Approximation methods for the Plateau-Bézier problem

Xiao-Diao Chen[a,b], Gang Xu[a], Yigang Wang[a]
[a] *College of Computer,Hangzhou Dianzi University, Hangzhou,P.R. China*
[b] *State Key Lab of CAD&CG, Zhejiang University, Hangzhou, P.R. China*
xiaodiao@nit.zju.edu.cn

Abstract

The stretching energy functional and the bending energy functional are widely used for approximating the solution of the Plateau-Béizer Problem. This paper presents another two simple methods by using the extended stretching energy functional and the extended bending energy functional. The resulting surface obtained by the new methods will have a smaller area. Comparisons are made with both the area and the mean curvature of the resulting surfaces.

1. Introduction

The Plateau Problem is to find a surface that minimizes the area with prescribed border [2, 11]. There is the fact that the resulting minimal surfaces have zero mean curvature. Most of these minimal surfaces are not in the polynomial form. When the prescribed border curves are constrained to the Bezier curves and the required resulting surface is restricted within rectangular or triangular Bezier surfaces, the corresponding Plateau Problem is called the Plateau-Bézier Problem [8, 11]. The Plateau-Bézier Problem can be described as follows: given the control points of the boundary curves of a Bézier surface, to find the inner control points of the surface such that the resulting Bézier surface has a minimal area.

The constraint that the mean curvature is equal to zero is too strong. In most of the cases, there are none of such a rectangular Bézier surface [9] with zero mean curvature. In fact, among all bicubical Bézier surfaces, only the Enneper's surface has zero mean curvature [4, 9].

The Plateau-Bézier Problem is equivalent to minimize the area functional, which is highly nonlinear. Several energy functionals are used to approximate the area functional, which lead to easy management for the Plateau-Bézier problem. The first one is a stretching energy functional, which is also called Dirichlet functional in the mathematical literature [9]. The extremals of a Dirichlet functional can be obtained by solving linear systems [1, 5, 9]. As the degrees of the boundary curves increase, this extremals of the resulting surface converge to that of the exact minimal Bézier surface [8, 9]. Farin and Hansford proposed a mask derived from the discretization of the Laplacian operator for generating the control net of the resulting Bézier surface [3], which is also to solve a linear system. Bending energy functional [7] and mean curvature energy functional [10, 12, 13] are also used for approximating the solution of the Plateau-Bézier Problem. Most of the methods discuss both the rectangular case and the triangular case [6, 9, 10].

This paper focuses on the Plateau-Bézier Problem for rectangular Bézier surfaces. Given a surface $\mathbf{X}(u,v)$, we propose an extended stretching energy functional

$$||\mathbf{X}_u||^2 + ||\mathbf{X}_v||^2 + \lambda < \mathbf{X}_u, \mathbf{X}_v >$$

and an extended bending energy functional

$$||\mathbf{X}_{uu}||^2 + ||\mathbf{X}_{vv}||^2 + \alpha||\mathbf{X}_{uv}||.$$

Simple methods are provided to estimate the values of λ and α. When α or λ is determined, the resulting surface can be computed by solving a linear system. Examples show that the resulting surfaces from the new methods have smaller areas than those in the previous methods. Comparisons with mean curvature are also made among the resulting surfaces from different methods.

2. The extended Dirichlet method

2.1. The extended Dirichlet functional

A Bézier surface can be described by

$$\mathbf{X}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m}\mathbf{P}_{ij}B_n^i(u)B_m^j(v), \quad \text{u,v} \in [0,1], \qquad (1)$$

where $\{\mathbf{P}_{ij}\}$ are the control points. The area of the Bézier surface is

$$A(\mathbf{X}) = \int\int_R \sqrt{EG - F^2}dudv,$$

where $R = [0,1] \times [0,1]$, and E, F, G are the coefficients of the first fundamental form of the surface $\mathbf{X}(u,v)$. The corresponding Dirichlet functional is

$$D(\mathbf{X}) = \int\int_R (\frac{E+G}{2})dudv. \qquad (2)$$

We have that

$$(E+G)/2 \geq \sqrt{EG} \geq \sqrt{EG - F^2}. \qquad (3)$$

978-1-4244-3701-6/09/$25.00 ©2009 IEEE

From Eq.(3), $F = 0$ is a necessary condition such that $D(\mathbf{X})$ is equal to $A(\mathbf{X})$. The condition $F = 0$ may imposes too many restrictions on the resulting Bézier surface. We take the condition $F = 0$ into account to obtain an extended Dirichlet functional

$$E(\mathbf{X}) = \int\int_R (\frac{E+G}{2} + \lambda F) du dv, \quad -1 \leq \lambda \leq 1. \quad (4)$$

Note that $\sqrt{EG - F^2} \geq 0$, here we restrict $\lambda \in [-1, 1]$ to ensure that $(E + G)/2 + \lambda F \geq 0$. When λ is set to zero, then Eq. (4) is degenerated into Eq. (2), which is just a Dirichlet one. When the value of λ is determined, all the corresponding inner control points can be represented as functions in λ by directly solving a system of linear equations as the Dirichlet method does.

2.2. Estimating the value of λ

The main idea of the extended Dirichlet method is to find a suitable value of λ such that $|E(\mathbf{X}) - A(\mathbf{X})|$ is very close to zero. Then the minimal value of $E(\mathbf{X})$ is also close to the minimal value of $A(\mathbf{X})$, which may lead to a better result than that of the Dirichlet method. Given a value of λ, we obtain the resulting surface $\mathbf{X}_\lambda(u, v)$ by minimizing the functional $E(\mathbf{X})$. Let the area of $\mathbf{X}_\lambda(u, v)$ be $h(\lambda)$. Then we will estimate the value of λ such that $h(\lambda)$ reaches the minimal in the interval $[-1, 1]$. To simplify the estimation process, we approximate $h(\lambda)$ by its Taylor expansion approximation

$$\bar{h}(\lambda) = h(0) + h'(0)\lambda + h''(0)\lambda^2/2.$$

Suppose that the possible root of $\bar{h}'(\lambda)$ in the interval $[-1, 1]$ is

$$t_1 = -h'(0)/h''(0).$$

Then $\bar{h}(\lambda)$ must reach its minimal in the interval $[-1, 1]$ at -1, 1 or t_1. Finally, we set λ to be λ_0 such that $\bar{h}(\lambda)$ reaches its minimal in the interval $[-1, 1]$. It is obvious that

$$\bar{h}(\lambda_0) \leq \bar{h}(0) = h(0),$$

which means that the area of the resulting surface from the extended Dirichlet method is equal to or less than that of the resulting surface from the Dirichlet method.

3. The extended bending energy method

Suppose that the Bézier surface is determined by Eq.(1) and the extended bending energy functional is given by

$$B(\mathbf{X}) = ||\mathbf{X}_{uu}||^2 + ||\mathbf{X}_{vv}||^2 + \alpha||\mathbf{X}_{uv}||, \quad \alpha \in [-2, 2]. \quad (5)$$

Given a value of α, we obtain the resulting surface $\mathbf{X}_\alpha(u, v)$ by minimizing the functional $B(\mathbf{X})$. Let the area of $\mathbf{X}_\alpha(u, v)$ be $g(\alpha)$. Then we will estimate the value of α such that $g(\alpha)$ reaches the minimal in the interval $[-2, 2]$. $g(\alpha)$ can be simply approximated by its Taylor expansion approximation

$$\bar{g}(\alpha) = g(2) + g'(2)(\alpha - 2) + g''(2)(\alpha - 2)^2/2.$$

$\bar{g}(\alpha)$ is a quadratic polynomial in α. Suppose that α_0 is the place where the minimum value of $\bar{g}(\alpha)$ in the interval $[-2, 2]$ occurs. Then α_0 is possibly one of the three values, i.e., -2, 2 or $-g'(2)/g''(2)$. And we have that

$$\bar{g}(\alpha_0) \leq \bar{g}(2) = g(2),$$

which means that the area of the resulting surface from the extended bending energy method is equal to or less than that of the resulting surface from the bending energy method.

4. Examples and comparisons

The Dirichlet method obtains the resulting surface with the minimal stretching energy, while the bending energy method reaches the minimal bending energy. Both of them can be used for approximating the solution the Plateau-Bézier problem, which is to minimize the area of the resulting surface. Since the minimal surface in the Plateau problem has both the minimal area and zero mean curvature, we compare these methods with both area and mean curvature in the approximation problem of the Plateau-Bézier problem.

Fig. 1 shows four bicubic cases, and the comparisons with area and mean curvature are shown in Tables 1 and 2. In Tables 1 and 2, $\mathbf{M}_D$, $\mathbf{E}_D$, $\mathbf{M}_B$ and $\mathbf{E}_B$ denote the Dirichlet method, the extended Dirichlet method, the bending energy method and the extended bending energy method, respectively. $\text{Max}_{|H|}$ and $\text{Aver}_{|H|}$ denote the maximum and the average of the absolute value of the mean curvature of the resulting surface, respectively. As shown in Tables 1 and 2, we may be able to say:

1) the Dirichlet method seems to reach the resulting surface of a smaller area than that of the bending energy method;
2) the extended Dirichlet method could reduce the area of the resulting surface, but it seems to be not effective, for it reduces at most 0.02% in these four cases;
3) when the boundary curves are extracted from a Enneper's surface, the Dirichlet method and the extended Dirichlet method are able to reach the corresponding Enneper's surface (see Table 1, Fig.1(c));
4) in some cases, the extended bending energy method may be very effective in reducing the area of the resulting surface (see Table 1, Fig.1(d));
5) the resulting surfaces with smaller areas may have larger absolute values of mean curvature (see Table 2, Fig.1(d) and Fig.2).

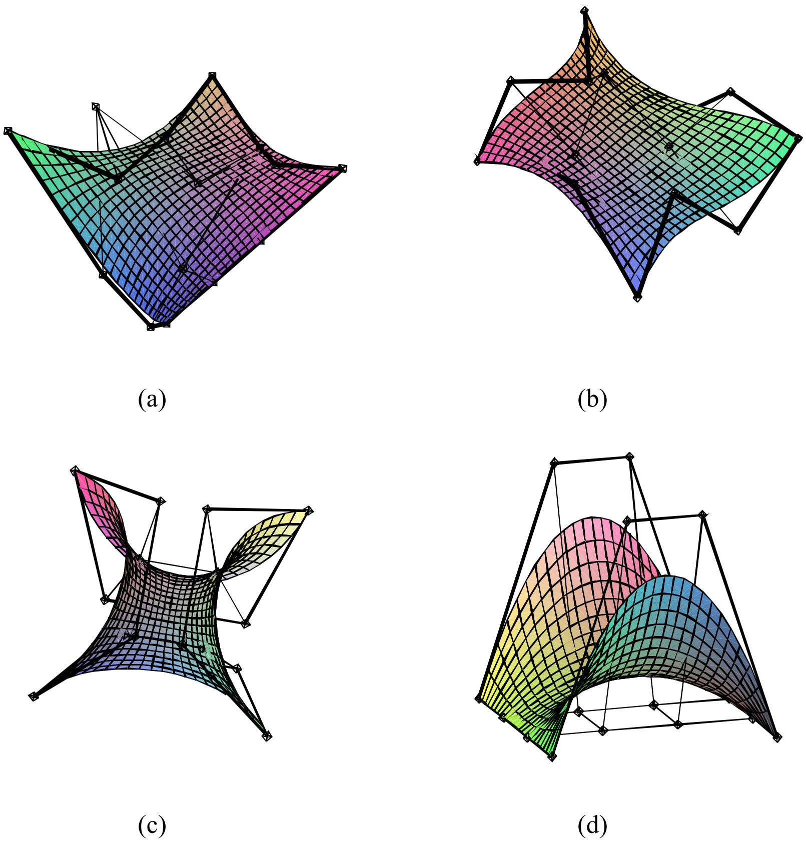


Figure 1. Examples of the extended Dirichlet method.

Table 1. Comparisons on areas from different methods

Example	$\mathbf{M}_D$	$\mathbf{E}_D$	$\mathbf{M}_B$	$\mathbf{E}_B$
Fig. 1(a)	180.8742	180.8316	181.2321	180.9807
	100%	99.98%	100.20%	100.05%
Fig. 1(b)	3303.0558	3302.3273	3307.4998	3304.8128
	100%	99.98%	100.13%	100.05%
Fig. 1(c)	110.4390	110.4390	111.2089	111.1068
	100%	100%	100.69%	100.60%
Fig. 1(d)	91.1152	91.1102	97.8972	87.4718
	100%	99.99%	107.44%	96.00%

5. Conclusions

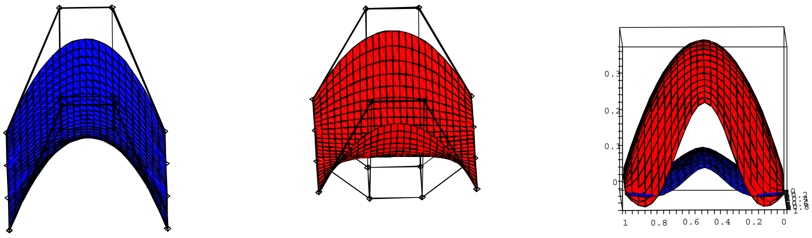
This paper discusses the approximation solution of the Plateau-Bézier Problem and presents two new methods, i.e., the extended Dirichlet method and the extended bending energy method. The resulting surfaces in the new methods are dependent on the parameters λ and α. Simple methods are provided to estimate the values of λ and α. Comparisons with both areas and mean curvatures of the resulting surfaces are also made in this paper. It shows that the resulting surfaces from the new methods could have smaller areas than those of the previous relative methods. In our future work, we will do more tests on the Bézier surfaces of higher degrees, and discuss the approximation solution of the Plateau-B-spline problem.

Acknowledgement

The research was partially supported by Chinese 973 Program (2004CB719400), the National Science Foundation of China (60803076), Zhejiang Provincial Science Foundation of China (2008C24014) and the Open Project Program of the State Key Lab of CAD&CG (A0804), Zhejiang University.

Table 2. Comparisons on mean curvatures from different methods

Example	$\mathbf{M}_D$		$\mathbf{E}_D$		$\mathbf{M}_B$		$\mathbf{E}_B$	
	$\mathrm{Max}_{\lvert H\rvert}$	$\mathrm{Aver}_{\lvert H\rvert}$	$\mathrm{Max}_{\lvert H\rvert}$	$\mathrm{Aver}_{\lvert H\rvert}$	$\mathrm{Max}_{\lvert H\rvert}$	$\mathrm{Aver}_{\lvert H\rvert}$	$\mathrm{Max}_{\lvert H\rvert}$	$\mathrm{Aver}_{\lvert H\rvert}$
Fig. 1(a)	0.0559	0.0066	0.0877	0.0127	0.0559	0.0066	0.0559	0.0123
Fig. 1(b)	0.0659	0.0056	0.0525	0.0053	0.0128	0.0060	0.0389	0.0048
Fig. 1(c)	0	0	0	0	0.0573	0.0247	0.0530	0.0226
Fig. 1(d)	0.1770	0.0744	0.1773	0.0749	0.0587	0.0286	0.3630	0.1784



(a) (b) (c)

Figure 2. Illustrations of Fig.1(d): (a) resulting surface from the bending energy method (b) resulting surface from the extended bending energy method (c) curvatures of the two methods.

References

[1] A Arnal, A Lluch, J Monterde. Triangular Bézier surfaces of minimal area. In *Proceedings of the Int. Workshop on Computer Graphics and Geometric Modeling, CG GM'2003, Montreal. In: Lecture Notes in Comput. Sci*, volume 2669, pages 366–375, 2003.

[2] M.P. do Carmo. Differential geometry of curves and surfaces. Prentice-Hall International, Englewood Cliffs, NJ, 1976.

[3] G. Farin, D Hansford. Discrete Coons patches. *Computer Aided Geometric Design*, 16(7):691–700, 1999.

[4] J Gallier. Curves and surfaces in geometric modeling. Morgan Kaufmann publishers, S. Francisco, California, 2000.

[5] GL Xu. Convergence analysis of a discretization scheme for Gaussian curvature over triangular surfaces. *Computer Aided Geometric Design*, 2:193–207, 2006.

[6] XF Gu, M Jin, F Luo. Computing general geometric structures on surfaces using Ricci flow. *Computer-Aided Design*, 8:663–675, 2007.

[7] YW Miao, HH Shou, JQ Feng, QS Peng, AR Forrest. Bézier surfaces of minimal internal energy. In *The Proceedings of the XI Conference on Mathematics of Surfaces*, volume 3604, pages 318–335, 2005.

[8] J Monterde. The Plateau-Bézier problem. In *The Proceedings of the X Conference on Mathematics of Surfaces, Leeds, UK. In: Lecture Notes in Comput. Sci.*, volume 2768, pages 262–273, 2003.

[9] J Monterde. Bézier surfaces of minimal area: The Dirichlet approach. *Computer Aided Geometric Design*, 21(2):117–136, 2004.

[10] G Xu. Researches on minimal surfaces and optimal method in CAD systems. PhD Thesis, Zhejiang University, Hangzhou, P.R. China, 2008.

[11] GL Xu. Geometric partial differential equation methods in computational geometry. Science Press, Beijing, P.R. China, 2008.

[12] GZ Xu, G Wang. Harmonic B-B surfaces over triangular domain. *Journal of Computers*, 12:2180–2185, 2006.

[13] Q Zhang, GL Xu. G^2 surface modeling using minimal mean-curvature-variation flow. *Computer-Aided Design*, 39(5):342–351, 2007.