Isogeometric methods for shape modeling and numerical simulation

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ABSTRACT

The topic of this paper is to give an overview of a recent approach, called isogeometric analysis, that aims at a seamless integration of geometric modeling and numerical computational. It is a uniform framework to describe both the geometry representation and approximate solutions of a simulation problem on this geometry. It rises interesting geometric problems, that we review. We describe the general framework of this approach, the interesting properties of B-spline bases, that it exploits. After showing on an example of 3D heat conduction problem hwo it works, we discuss geometric issues including the parametrization of computational domains and its impact on quality of approximation, refinement techniques which allow to extend the function basis and to develop adaptive methods to improve efficiently the accurancy of approximation and geometric issues related to complex topologies.

KEYWORDS

Isogeometric analysis; geometric modeling; physical simulation; B-spline; T-spline ; simplex spline.

I.INTRODUCTION

In engineer design and simulation of physical phenomena, geometry plays an important role. The shape of an object directly influences the functionalities that we expect from it. Consider just as examples the properler of a ship, the wing of a plane or the structure of a mechanical piece in a car engine. Its performances (force, drag, resistance) are directly related to its shape. To analyze and optimize these performances, numerical simulations are usually performed. In the design process, these objects are usually described by CAGD tools, which involve parametric non-linear models using bspline functions. But in the simulation process, usually surface or volume discrete meshes are used to approximate the solutions of partial differential equations that describe the physical phenomena we want to analyse.

This has two important consequences. Firstly, a conversion step is needed to go from one representation to another, which might deviate corresponding performance analysis. Secondly, this transformation needs to be tightly connected with design parameters when one want to optimize the geometry with respect to performance analysis.

The topic of this paper is to give a brief overview of recent developments, which tackle these problems. The approach uses the same type of mathematical (piecewise non-linear) representation, both for the geometry and for the physical solutions, and thus avoid this costly forth and back transformations. Moreover its reduces the number of parameters needed to describe the geometry, which is of particular interest for shape optimisation.

This approach was introduced by T. Hughes and his collaborators under the name of isogeometry in the context of PDE problems [1]. This uniform framework provides more accurate and efficient ways to deal with complex shapes and to approximate the solutions of physical simulation problems. But it also rises interesting geometric problems for the representation of shapes and functions on shapes that we will describe.

To present the general idea of this approach, we simplify the context and consider a surface patch Ω of $\mathbb{R}^3$ on which we want to solve a differential equations of the form

$D(f(\boldsymbol{x}))=0 \; for \; \boldsymbol{x} \in \Omega$

with boundary conditions:

$N(f(\boldsymbol{x}))=f_0(x) \; for \; \boldsymbol{x} \in \partial OMEGA.$

Instead of directly approximating the function $f(\boldsymbol{x})$ on the domain Ω ,

- we first parametrize the physical domain by a computational domain D (here a rectangle) by a map $\sigma : D \to \Omega$
- And then we compute the solution $\phi = f \circ \sigma$ induced by partial differential equations with boundary conditions on D through the map σ .

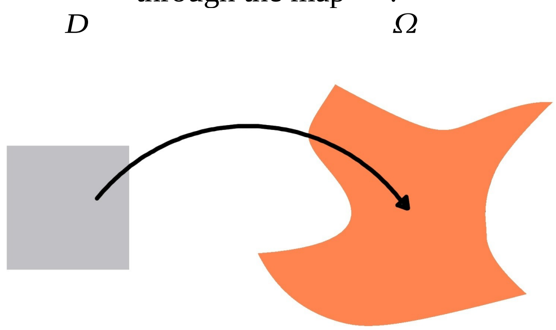


Figure 1. **Parametrization of the physical domain.**

Therefore the solution f is defined implicitly on Ω by $f(\boldsymbol{x}) = \phi \circ \sigma^{-1}(\boldsymbol{x})$. This method naturally extends to cases where the physical domain Ω is a volume parametrized by a cube D in $\mathbb{R}^3$.

The isogeometric approach consists in choosing the same type of representation for the parametrization map σ and the actual solution function ϕ. Because we are interested by representing geometry objects which are coming from CAGD, a natural choice is to use B-spline basis functions. In the next section, we will recall their definition and give their main properties. In the next section, we will show on an example, how this is done in practice. In section IV, we will discuss some of the geometric issues related to this approach before the concluding section.

II.SPLINE REPRESENTATION

Given a nondecreasing sequence of knots

$$\tau = \{t_0, t_1, \ldots, t_h\} \; t_i \le t_{i+1} \in \mathbb{R}$$

the B-spline basis of degree n can be defined using the Cox-de Boor recursion formula:

$$N_{j,0}(t) = \begin{cases} 1 & if \; t_j \le t < t_{j+1} \\ 0 & otherwise \end{cases}$$

$$N_{j,n}(t) = \frac{t - t_j}{t_{j+n} - t_j} N_{j,n-1}(t) + \frac{t_{j+n+1} - t}{t_{j+n+1} - t_{j+1}} N_{j+1,n-1}(t)$$

B-spline cuves can be defined as follows,

$$\boldsymbol{P}(t) = \sum_{i=0}^{m} \boldsymbol{P_i} N_{i,n}(t)$$

where $\boldsymbol{P}_i$ are called control points. B-spline surfaces and volumes can be defined in a tensor product way,

$$\boldsymbol{P}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{l} \boldsymbol{P_{i,j}} N_{i,n}(u) N_{j,n}(v)$$

$$\boldsymbol{P}(u,v,w) = \sum_{i,j,k=0}^{m,l,q} \boldsymbol{P_{i,j,k}} N_{i,n}(u) N_{j,n}(v) N_{k,n}(w)$$

B-spline representation has many interesting properties, such as local support, partition of unity, C^{n-1} continuity and refinement prop erties [3], which are desired for numeric analysis.

III. EXAMPLE

In this section, we show how to solve a 3D heat conduction problem with boundary conditions, by using an isogeometric method. Given a domain Ω closed by the boundary $\partial \Omega$, this physical problem can be described as the following PDE,

$\Delta T(\boldsymbol{x}) = f(\boldsymbol{x})$ in Ω

$N(T(\boldsymbol{x})) = T_0(\boldsymbol{x})$ on $\partial \Omega$

where $\boldsymbol{x}$ are the Cartesian coordinates, T represents the temperature field. These boundary conditions that could be of Dirichlet or

Neumann type are applied on the boundary $\partial\Omega$ of Ω, T_0 being the imposed temperature. f is a user-defined function that allows to generate problems with an analytical solution, by adding
a source term to the classical heat conduction equation.

According to a classical variational approach, we seek for a solution $T \in H^1(\Omega)$, such that $T(\boldsymbol{x}) = T_0(\boldsymbol{x})$ on $\partial\Omega$ and:

$$-\int_\Omega \nabla T(\boldsymbol{x}) \nabla \psi(\boldsymbol{x}) d\Omega = \int_\Omega f(\boldsymbol{x}) \psi(\boldsymbol{x}) d\Omega \qquad (1)$$

According to the isogemetric paradigm, the temperature field is represented using B-spline basis functions. For a 3D problem, we have:

$$T(u,v,w) = \sum_{i,j,k=0}^{m,l,q} T_{i,j,k} N_{i,n}(u) N_{j,n}(v) N_{k,n}(w)$$

where $N_{i,n}$ functions are B-spline basis functions and $\boldsymbol{p} = (u,v,w) \in D$ are domain parameters. Then, we define the test functions $\psi(\boldsymbol{x})$ in the physical domain bys:

$M_{ijk}(\boldsymbol{x}) = N_{ijk} \circ \sigma^{-1}(\boldsymbol{x})$ where

$N_{ijk}(\boldsymbol{p}) = N_{i,m}(u) N_{j,l}(v) N_{k,q}(w)$

The weak formulation Eq. 1 reads:

$$\sum_{r=0}^{n_r} \sum_{s=0}^{n_s} \sum_{t=0}^{n_t} T_{rst} \int_\Omega \nabla M_{rst}(\boldsymbol{x}) \nabla M_{ijk}(\boldsymbol{x}) d\Omega = -\int_\Omega f(\boldsymbol{x}) M_{ijk}(\boldsymbol{x}) d\Omega$$

Finally, we obtain a linear system similar to that resulting from the classical finite-element methods, with a matrix and a right-hand side defined as:

$$E_{ijk,rst} = \int_\Omega \nabla M_{rst}(\boldsymbol{x}) \nabla M_{ijk}(\boldsymbol{x}) d\Omega = \int_\Omega \nabla_{\boldsymbol{p}} N_{rst}(\boldsymbol{p}) B^T(\boldsymbol{p}) B(\boldsymbol{p}) \nabla_{\boldsymbol{p}} N_{ijk}(\boldsymbol{p}) J(\boldsymbol{p}) dP$$

$$S_{ijk} = -\int_\Omega f(\boldsymbol{x}) M_{rst}(\boldsymbol{x}) d\Omega = -\int_\Omega f(T(\boldsymbol{p})) N_{rst}(\boldsymbol{p}) J(\boldsymbol{p}) dP$$

where $J(\boldsymbol{p})$ is Jacobian of the transformation, B^T is the transposed of the inverse of the Jacobian matrix. The above integrations are performed in parametric space using classical Gauss quadrature rules.

An example is given in Figure 2 and Figure 3 with

$$f(\boldsymbol{x}) = \frac{-\pi^2}{3} \sin\left(\frac{\pi x}{3}\right) \sin\left(\frac{\pi y}{3}\right) \sin\left(\frac{\pi z}{3}\right).$$

Figure 2 shows the physical domain and its boundary surfaces. Figure 3 presents the color map of the solution field and corresponding 3D control points. The solution value is represented by the color information.

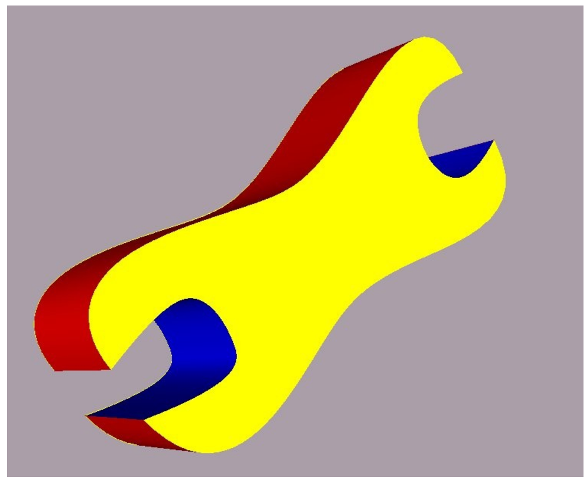
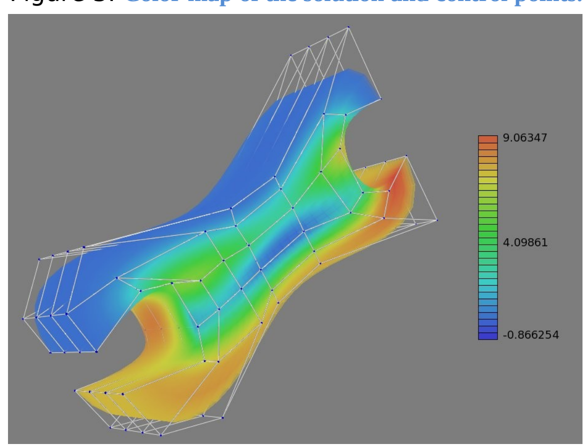


Figure 2. **Physical domain and its boundary surfaces.**

Figure 3. **Color map of the solution and control points.**



IV. GEOMETRIC ISSUES

The isogeometric approach provides a uniform framework to represent the geometry and the physical solutions. This simplifies significantly the computation process involved in a simulation or optimisation problem. But it also rises new questions, which are of geometric nature.

Injectivity Condition. The first geometric issue is to guarantee the injectivity of the paramaterisation map $\sigma : D \to \Omega$. In a usual context, this physical domain is described by boundary curves or surfaces (see Figure 2); The computational domain needs to be parametrised such that the parametrisation coincides on the boundary of D , with the parametrisation of the boundary surfaces. Using B-spline representations, the control coefficients are known on the boundary. In Figure 3, we find that the outer control points of the volume parameterisation, wich are deduced from the control points of the boundary surfaces. The problem reduces to find the interior control points such that the map σ is a bijection between the computation domain D and the physical domain Ω . The injectivity of σ is verified if its Jacobian does not vanish on D . A sufficient condition for injectivity can be deduced from the relative position of the control points of the parametrisation. For simple shapes, such a parameterisation can be constructed from so-called Coons pacthes [2]. For more complex shapes, a solution can be found using standard linear programming techniques on the position of the controlled points. Such an approach is described in [4]. As a matter of fact, the choice of the free inner control points has an influence on the quality of approximation of the physical solution. As shown in [4], the optimal position of the control points is not necessarily the natural (or regular) one. New types of strategies combining the optimisation of the position of the inner control points with approximation of the solution and the estimation of the error can be considered.

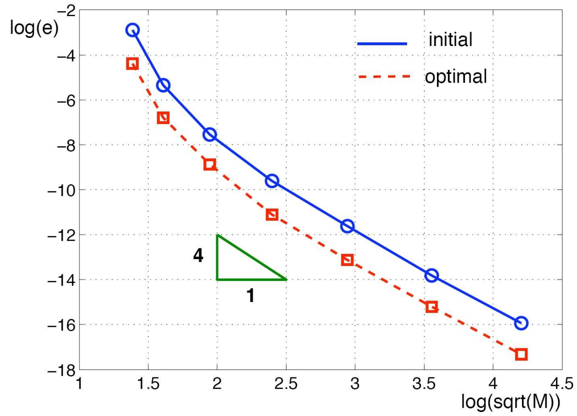
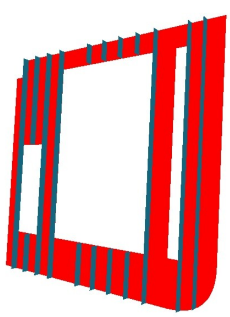


Figure 4. Error history during refinement for different parametrization of computational domain

Function space refinement. A standard and tranditional technique to improve the quality of approximation is to refine the computational domain. This process, also called h-refinement, consists to insert knots in the parametric domain, that is to add new control points. Note that it does not change the parametrisation map but increase the space of (B-spline) functions used to represent it. Thus it provides more freedom to better approximate the solution of our problem. An interesting characteristic of these approximation schemes is that their order of approximation is directly related to the degree of regularity of the bspline space. Figure 4 is an example of a planar heat conduction problem analysis, where the error is given in terms of the log of the square root of the number of control points: The two curves represent the L2 error of the approximated solution, when we use a natural position of the control points (plain blue curve) and we when we optimize their position (dash red curve). We observe that in both cases, the slopes of the curves tend to -4, which is the speed of approximation we expect using bicubic bspline functions.

Using tensor product B-spline functions has however some drawback in this context. When a knot is inserted in one direction of the parameter domain, we add not one but the number control points involved in the other parameter direction. This can lead to too many knot insertions if h-refinement operations are required in all parametric directions. Instead, we would like to have local parameter space refinements possibilities. To handle this problem, new types of B-spline basis are considered. So-called T-splines, introduced by T.W. Sederberg et al. [5], are a generalisation of rational bspline functions, which are associated to rectangular subdivisions of the parametric domain. These subdivisions allow to have T-junctions and to perform local refinements. They have interesting properties for the isogeometric approach [6]. Other types of T-splines which are piecewise polynomial [7] are also considered in isogeometric problems [8]. These spaces of T-splines are not completely under- stood. In particular, open questions remain on their dimension and the construction of explicit bases. Another family of B-spline functions is related to the triangular control meshes. These splines extend the concept of simplex splines to a triangular mesh, attaching a sequence of nodes to each vertex of the triangulation [9]. They allow to deal with 2D or 3D domains with arbitrary topology, and with arbitrary degree of regularity. Having the possibility to perform local refinement with these types of bspline functions is important in the isogeometric approach. To fully exploit these capacities, efficient local error estimators are however needed, which remains a difficult issue in numerical analysis, whatever the approach chosen to approximate the physical solution.

Multipatches. The geometry on which we want to perform the simulation may not be composed of one part that can be parametrized by a simple domain D. It may have holes or different pieces assembled in a non-manifold way:



This type of geometry requires a special treat-ment for the description of the parametrisations of the different parts of the object and the constraints that should be satisfied along the boundary of these different components. The topological structure and the geometric and functional basis description should be tightly linked in order to provide an efficient solution to the simulation problem. In particular, assembling the (stiffness) matrix E should be optimized according to the support of the basis functions.

V.CONCLUSION

The isogeometric approach is a promosing technique which represents in the same framework the geometry and for the physical functions on the geometry. By representing exactly the geometry, it avoids some numerical artefacts that can appear in finite element method with mesh approximation. It also leads to high order numerical approximation scheme, using basis functions such as splines. These piecewise polynomials functions which are heavily used in CAGD provide a uniform framework to describe the geometry and the solutions. Tranditional finite element techniques extend naturally to this new framework. Shape optimisation methods can be applied more efficiently by moving control points instead of nodes on the finite element mesh.

This recent approach rises interesting geometric modeling and representation challenges, that need to be addressed for further impact of isogeometry in scientific computing. This also implies some deep changes in the numerical tools and techniques involved in numerical computation, which is another challenge to address.

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