Improper colouring of weighted hexagonal graphs

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A basic problem in the design of mobile telephone networks is to assign sets of radio frequency bands (colours) to transmitters (vertices) to avoid interference. The number $p(v)$ of bands demanded at transmitter v may vary between transmitters. We assume also that adjacent vertices must not be assigned the same band, so as to avoid interference. There are many more refined versions of this channel assignment problem, see for example [3, 6], in which we insist on a minimum separation between channels assigned to two transmitters (where this minimum separation may depend on the proximity of the transmitters). But we consider only the most basic case here.

The channel assignment problem described above is a colouring problem of a weighted graph. A *weighted graph* is a pair (G, p), where G is a graph and p a weight function on the vertex set of G. A *proper t-colouring* of a weighted graph (G, p) is a mapping $C : V(G) \to \mathcal{P}(\{1, \ldots, t\})$ such that for every vertex $v \in V(G)$, $|C(v)| = p(v)$ and for all edges $uv \in E(G)$, $C(u) \cap C(v) = \emptyset$. The *chromatic number* of a weighted graph (G, p), denoted by $\chi(G, p)$, is the least integer t such that (G, p) admits a proper t-colouring. This is a natural generalization of the chromatic number of a graph since $\chi(G, \mathbf{1}) = \chi(G)$ (for every positive integer k, we denote by $\mathbf{k}$ the appropriate all k's function).

The *clique number* of a weighted graph (G, p), denoted by $\omega(G, p)$, is the maximum weight of a clique, that is $\max\{p(C) \mid C \text{ clique of } G\}$, where $p(C) = \sum_{v \in C} p(v)$.

Trivially, we have

$$\chi(G, p) \geq \omega(G, p).$$

The triangular lattice crops up naturally in radio channel assignment. It is sensible to aim to spread the transmitters out to form roughly a part of a triangular lattice, with hexagonal cells, since that will give the best "coverage", that is, for a given number of transmitters in a given area this pattern minimizes the maximum distance to a transmitter. The triangular lattice graph may be described as follows. The vertices are all integer linear combinations $a\mathbf{e_1} + b\mathbf{e_2}$ of the two vectors $\mathbf{e_1} = (1, 0)$ and $\mathbf{e_2} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Thus we may identify the vertices with the pairs (a, b) of integers. Two vertices are adjacent when the Euclidean distance between them is 1. See Figure 1. A *hexagonal*

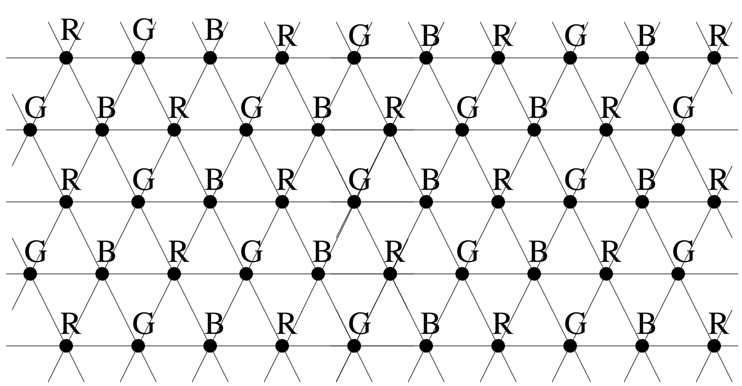

Figure 1: The triangular lattice.

graph is an induced subgraph of the triangular lattice.

McDiarmid and Reed [4] showed that it is NP-complete to decide whether the chromatic number of a weighted hexagonal graph is 3 or 4. Hence, there is no polynomial-time algorithm for finding the chromatic number of weighted hexagonal graphs (unless P=NP). Therefore, one has to find approximate algorithms. The better known so far has approximation ratio $4/3$ and is based on the following result:

Theorem 1 (McDiarmid and Reed [4])**.** *For any weigthed hexagonal graph G,*

$$\chi(G,p) \leq \frac{4\omega(G,p)+1}{3}.$$

A distributed algorithm which guarantees the $\frac{4}{3}\omega(G,p)$ bound is reported by Narayanan and Schende [5]. However, one expects to have approximate algorithms with ratios better than $4/3$. In particular, Reed and McDiarmid conjecture that, for big weights, the ratio may be decreased to almost $9/8$.

Conjecture 2 (McDiarmid and Reed [4])**.** There exists a constant c such that for any weigthed hexagonal graph (G,p),

$$\chi(G,p) \leq \frac{9}{8}\omega(G,p)+c.$$

Note that the ratio $9/8$ in the above conjecture is the best possible. Indeed, consider a 9-cycle C_9 with constant weight k. A colour can be assigned to at most 4 vertices, so $\chi(C_9,\mathbf{k}) \geq \frac{9k}{4}$. Clearly, $\omega(C_9,\mathbf{k}) = 2k$. So $\chi(C_9,k) \geq \frac{9}{8}\omega(C_9,\mathbf{k})$.

A first aim of the the internship would be to improve the ratio $4/3$ of Theorem 1 and to find a distributed algorithm which uses at most $\alpha \cdot \chi(G,p)+\beta$ colours with $\alpha < 4/3$ and β an arbitrary constant.

Another problem posed by Alcatel Space Technologies (see [1]) can be modelled into a similar colouring problem. A satellite sends informations to receivers on earth, each of which is listening several frequencies, one for each signal it needs to receive. Technically it is impossible to focus a signal sent by the satellite exactly on the destination receiver. So part of the signal is spread in an area around it, creating noise for the other receivers displayed in this area and listening the same frequency. Each receiver is able to distinguish the signal directed to it from the extraneous noises it picks up if the sum of the noises does not become too large, i.e. does not exceed a certain threshold T. The problem is to assign frequencies to the signals in such a way that each receiver gets its dedicated signals properly, while minimizing the total number of frequencies used.

Generally the "noise relation" is symmetric, that is if a receiver u is in the noise area of a receiver v then v is in the noise area of u. Hence, interferences may be modelled by a *noise graph* $G = (V(G),E(G))$ whose vertices are the receivers and where two vertices are joined by an edge if and only if they interfer. Moreover, to the graph is attached a *weight function* $p : V(G) \to \mathbb{N}$, where the weight $p(v)$ of the vertex v is equal to the number of signals it has to receive. Hence we have a *weighted graph*. We can model the frequencies by colours. Therefore, to each vertex, we associate a set $C(v)$ of $p(v)$ distinct colours. If in total l colours are used, C is an *l-colouring* of (G,p).

In a simplified version, the intensity I of the noise created by a signal is independent of the frequency and the receiver. Hence to distinguish its signal from noises, a receiver must be in the noise area of at most $k = \lfloor \frac{T}{I} \rfloor$ receivers listening signals on the same frequency. In terms of colouring this property is equivalent to say that for any colour c, the set of vertices having one colour c induces a graph of degree at most k. Such a colouring is called *k-improper*. The *k-improper chromatic number* of (G,p), denoted $\chi_k(G,p)$, is the smallest l such that (G,p) admits a k-improper l-colouring. Note that a 0-improper colouring corresponds to a proper colouring.

In this problem too, hexagonal graphs are interesting because the transmitters are often spread out like the vertices of the triangular lattice.

First results on improper colouring of hexagonal graphs are reported in [2]. A second goal of the internship will be to design distributed algorithms to k-improper colour hexagonal graphs. A first step could be to prove results analogous to Theorem 1.

References

[1] S. Alouf, E. Altman, J. Galtier, J.-F. Lalande, and C. Touati, Quasi-optimal bandwidth allocation for multi-spot MFTDMA satellites, in *Proc. IEEE INFOCOM* 2005, Miami, FL, 2005.

[2] J. Araujo, J-C. Bermond, F. Giroire, F. Havet, D. Mazauric, and R. Modrzejewski. Weighted improper colouring. *Journal of Discrete Algorithms*, 16:53-66, 2012.

[3] D. Král. The channel assignment problem with variable weights. *SIAM J. Discrete Math.*, 20(3):690–704 (electronic), 2006.

[4] C. McDiarmid and B. Reed. Channel assignment and weighted coloring. *Networks* 36:114–117, 2000.

[5] L. Narayanan and S. Schende. Static Frequency Assignment in Cellular Networks. In *Sirocco 97, (Proceedings of the 4th international Colloqium on structural information and communication complexity, Ascona, Switzerland)*, D. Krizanc and P. Wildmayer (eds.), Carleton Scientific pp. 215–227, 1997.

[6] R. K. Yeh. A survey on labeling graphs with a condition at distance two. *Discrete Math.*, 306(12):1217–1231, 2006.