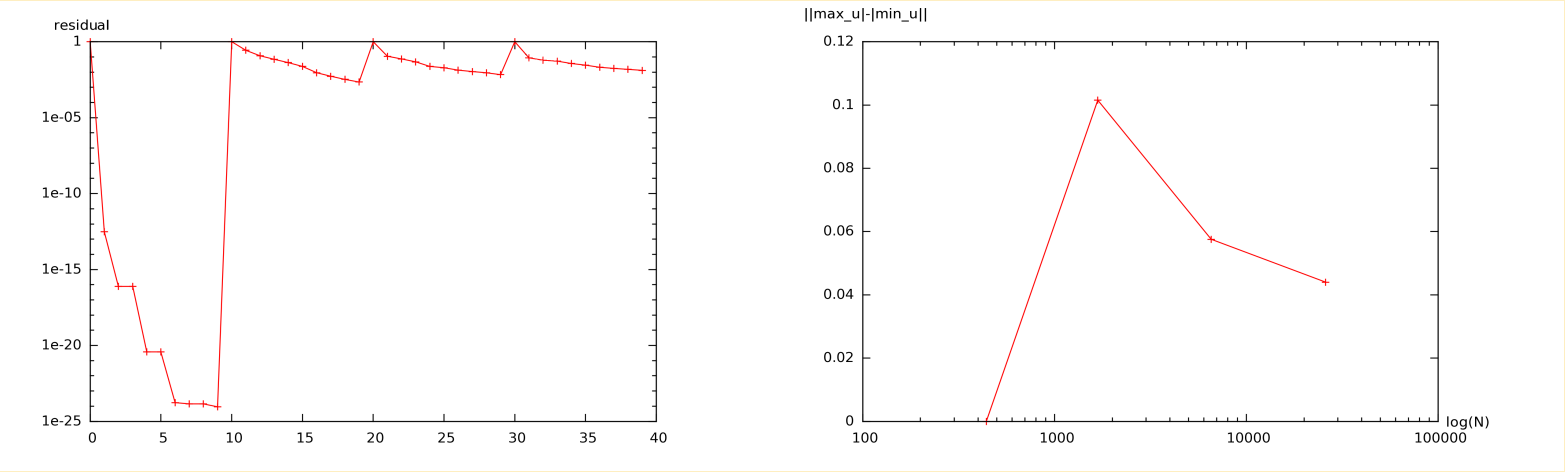
On each phase, we solve the equation on the finest mesh of the phase using the GMRES method.
Multigrids are used as the preconditionner of GMRES (Tatebe, 1993).
The smoothing is done with ten Jacobi sweeps (only passing from fine to coarse).
The transfers between the meshes are done with interpolation (for a transfer from a coarse mesh to a fine mesh) and accumulation (for a transfer from a fine mesh to a coarse mesh).

Tatebe case:
$-div(\frac{1}{\rho}\nabla u) = rhs$

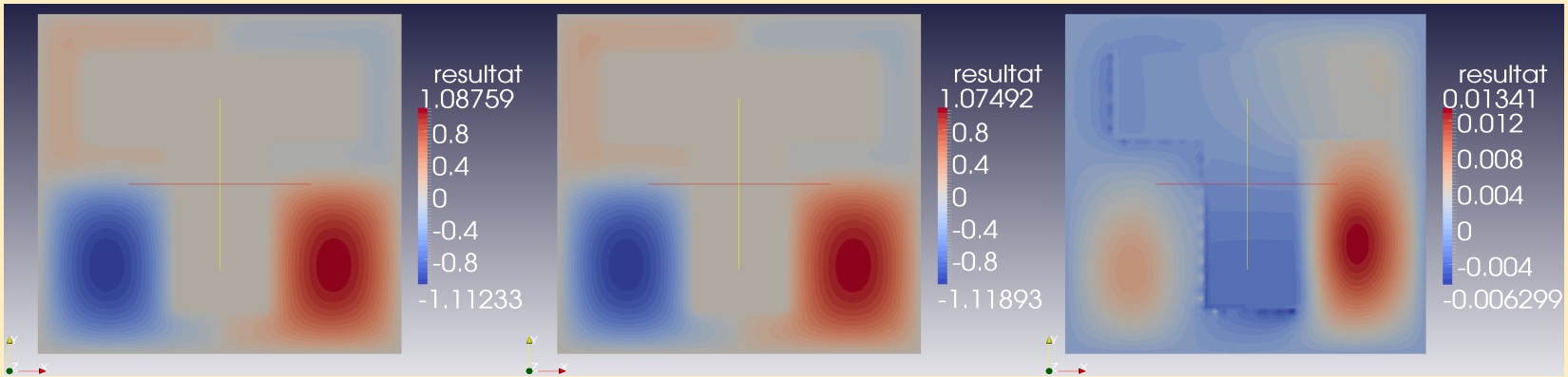


Results

10 cycles per phase



Converged solution, solution for 10 cycles per phase and difference between the two solutions.



$|min(u)| \approx 1.1$. The approximation is wrong by 1.1% and deviation between FMG and converged is about the same order.

Continuous mesh

Metric $\mathcal{M}$: $\mathcal{M}(\mathbf{x})$ a matrix $\forall \mathbf{x} \in \Omega$.

Number of vertices: $\mathcal{C}(\mathcal{M}) = \int_\Omega \sqrt{det(\mathcal{M}(\mathbf{x}))}\, \mathrm{d}\mathbf{x}$

Defining a Riemannian distance between two points:

$$dist(\mathbf{a}, \mathbf{b}) = length_{\mathcal{M}}(\mathbf{ab}) = \int_0^1 \sqrt{{}^t\mathbf{ab}\mathcal{M}(\mathbf{a} + \theta\mathbf{ab})\mathbf{ab}}\, \mathrm{d}\theta$$

$\mathcal{H}_{\mathcal{M}}$ =**unit mesh for** $\mathcal{M} \Leftrightarrow \forall$ edge $\mathbf{e} \in \mathcal{H}_{\mathcal{M}}$, $length_{\mathcal{M}}(\mathbf{e}) \approx 1$

Building the metric

An approximation u_h of u, computed on a given mesh. H_{u_h} the Hessian.
$h_i(\mathbf{x}) = (\lambda_i(\mathbf{x}))^{-1/2}$, with $(\lambda_i(\mathbf{x}))_{i=1,dim}$ eigenvalues of $H_{u_h}(\mathbf{x})$.
$(\mathbf{v}_i(\mathbf{x}))_{i=1,dim}$ the eigenvectors of $H_{u_h}(\mathbf{x})$.
Minimizing the error

$$\epsilon_{\mathcal{M}} = ||u - \Pi_h u|| \approx \int_\Omega \sum_{i=1}^{N} h_i(\mathbf{x})^2(\mathbf{x}) |^t\mathbf{v}_i(\mathbf{x}) H_{u_h}(\mathbf{x}) \mathbf{v}_i(\mathbf{x})| \, \mathrm{d}\mathbf{x}$$

under the constraint: $N = \int_\Omega \sqrt{\det(\mathcal{M}_{L^1}(\mathbf{x}))}\, \mathrm{d}\mathbf{x}$.
The **optimal metric field** $\mathcal{M}_{L^1}(\mathbf{x})$ is given by:

$$\mathcal{M}_{L^1}(\mathbf{x}) = \mathcal{D}_{L^p} det(|H_{u_h}(\mathbf{x})|)^{\frac{-1}{5}} |H_{u_h}(\mathbf{x})|$$

where $\mathcal{D}_{L^1} = N^{\frac{2}{3}} (\int_\Omega det(|H_{u_h}(\mathbf{x})|)^{\frac{2}{5}} \, \mathrm{d}\mathbf{x})^{\frac{-2}{3}}$.

Creation of a *unit mesh* for this metric.

Fixed-point mesh adaptation loop

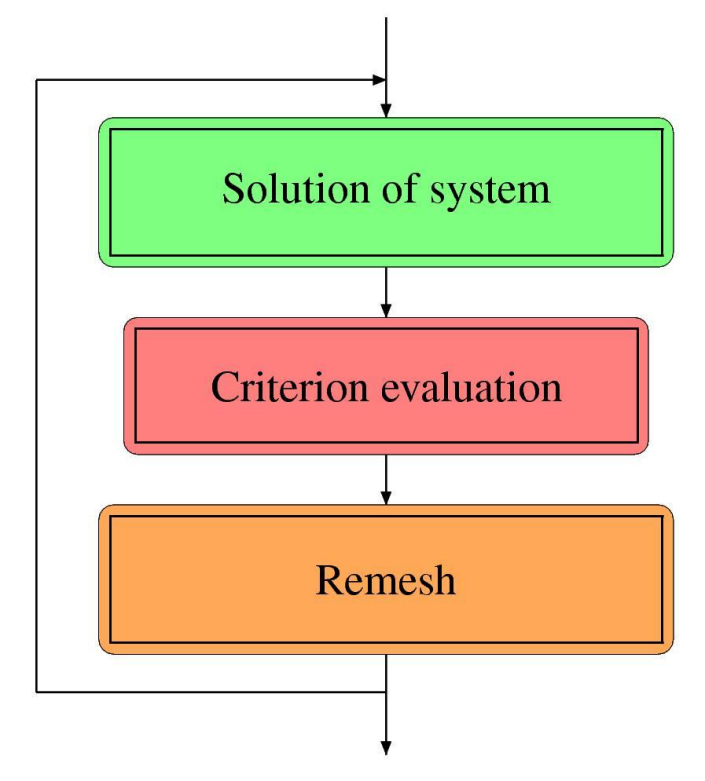
We apply the Hessian-based adaptation until we find the mesh which is the best for the solving of this equation, the optimal mesh:

1- compute the PDE approximate solution on current mesh

2- compute approximate Hessian and optimal metric

3- build new mesh according to the metric

4- go to 1.

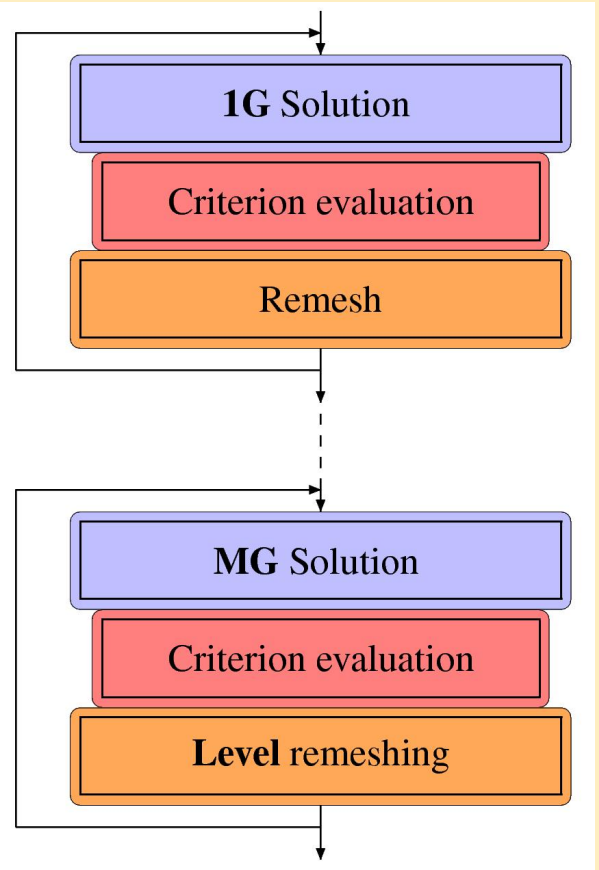


III. Full-multigrid anisotropic adaptive algorithm

We use FMG algorithm but, *for each FMG phase*, we apply the above mesh adaptation loop. Inside it, solution is obtained by MG cycling. Between phases, the number of nodes is increased in the optimal metric.

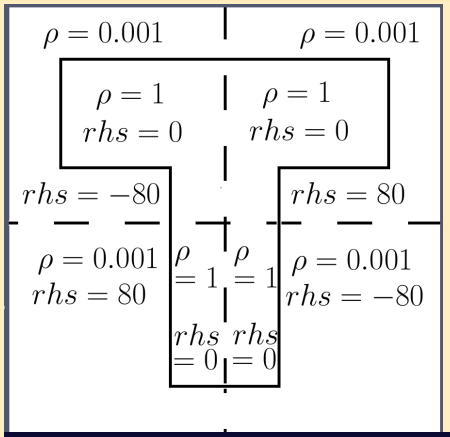
Adaptation carries a better convergence property.

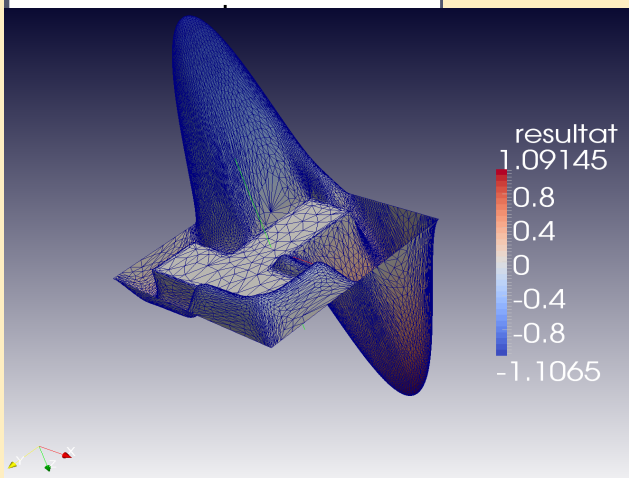
Adaptive FMG should have a **complexity O(N)**.

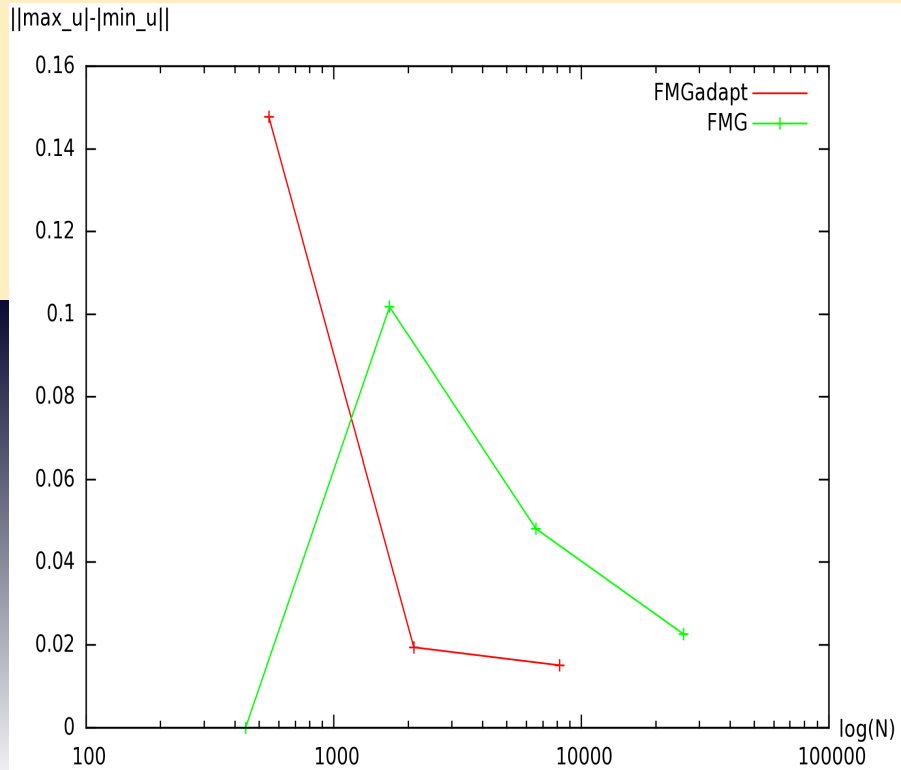


IV. Test cases

Tatebe test case

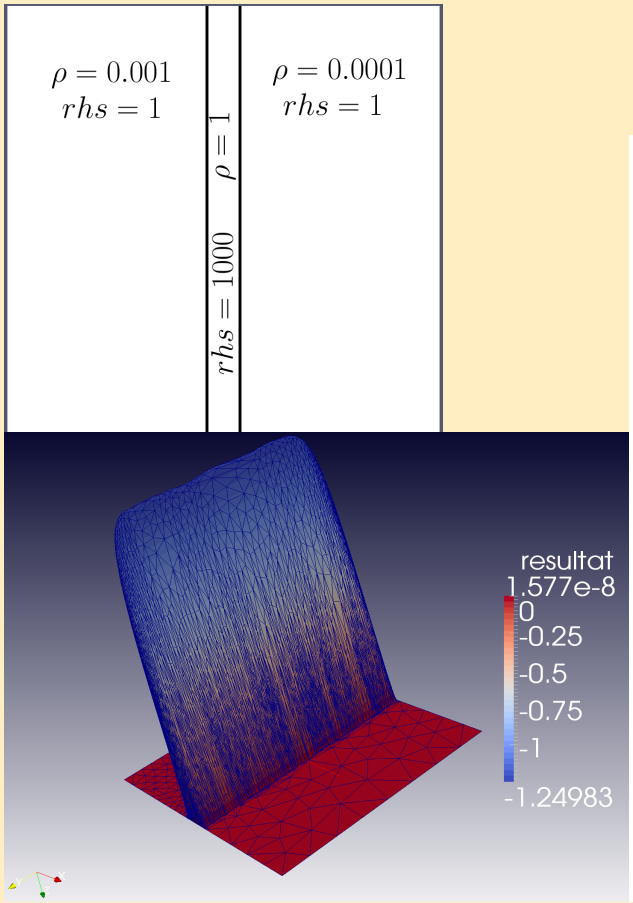


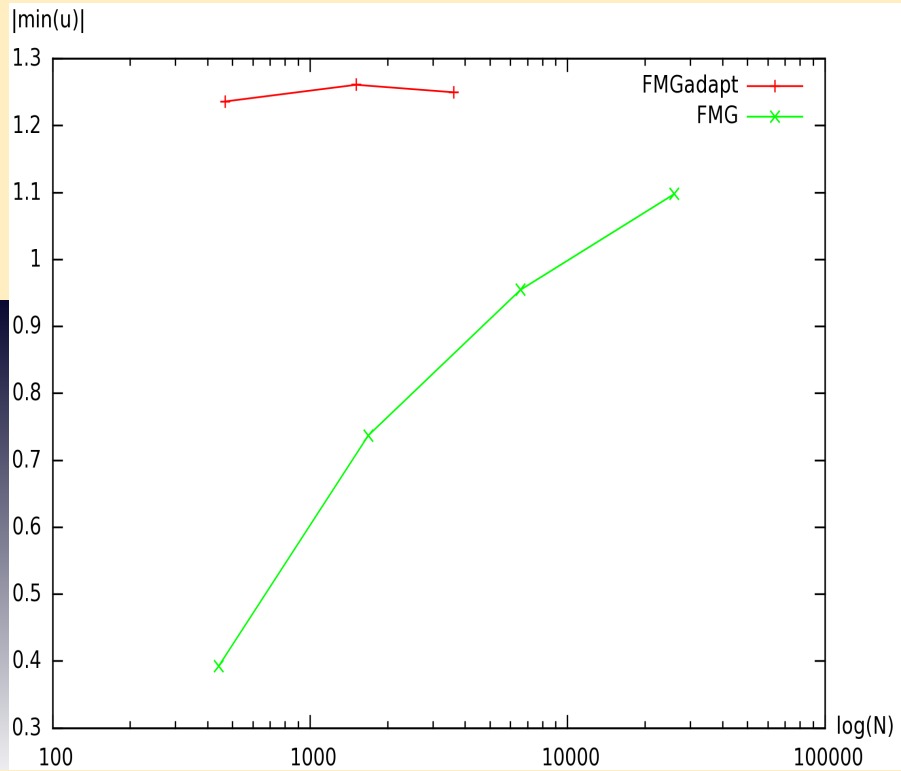




This test case is symmetric. After convergence, the difference between the maximum and the minimum must be equal to 0. We observe this difference decrease more quickly with adaptation.

A stiff layer case





We observe the minimun tends to a limit value around 1.25 more quickly achieved with adaptation.

Conclusion

Synthesis

- MG combined with new mesh adaptation technology.
- Introducing this more complex algorithm brings
  - a higher safety in the accuracy of results and
  - a better control of computational cost:
  $N = \varepsilon^{-\frac{dim}{\alpha}}$ for obtaining a prescribed error ε.

Prospects

- Convergence control of MG.
- Convergence control of the adaptation loop.
- A posteriori error estimator and corrector : $u_h \pm \delta u_h$.