# Topological test graphs and the geometry of stable Kneser graphs

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SGT 2015

# Schrijver graphs

$k \geq 0,\ \ell \geq 1,\ m = 2\ell + k.$

- $V(KG_{m,\ell}) = \{x \subset \mathbb{Z}_m \colon |x| = \ell\}$, $S \sim T \iff S \cap T = \emptyset$.
- $V(SG_{m,\ell}) = \{S \in V(KG_{m,\ell}) \colon \{i, i+1\} \not\subset S$ for all $i \in \mathbb{Z}_m\}$.
- $KG_{m,1} = SG_{m,1} \cong K_m$.
- $k = 0$: $SG_{2\ell,\ell} \cong K_2$.
- $k = 1$: $SG_{2\ell+1,\ell} \cong C_{2\ell+1}$.
- $\chi(SG_{m,\ell}) \leq \chi(KG_{m,\ell}) \leq k + 2$.
- Lovász: $\chi(KG_{m,\ell}) \geq k + 2$.
- Schrijver: $\chi(SG_{m,\ell}) \geq k + 2$.

# The Bárány-Schrijver construction

Given vectors $w_0, \dots, w_{m-1}$, set

$$U_S = \left\{ x \in \mathbb{S}^k \colon \langle x, w_j \rangle > 0 \text{ f.a. } j \in S \right\}, \quad S \in V(KG_{m,\ell}).$$

Then

- $U_S \cap -U_T \neq \emptyset \Rightarrow S \sim T$,
- if $G$ is an induced subgraph and

  $$\bigcup_{S \in V(G)} U_S = \mathbb{S}^k$$

  then $\chi(G) \geq k + 2$.

# The moment curve

Define

$$f\colon \mathbb{R} \to \mathbb{R}^{k+1},$$
$$f(t) = (1, t, \dots, t^k).$$

For $x \in \mathbb{R}^{k+1}$,

$$\langle x, f(t) \rangle = x_0 + x_1 t + \cdots + x_k t^k$$

is a polynomial in $t$ of degree $\leq k$.

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For $m > k$ let

$$t_0 < \cdots < t_{m-1}, \quad v_j := f(t_j).$$

Study

$$\mathcal{L} := \left\{(\operatorname{sgn}\langle x, v_0\rangle, \ldots, \operatorname{sgn}\langle x, v_{m-1}\rangle)\colon x \in \mathbb{R}^{k+1} \setminus \{0\}\right\} \subset \{+, -, 0\}^m.$$

# The alternating oriented matroid.

## Definition

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partially ordered set (poset) with $0 < +$, $0 < -$.

## Proposition

*Let $\mathcal{M}$ be the set of sign vectors such that*

- $|\{j \colon s(j) = 0\}| = k,$
- *Whenever $i < j$, $s(i) \neq 0 \neq s(j)$,*

$$s(i)s(j) = (-1)^{|\{r \colon i < r < j,\ s(r) = 0\}|}$$

.

*Then*

$$\mathcal{L} = \left\{ s \colon \mathit{Ex.}\ s' \in \mathcal{M}\ \mathit{s.t.}\ s \geq s' \right\}.$$

# Back to Schrijver. . .

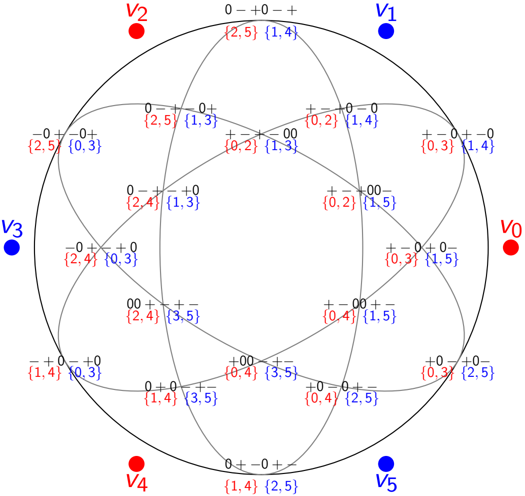
**Proposition**

*$s \in \mathcal{L}$ minimal,*

$$S_\pm := \left\{ j \colon (-1)^j s(j) = \pm \right\},$$

*then $(S_+, S_-) \in E(SG_{m,\ell})$.*

# The Bárány-Schrijver construction




$$\ell = 2,\ k = 2,\ m = 2\ell + k = 6$$

# Consequences

Set $w_j := (-1)^j v_j \in \mathbb{R}^{k+1}$.

$$U_S = \left\{ x \in \mathbb{S}^k \colon \langle x, w_j \rangle > 0 \text{ f.a. } j \in S \right\}.$$

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## Proposition

*Let $M \subset V(SG_{m,\ell})$. TFAE*

1. $\bigcap_{S \in M} U_S \neq \emptyset$.
2. *Ex. $T \in V(SG_{m,\ell})$ s.t. $T \sim S$ f.a. $S \in M$.*

Lecture one ends here.

# Outline

Lovász' original proof:

1. Definition *neighbourhood complex* $\mathcal{N}(G)$.
2. $\mathcal{N}(G)$ is $(k-1)$-connected $\Rightarrow \chi(G) \geq k+2$.
3. $\mathcal{N}(KG_{m,\ell})$ is $(k-1)$-connected, $m = 2\ell + k$.

Later:

4. Definition *box complex* $B(G)$.
5. $B(G) \simeq \mathcal{N}(G)$.

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Goals:

- (1)
- (3) for $SG_{m,\ell}$
- (4)
- (5)
- (2) for $B(G)$

# The neighbourhood complex

## Definition

Let $G$ be a graph.

$$\nu\colon \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$$
$$E \mapsto \{u\colon u \sim v \text{ f.a. } v \in E\}$$

The *neighbourhood complex of G* is the abstract simplicial complex

$$\mathcal{N}(G) = \{S \subset \mathcal{P}(V(G))\colon S \neq \emptyset,\, \nu(S) \neq \emptyset\}\,.$$

## Examples

- $\mathcal{N}(K_n) \approx$

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- $\mathcal{N}(K_{m,n}) \approx$

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- $\mathcal{N}(K_n) \approx \mathfrak{d}\Delta^{n-1} \approx \mathbb{S}^{n-2}$.
- $\mathcal{N}(K_{m,n}) \approx \Delta^{m-1} + \Delta^{n-1} \simeq \mathbb{S}^0$.

# The Nerve Theorem

**Theorem (Nerve Theorem)**

*Let $X$ be a metric space $\mathcal{C}$ a finite open cover of $X$ and*

$$\bigcap M \text{ empty or contractible for all } M \subset \mathcal{C},\ M \neq \emptyset$$

*($\mathcal{C}$ is a good cover). Let $\mathcal{S}$ be the abstract simplicial complex*

$$\mathcal{S} := \left\{ M \in \mathcal{P}(\mathcal{C}) \setminus \{\emptyset\} : \bigcap M \neq \emptyset \right\},$$

*the nerve of the cover. Then*

$$|\mathcal{S}| \simeq X.$$

# The neighbourhood complex of a Schrijver Graph

$m = 2\ell + k$.

Earlier we had for suitable vectors $w_0, \ldots, w_{m-1}$ and
$U_S := \left\{x \in \mathbb{S}^k \colon \langle x, w_j\rangle > 0 \text{ f.a. } j \in S\right\}, \quad S \in V(SG_{m,\ell})$:

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So $(U_S)_{S \in V(SG_{m,\ell})}$ is a good cover of $\mathbb{S}^k$ and its nerve is $\mathcal{N}(SG_{m,\ell})$.

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**Theorem (Björner, de Longueville (1999))**

$\mathcal{N}(SG_{m,\ell}) \simeq \mathbb{S}^k$.

In particular, $\mathcal{N}(SG_{m,\ell})$ is $(k-1)$-connected.

# The box complex

## Definition

$G$ a graph. $B(G)$ is a completely regular cell complex with face poset

$$\left\{(A,B) \in (\mathcal{P}(V(G)) \setminus \{\emptyset\})^2 \colon \nu A \supset B\right\},$$

where $(A,B) \leq (A',B') \iff A \subset A', B \subset B'$.

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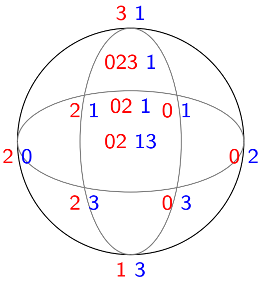


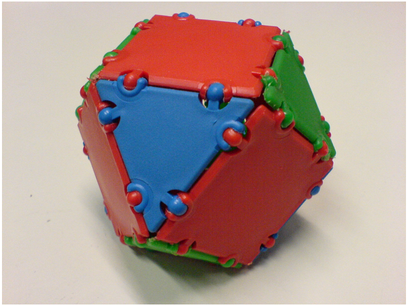

$B(K_4)$

# Barycentric subdivision

## Definition

The *order complex* of a poset $P$ is the abstract simplicial complex

$$\Delta(P) := \{C \subset P \colon C \neq \emptyset,\ C \text{ a chain.}\}$$

## Proposition and definition

*Let $K$ be a completely regular cell complex with face poset $FK$, then $\Delta(FK)$ is called its barycentric subdivision and*

$$|\Delta(FK)| \simeq |K|.$$

.

# Functorial properties of the box complex

Let $C_2 = \{e, \tau\}$ be the group with two elements.

## Observation and definition

*On $B(G)$ a $C_2$-action is defined by $\tau \cdot (A, B) := (B, A)$. If $G$ has no loops, then this is a free action.*

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*If $f\colon G \to H$ is a graph homomorphism then*

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Conclusion:
If there is no $C_2$-map $|B(G)| \to |B(H)|$, then there is no graph homomorphism $G \to H$.

# Connectivity and equivariant maps.

## Proposition

*Let $K$ be a free $C_2$-cell complex of dimension $\leq k$, and $X$ a $(k-1)$-connected $C_2$-space. Then there is a $C_2$-map*

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Assuming for a minute that we know that $B(K_n)$ is an $(n-2)$-sphere (or at least $(n-3)$-connected) we obtain:

## Theorem (The Lovász criterion)

*Let $G$ be a graph, $k \geq 0$. If $|B(G)|$ is $(k-1)$-connected, then $\chi(G) \geq k+2$.*

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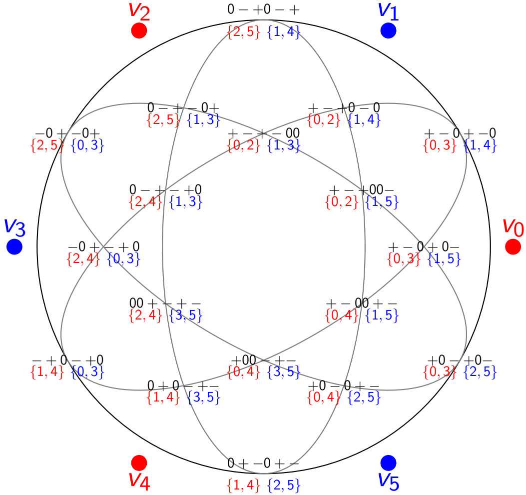
and again:

**Proposition**

*Let $m = 2\ell + k$. Then $\chi(SG_{m,\ell}) = k+2$.*

# And yet again...

We can see a $C_2$-map $\mathbb{S}^k \to SG_{m,\ell}$ directly.

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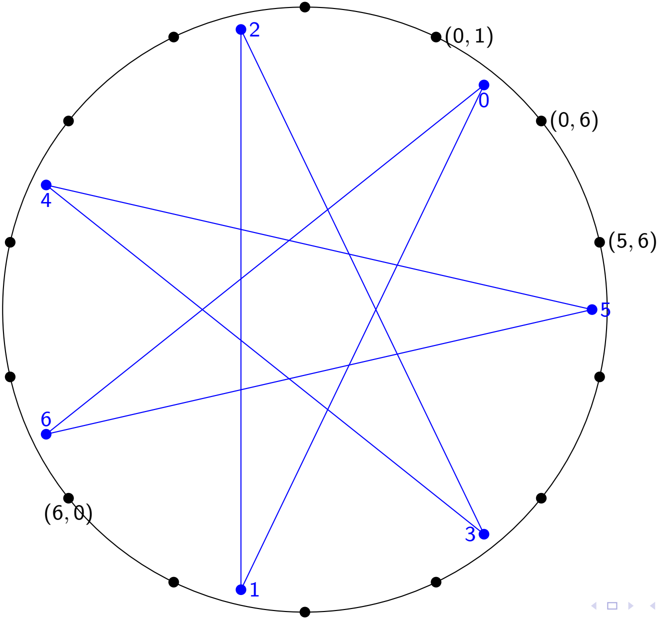
# The box complex and the neighbourhood complex

**Proposition**

*Let $G$ be a graph. Then $|B(G)| \simeq |\mathcal{N}(G)|$.*

# End of part 2

# $\mathrm{Hom}(K_2, C_{2n+1})$



End of part 3.

# Test graphs.

Is there a map

$$\mathbb{S}^1 \times \mathbb{S}^r \xrightarrow{\gamma} \mathbb{S}^r$$

as follows?
With $\rho(x_0, x_1) = (-x_0, x_1)$,

$$\gamma(-x, y) = -\gamma(x, y), \qquad \gamma(\rho(x), y) = \gamma(x, -y).$$

# Test graphs.

**Proposition**

*With* $(x_0, \ldots, x_k) \cdot \tau = (-x_0, x_1, \ldots, x_k)$ *there is no map*

$$\mathbb{S}^k \times_{C_2} \mathbb{S}^r \to_{C_2} \mathbb{S}^{r+k-1}.$$

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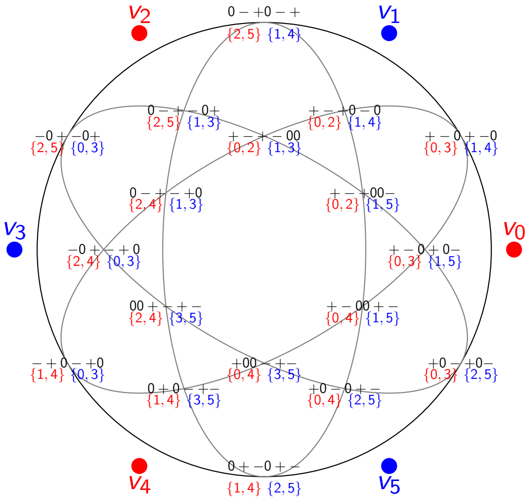
**Proposition**

$SG_{2\ell+2,\ell}$ *is a test graph.*

# One more time

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# Prettier pictures = more symmetry

Let $k = 2r$.
We consider vectors of the form

$$v = (1, z, z^2, \ldots, z^r) \in \mathbb{R} \times \mathbb{C}^r = \mathbb{R}^{k+1}, \quad z \in \mathbb{S}^1 \subset \mathbb{C}.$$

Then for $x \in \mathbb{R} \times \mathbb{C}^r$,

$$2\langle x, v\rangle_{\mathbb{R}} = 2x_0 + x_1\bar{z} + \bar{x}_1 z + \cdots + x_r\bar{z}^r + \bar{x}_r z^r$$
$$= 2\langle x, v\rangle_{\mathbb{R}} = 2x_0 + x_1 z^{-1} + \bar{x}_1 z + \cdots + x_r z^{-r} + \bar{x}_r z^r$$
$$= z^{-r}(x_r + x_{r-1}z + \cdots + x_1 z^{r-1} + 2x_0 z^r + \bar{x}_1 z^r + \cdots + \bar{x}_r z^{2r}) = z^{-r} p_x(z).$$

Again, $p_x$ is a polynomial of degree $k \Rightarrow$ at most $k$ zeros.
And

$$\frac{d}{dt}\exp(it)^{-r} p_x(\exp(it)) = i\exp(-irt)(-rp_x(\exp(it)) + \exp(it)p_x'(\exp(it)))$$

$\Rightarrow$ sign change at every simple root of $p'$.

# Symmetry.

$k = 2r$, $m = 2\ell + k$.

Setting $v_j = (1, \xi^j, \xi^{2j}, \ldots, \xi^{rj})$ with $\xi = \exp(2\pi i/m)$ we obtain the same sign vectors as before, but we can write down $S, R \in \mathrm{O}(k+1)$ such that

- $S(v_j) = S(v_{j+1}), \quad (\mathrm{mod}\ m)$,
- $R(v_j) = v_{-j}$.

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$k = 2r + 1$, $m = 2\ell + k$.

Set $v_j = (\xi^j, \xi^{3j}, dots, \xi^{(2r+1)j})$ with $\xi = \exp(\pi i/m)$.
Proceed similarly.

# Detecting 1-test graphs

**Definition**

$T$ is a 1-*test graph* if for every graph $G$ such that $\mathrm{Hom}(T, G)$ is 0-connected the inequality $\chi(G) \geq \chi(T) + 1$ holds.

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## Proposition

*Let* $\chi(T) = k + 2$*. If there exist* $f : \mathbb{S}^k \to_{C_2} \mathrm{Hom}(K_2, T)$, $A \in \mathrm{O}(k+1)$, $\det A = -1$ *and* $\gamma \in \mathrm{Aut}(T)$ *s.t.* $\mathrm{Hom}(K_2, \gamma) \circ f = f \circ A$ *then* $T$ *is a* 1*-test graph.*

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## Proposition

*Let* $k \geq 0$, $\ell \geq 1$, $m = 2\ell + k$.
*If* $k \not\equiv 3 \pmod 4$ *then* $SG_{m,\ell}$ *is a* 1-*test graph.*

# Proof by inspection

## Definition

Let $\ell \geq 1,\ k \geq 0,\ m = 2\ell + k$,

$$R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

For $k = 2r$ we have

$$w_{j+1} = -\operatorname{diag}(1, R_{2\pi/m}, R_{4\pi/m}, \dots, R_{k\pi/m}) \cdot w_j,$$
$$w_{m-j} = \operatorname{diag}(1, 1, -1, \dots, 1, -1) \cdot w_j.$$

For $k = 2r + 1$ we have

$$w_{j+1} = -\operatorname{diag}(R_{\pi/m}, R_{3\pi/m}, \dots, R_{k\pi/m}) \cdot w_j,$$
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# Total test graph failure

## Claim

*Let* $k \geq 0$, $\ell > 1$, $m = 2\ell + k$, $k \equiv 3 \pmod 4$ *then* $SG_{m,\ell}$ *is not a* 1*-test graph.*
*I.e. there exists a graph* $G$ *with* $\chi(G) = \chi(SG_{m,\ell})$ *such that* $\mathrm{Hom}(SG_{m,\ell}, G)$ *is non-empty and connected.*

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**Proposition**

*Let* $k \geq 0$, $\ell > 1$, $m = 2\ell + k$, $k \equiv 3 \pmod 4$ *then there is a colouring* $c \colon SG_{m,\ell} \to K_{k+2}$ *such that all* $c \circ \gamma$, $\gamma \in \mathrm{Aut}(SG_{m,\ell})$ *are in the same component of* $\mathrm{Hom}(SG_{m,\ell}, K_{k+2})$.

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*I.e. there exists a graph* $G$ *with* $\chi(G) = \chi(SG_{m,\ell})$ *such that* $\mathrm{Hom}(SG_{m,\ell}, G)$ *is non-empty and connected.*

## Proposition

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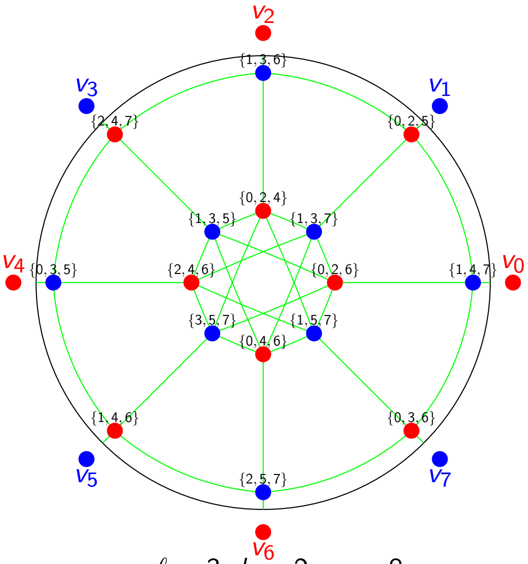
- An elementary proof exists.
- A proof for large enough $\ell$ follows.

# A dual construction




$\ell = 3,\ k = 2,\ m = 8$

$$v\colon V(SG_{m,\ell}) \to_{D_{2m}} \mathbb{S}^k,$$

$$S \mapsto \frac{\sum_{j\in S} w_j}{\left\|\sum_{j\in S} w_j\right\|}.$$

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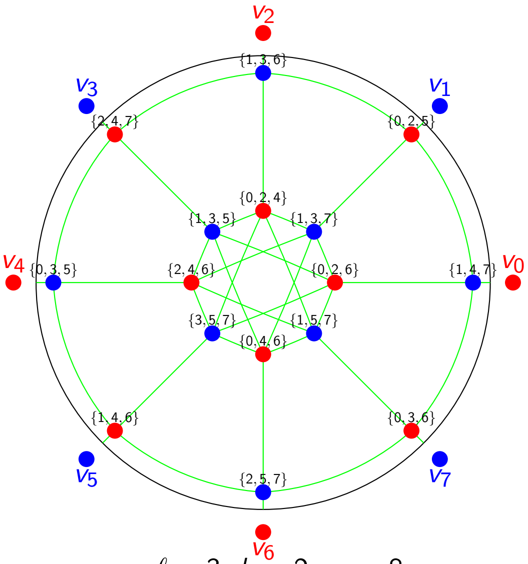


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f. a. $(S, T) \in E(SG_{m,\ell})$.

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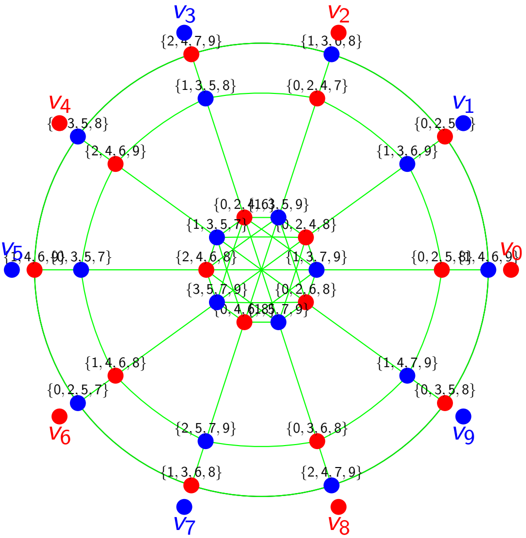


$\ell = 4,\ k = 2,\ m = 10$

$$v\colon V(SG_{m,\ell}) \to_{D_{2m}} \mathbb{S}^k,$$

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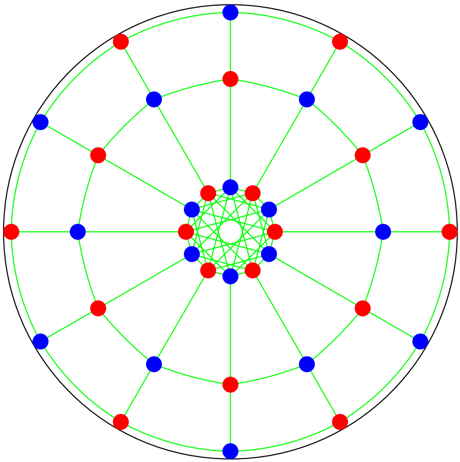

$\ell = 5,\ k = 2,\ m = 12$

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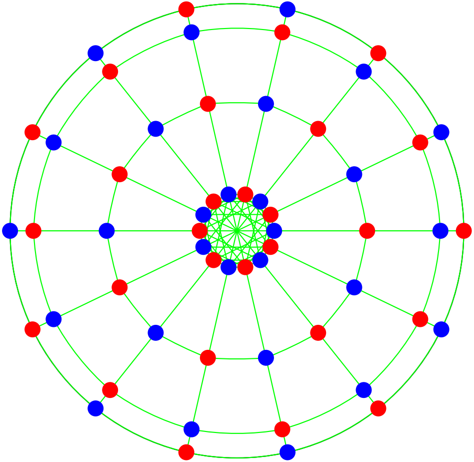

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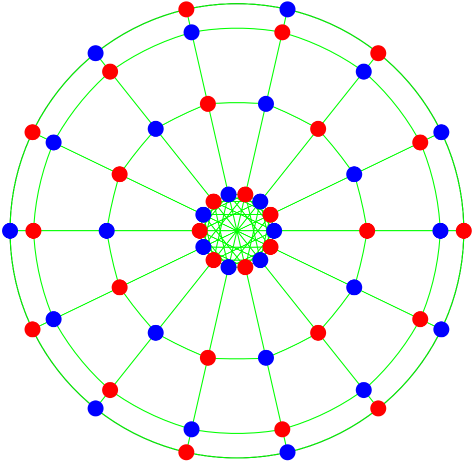

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Any closed covering $(A_j)_{j\in J}$ of $\mathbb{S}^k$ with $\operatorname{dist}(A_j, -A_j) > \varepsilon$ f.a. $j$ yields $SG_{m,\ell} \to K_{|J|}$.

# Constructing a counter-example.

**Proposition**

*Let $T$ be a finite graph such that $\mathrm{End}(T) = \mathrm{Aut}(T)$. If there is a component of $\mathrm{Hom}(T, K_{\chi(T)})$ which is $\mathrm{Aut}(T)$-invariant, then there exists a graph $G$ with $\chi(G) = \chi(T)$ such that $\mathrm{Hom}(T, G)$ is non-empty and connected.*

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- $\mathrm{Aut}(SG_{m,\ell}) = D_{2m}$ (Braun)

In higher dimensions. . .

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# What else we know

**Theorem**

*Let $\ell > 0$, $k \geq 0$. If $k \in \{0, 1, 2\}$ or if $k = 4$ and $\ell$ is even, then $SG_{2\ell+k,\ell}$ is a homotopy test graph.*

**Theorem**

*Let $k \geq 0$, $k \notin \{0, 1, 2, 4, 8\}$. Then there is an $N > 0$ such that for no $\ell \geq N$ the graph $SG_{2\ell+k,\ell}$ is a homotopy test graph.*
*Also, there is an $N > 0$ such that for no odd $\ell \geq N$ the graph $SG_{2\ell+8,\ell}$ is a homotopy test graph.*

# Thank you!

Please write to me for comments and questions.