# MAIN ISSUES IN ANISOTROPIC MESH ADAPTIVE FMG

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# Motivations

The job of Numerical methods designer seems to concentrate on the reach of a given accuracy for a computational effort as small as possible.
This is not because we are avaricious... The challenge is to compute systems which were unreachable up to now.
Computational effort is -in short- related to the discrete system size, *i.e.* the number of degrees of freedom:

- Reduce the number of operations per degree of freedom: FMG,
- Reduce the number of degrees of freedom: mesh adaptation.

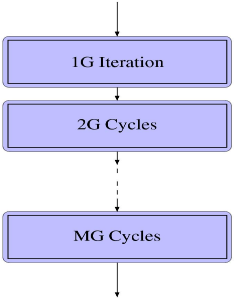
# Talk contents

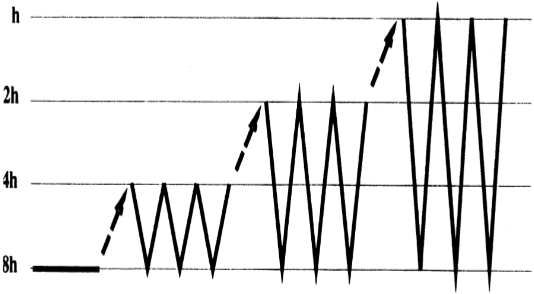
- Error-based FMG
- Hessian-based Adaptive FMG
- Norm-oriented adaptive FMG

# 1. Error-based FMG

## Conditions for FMG

- smoothing property
- approximation property
- stopping criterion: $||u_h^{k_{cycle}} - u_h|| \leq ||u - u_h||$



# Error-based FMG (2)

## Theoretical stopping criterion

Let $\alpha$ be the order of accuracy,
**If** we have the **convergence property**:

$$\forall i_{phase} > 0, ||u^{i_{phase}+1} - Pu^{i_{phase}}|| \leq C_1(h_{i_{phase}+1})^{\alpha} \text{ (*)},$$

**Then** there exists a fixed number of cycles $k_{cycle}$ (independant of mesh size) such that:

$$||u_h^{k_{cycle}} - u|| < (1+\epsilon)||u - u_h||, \ k_{cycle}.$$

Typically, for second order accuracy, divide the MG residual by a factor 10 at each FMG phase.

**This strategy may be unaccurate for non-trivial problems and need be replaced**.

# Error-based FMG (3)

The finite-element discretisation is written:

$$u_h \in V_h \;\; \forall\ \phi_h \in V_h \;\; a(u_h, \phi_h) = (f, \phi_h)$$

Let $(N_i)$ denote the finite-element basis:

$$u_h = \sum \mathbf{u}_{h,i} N_i \quad \Leftrightarrow \quad u_h = T\mathbf{u}_h$$

$$T^* : V' \to \mathbb{R}^n \quad [T^* f]_i = (f, N_i).$$

Then:
$u_h - u_h^{k_{cycle}} = T(\mathbf{u}_h - \mathbf{u}_h^{k_{cycle}}) = T\mathbf{A}_h^{-1}(\mathbf{f}_h - \mathbf{A}_h \mathbf{u}_h^{k_{cycle}})$
$u - u_h^{k_{cycle}} = A^{-1}(f - Au_h^{k_{cycle}}) \approx T\mathbf{A}_h^{-1} T^*(f - Au_h^{k_{cycle}})$

# Error-based FMG (4)

Instead of comparing:

$$||T\mathbf{A}_h^{-1}(\mathbf{f}_h - \mathbf{A}_h\mathbf{u}_h^{k_{cycle}})||$$

and

$$||T\mathbf{A}_h^{-1}T^*(f - Au_h^{k_{cycle}})||$$

we choose to compare

$$||\mathbf{f}_h - \mathbf{A}_h\mathbf{u}_h^{k_{cycle}}||_{l^1}$$

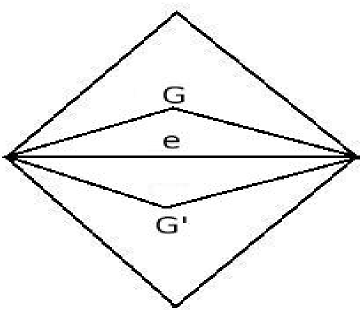
and

$$||T^*(f - Au_h^{k_{cycle}})||_{l^1}.$$

# Evaluation of the continuous residual

$$(Au_h^{k_{cycle}}, \phi_h^i) = \sum_T \sum_{ijedgesT} \nabla u_h^{k_{cycle}}(X).n_{ij} \int_{ij} \phi_h^i(X)\,\mathrm{d}X$$

$$\approx \sum_T \sum_{ijedgesT} \nabla u_h^{k_{cycle}}(X).n_{ij} \frac{mes(ij)}{mes(D_{ij})} \int_{D_{ij}} \phi_h^i(X)\,\mathrm{d}X$$



# Error-based FMG': stopping criterion

Let $C_{MG}$ be the MG preconditioner of the GMRES iteration.

1. Let:
$resPreCondInit = ||C_{MG}^{-1}\mathbf{f}_h - C_{MG}^{-1}A_h\mathbf{u}_h^1||_{L^2}$
$resInit = ||\mathbf{f}_h - A_h\mathbf{u}_h^0||_{L^2}$

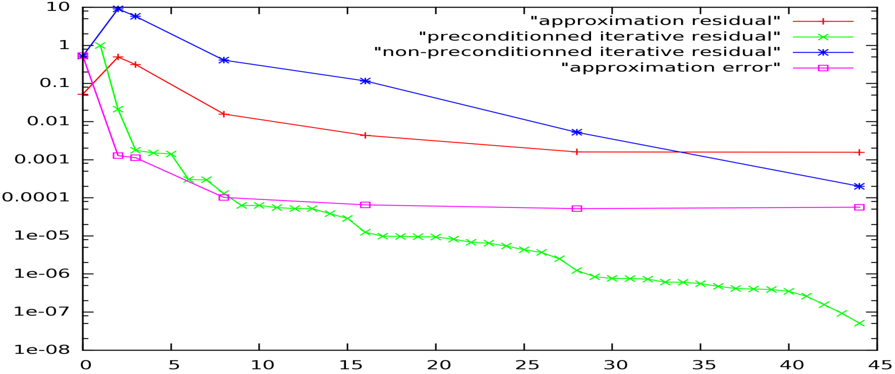
2. Iterate preconditioned-GMRES while:
$||C_{MG}^{-1}\mathbf{f}_h - C_{MG}^{-1}A_h\mathbf{u}_h^k||_{L^2} \leq 0.1 resPreCondInit$.

3. If $0.1^*||\mathbf{f}_h - A_h\mathbf{u}_h^0||_{L^2} < ||\mathbf{f}_h - A_h\mathbf{u}_h^k||_{L^2}$,
$resPreCondInit = ||C_{MG}^{-1}\mathbf{f}_h - C_{MG}^{-1}A_h\mathbf{u}_h^k||_{L^2}$ go to 1.

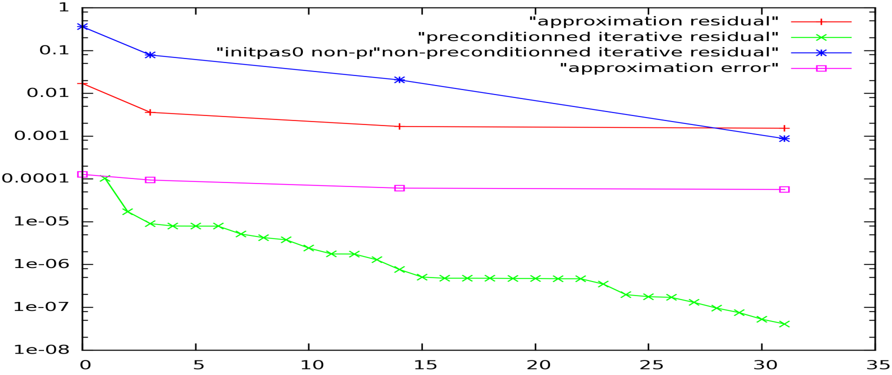
4. If $0.1||T^*(f - Au_h^k)||_{L^2} \leq ||\mathbf{f}_h - A_h\mathbf{u}_h^k||_{L^2}$,
$resInit = ||\mathbf{f}_h - A_h\mathbf{u}_h^k||_{L^2}$ go to 1.

5. Stop iteration.

## First example: starting with a zero initial iterate:



Second example: starting from a just-coarser approximate solution as initial iterate:



# II. Hessian-based mesh adaptation

## Mesh parameterization

**Metric $\mathcal{M}$**: $\mathcal{M}(\mathbf{x}) = \mathcal{R}^t(x,y) \begin{pmatrix} \frac{1}{\Delta\xi^2(x,y)} & 0 \\ 0 & \frac{1}{\Delta\eta^2(x,y)} \end{pmatrix} \mathcal{R}(x,y)$

with:

$\Delta\xi(x,y) =$ mesh size in the first characteristic direction
$\Delta\eta(x,y) =$ mesh size in the second characteristic direction
$\mathcal{R}(x,y) =$ matrice of eigenvectors.

Number of vertices: $\mathcal{C}(\mathcal{M}) = \int_\Omega \sqrt{det(\mathcal{M}(\mathbf{x}))}\,\mathrm{d}\mathbf{x}$

Riemannian distance between two points:

$dist(\mathbf{a},\mathbf{b}) = length_{\mathcal{M}}(\mathbf{ab}) = \int_0^1 \sqrt{{}^t\mathbf{ab}\mathcal{M}(\mathbf{a}+\theta\mathbf{ab})\mathbf{ab}}\,\mathrm{d}\theta$

$\mathcal{H}_{\mathcal{M}}$ =**unit mesh for $\mathcal{M}$** $\Leftrightarrow \forall$ edge $\mathbf{e} \in \mathcal{H}_{\mathcal{M}}$, $length_{\mathcal{M}}(\mathbf{e}) \approx 1$

# II. Adaptation basée Hessien

## Construction de la métrique

Une approximation $u_h$ de $u$, calculée sur un maillage donné. $H_{u_h}$ le Hessien. $H_{u_h} = v_i^{u_h} \Lambda_{u_h}^t v_i^{u_h}$

Minimize the interpolation error:

$$\epsilon_{\mathcal{M}} = ||u - \Pi_h u|| \approx \int_\Omega \sum_{i=1}^{dim} h_i^{u_h}(\mathbf{x})^2(\mathbf{x}) |{}^t\mathbf{v}_i^{u_h}(\mathbf{x}) H_{u_h}(\mathbf{x}) \mathbf{v}_i^{u_h}(\mathbf{x})| \, \mathrm{d}\mathbf{x}$$

under the constraint: $\mathcal{C}(\mathcal{M}) \int_\Omega \sqrt{\det(\mathcal{M}_{L^1}(\mathbf{x}))} \, \mathrm{d}\mathbf{x} = N$.

Le **champ de métrique optimal** $\mathcal{M}_{L^1}(\mathbf{x})$ est donné par:

$$\mathcal{M}_{L^1,opt}(\mathbf{x}) = \mathcal{K}_1(1, u_h)$$

$$\mathcal{K}_1(k, u_h) = \mathcal{D}_{L^1} det(|kH_{u_h}(\mathbf{x})|)^{\frac{-1}{5}} |kH_{u_h}(\mathbf{x})|$$

$$\mathcal{D}_{L^1} = N^{\frac{2}{3}} (\int_\Omega det(|kH_{u_h}(\mathbf{x})|)^{\frac{2}{5}} \, \mathrm{d}\mathbf{x})^{\frac{-2}{3}}.$$

# II. Adaptation basée Hessien

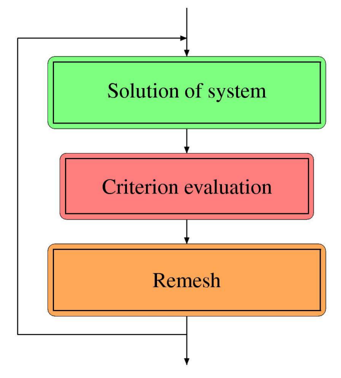
## Boucle point fixe d'adaptation de maillage

1- calcul d'une solution approchée de l'EDP sur le maillage courant

2- calcul du Hessien approché et de la métrique optimale $\mathcal{M}_{L^1}$

3- construction d'un nouveau maillage suivant la métriqu optimale

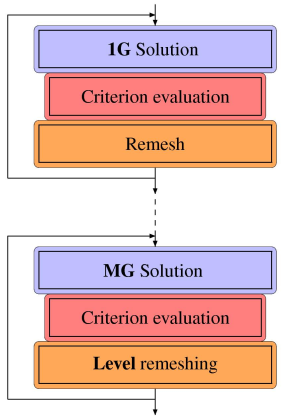
4- retour à 1.



# III. Algorithme Full-Multigrid adaptatif anisotrope

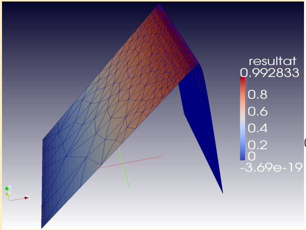
On utilise l'algorithme FMG mais, *pour chaque phase FMG*, on applique la boucle d'adaptation ci-dessus. A l'intérieur, la solution est obtenue par cyclage MG. Entre les phases, le nombre de noeuds est augmenté dans la métrique optimale. L'adaptation assure une meilleure propriété de convergence.

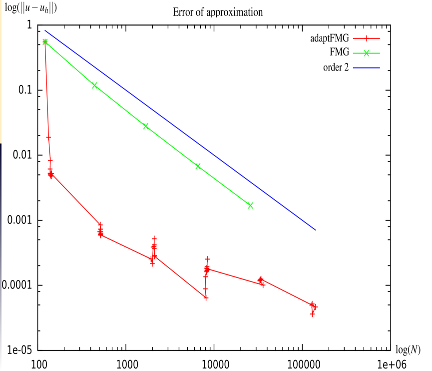
Le FMG adaptatif doit avoir une **complexité de O(N)**.



## Cas de la couche limite

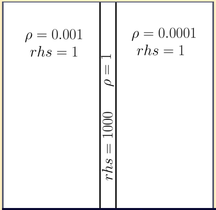
On résout $-\Delta u = rhs$ où $rhs(x,y) = \frac{1}{\alpha^2(exp(1/\alpha)-1)} exp(x/\alpha)$ avec $\alpha = 0.006$.

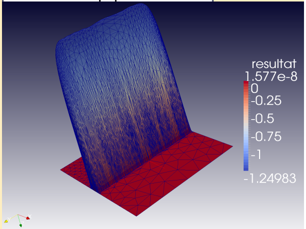


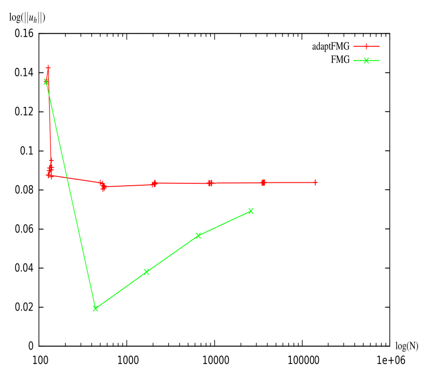




Le résultat tend vers une limite $\approx 0.99$ qui est atteinte beaucoup plus vite avec l'adaptation que sans.

## Cas de la bande centrale



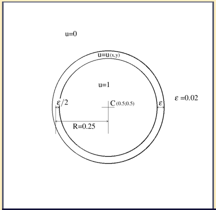

On observe que le minimum tends vers une limite $\approx 1.25$ qui est plus vite atteinte avec l'adaptation.

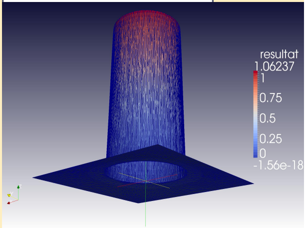
# IV. Cas test

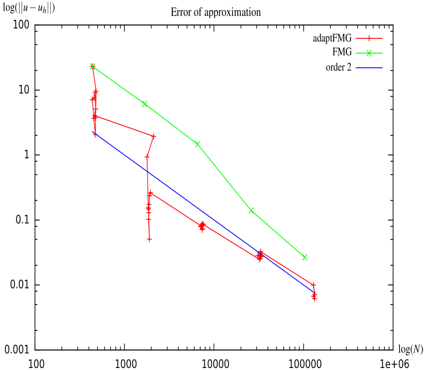
## Cas de la discontinuité circulaire

On résout $-\Delta u = rhs$ où $u(x,y) = \frac{1}{2}[1 + \frac{2\psi}{\epsilon} + \frac{1}{\Pi} sin(\frac{2\Pi\psi}{\epsilon})]$

avec $\psi = R - \sqrt{(x_C - x)^2 + (y_C - y)^2}$




La norme, qui doit être égale à 0 pour une bonne approximation, décroit plus vite avec l'adaptation.

# IV. Hessian-based mesh adaptation, concluded

Provides a better solution for difficult cases.

But do not take into account the PDE except through ist approximate solution.

The goal-oriented formulation is probably the first formulation in which the mesh adaptation problem is completely set on a mathematical form:
*Find the mesh which minimizes the error committed on a specified scalar output:*

$$\min_{\mathcal{M}} \delta j_{goal}(\mathcal{M}) = \left|(g, \Pi_{\mathcal{M}} u - u_{\mathcal{M}})\right| \ , \ \mathcal{C}(\mathcal{M}) = N.$$

$$\frac{\partial}{\partial \mathcal{M}} \delta j_{goal}(\mathcal{M}).\delta\mathcal{M} = 0 \quad \mathcal{C}(\mathcal{M}) = N, \ \mathcal{C}(\delta\mathcal{M}) = 0.$$

# V. Goal-Oriented mesh adaptation

$$\delta j_{goal}(\mathcal{M}) = |(g, \Pi_{\mathcal{M}} u - u_{\mathcal{M}})|.$$

Expression $\Pi_{\mathcal{M}} u - u_{\mathcal{M}}$ needs be replaced by an **estimate**:

$$|\int_\Omega \frac{\partial}{\partial x_i}(u_{\mathcal{M}} - \Pi_{\mathcal{M}} u)\frac{\partial}{\partial x_j}\Pi_{\mathcal{M}}\varphi \mathrm{d}\mathbf{x}| \ \preceq \ K \int_\Omega |\rho(H(\varphi)| \ |u - \Pi_{\mathcal{M}} u| \mathrm{d}\mathbf{x}$$

After introduction of an adjoint

$$\delta j_{goal}(\mathcal{M}) \approx |a(u^*_{g,\mathcal{M}}, \Pi_{\mathcal{M}} u - u)|$$

$$\frac{\partial}{\partial \mathcal{M}} \delta j_{goal,approx}(\mathcal{M}).\delta\mathcal{M} \approx |a(u^*_{g,\mathcal{M}}, \frac{\partial}{\partial \mathcal{M}}(\Pi_{\mathcal{M}} u - u)\delta\mathcal{M})|$$

by *freezing* $a(u^*_{g,\mathcal{M}}, .)$ with respect to $\mathcal{M}$. Then:

$$\mathcal{M}_{opt,goal} = \mathcal{K}_1(|\rho(H(u^*_{g,\mathcal{M}_{opt,goal}}))|, u_{\mathcal{M}}).$$

**But:**

**- only features influencing the functional will be refined,**

**- then we have lost the convergence to the PDE solution!**

# VI. Norm-Oriented mesh adaptation

Let us try to minimize:

$$j(\mathcal{M}) = ||\Pi_{\mathcal{M}} u - u_{\mathcal{M}}||^2_{L^2(\Omega)}$$

$$j'(\mathcal{M}).\delta\mathcal{M} = (\Pi_{\mathcal{M}} u - u_{\mathcal{M}}, \frac{\partial}{\partial \mathcal{M}}(\Pi_{\mathcal{M}} u - u_{\mathcal{M}}).\delta\mathcal{M}) = 0 \ \forall \delta\mathcal{M}.$$

- Term $\frac{\partial}{\partial \mathcal{M}}(\Pi_{\mathcal{M}} u - u_{\mathcal{M}}).\delta\mathcal{M}$ is the same as in the goal-oriented analysis.
- Term $\Pi_{\mathcal{M}} u - u_{\mathcal{M}}$ is approximated by a *corrector*:

$$a(u'_{prio}, \phi_h) = K(\phi_h, u_h) \qquad \text{with}$$

$$K(\phi_h, u_h) \ = \sum_{\partial T_{ij}} \ (\nabla \phi_h|_{T_i} - \nabla \phi_h|_{T_j}) \cdot \mathbf{n}_{ij} \ \int_{\partial T_{ij}} (\pi_h u_h - u_h) \, \mathrm{d}\sigma.$$

# V. Adaptation Norm-Oriented

$$j'(\mathcal{M}).\delta\mathcal{M} \approx (u'_{prio}, u_{aux}.\delta\mathcal{M}) \quad \text{with}$$

$$a(u_{aux}.\delta\mathcal{M}, \phi) =$$

$$\sum_{\partial T_{ij}} (\nabla\phi|_{T_i} - \nabla\phi|_{T_j}) \cdot \mathbf{n}_{ij} \int_{\partial T_{ij}} \frac{\partial}{\partial\mathcal{M}}(\pi_{\mathcal{M}} u - u).\delta\mathcal{M} \, \mathrm{d}\sigma$$

It is now useful to introduce a new adjoint system:

$$a(\psi, u^*_{prio}) = (u'_{prio}, \psi)$$

Then

$$j'(\mathcal{M}).\delta\mathcal{M} \approx= a(u_{aux}.\delta\mathcal{M}, u^*_{prio})$$

$$|j'(\mathcal{M}).\delta\mathcal{M}| \preceq \int_\Omega |\rho(H(u^*_{prio}))| |\frac{\partial}{\partial\mathcal{M}}(\pi_{\mathcal{M}} u - u).\delta\mathcal{M}| \, \mathrm{d}\Omega.$$

# V. Adaptation Norm-Oriented

$$\mathcal{M}_{opt,norm} = \mathcal{K}_1(|\rho(H(u^*_{prio}))|, u)$$

**Step 1**:

$$a(u'_{prio}, \phi_h) = \sum_{\partial T_{ij}} \left(\nabla\phi_h|_{T_i} - \nabla\phi_h|_{T_j}\right) \cdot \mathbf{n}_{ij} \int_{\partial T_{ij}} (\pi_h u_h - u_h)\, \mathrm{d}\sigma.$$

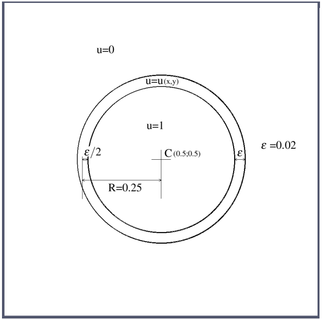
**Step 2**:

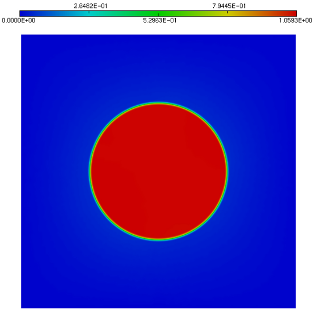
$$a(\psi, u^*_{prio}) = (u'_{prio}, \psi)$$

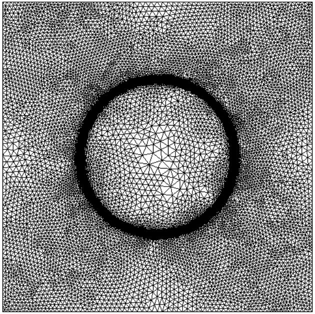
**Step 3**:

$$\mathcal{M}_{opt,norm} = \mathcal{K}_1(|\rho(H(u^*_{prio}))|, u)$$

# VI Numerical examples



# VI Numerical examples

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# Conclusions ans perspectives

### Conclusion(1)

- Combination of FMG and anisotropic mesh adaption.
- Proposal of a stopping criterion.
  $N = \varepsilon^{-\frac{dim}{\alpha}}$ pour obtenir une erreur $\varepsilon$.

### Conclusion(2)

- Proposal for a new mesh adaptation criterion.
- The new criterion seems to be more accurate than standard Hessian.

### Perspectives

- The principle extends naturally to systems.