Simplicial complexes and connectivity

Basics

A hypergraph H=(V,E) is a pair, where V is a set of elements, and E a collection of (finite) subsets of V, called *edges*.

We sometimes identify H with E.

H is called r-uniform if all edges are of the same size r. We also say then that H is an *r-graph.*

A wonderful fact: hypergraphs can be realized geometrically

And the geometric properties reflect on combinatorial properties.

In particular – the holes.

A hypergraph with no holes behaves very nicely.

For example, a graph with no holes is a tree.

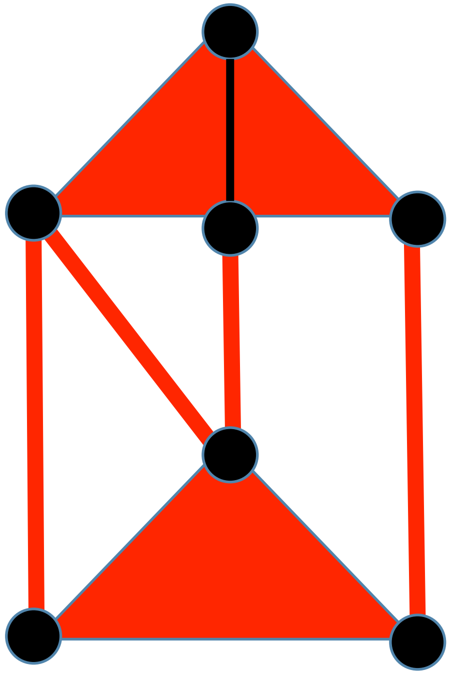
Simplicial complexes

- Fact: two affine subspaces of dimension at most k in R^{2k-1} in general position do not meet.
- Example: two random lines in R^3 do not meet.

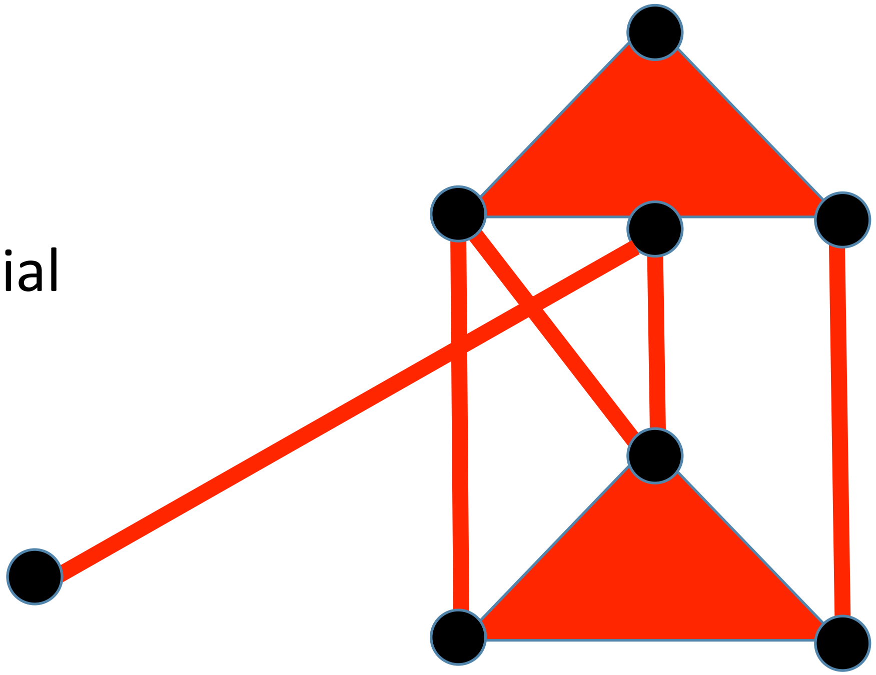
Geometric realization of hypergraphs

- Given a hypergraph H of rank (=maximal size of an edge) r, put its points in general position in R^{2r-1} and for every edge e of H take the convex hull of its points.
- The resulting collection of simplices (=lines, triangles, tetrahedra…) is called a “simplicial complex”

A simplicial complex in R^2



A non- simplicial complex in R^2



Holes

A hole of dimension *n* in *X* is an empty S^{n-1}

Namely an image of a sphere, that cannot be filled in X.

Formally: a function $f : S^{n-1} \to X$

that cannot be extended to a function $\tilde{f} : B^{n+1} \to X$

Connectivity

The topological connectivity

of a set X in Euclidean space is the smallest dimension of a hole in *X.*

It is denoted by $\eta(X)$.

Example: $\eta(S^n) = n + 1$

Connectivity

The connectivity of a complex C, denoted by $\eta(C)$ is the smallest dimension of a hole in C.

Examples:

C=  $\eta(C) = \infty$

C =  $\eta(C) = 1$

Examples

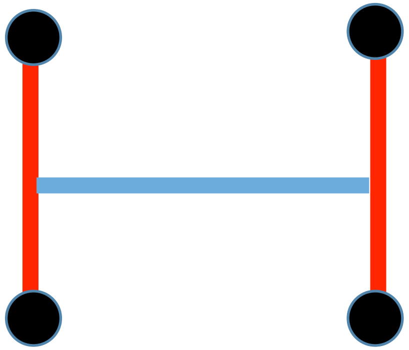
- $\eta(X) \geq 1$ if X is non empty (convention).

- $\eta(X) \geq 2$ if X is connected (usual sense) – for every two points (image of S^0) there is a filling line between them.

- $\eta(X) \geq 3$ means that X is simply connected.

- $\eta(X) = \infty$ if X has no holes (namely it is contractible).

Example:

- C=  $\eta(C) = 1$

Joins

- The “join” of two complexes A,B, is:

$$A * B := \sigma \cup \tau \sigma \in A,\ \tau \in B$$

Fact:

$$\eta(A * B) = \eta(A) + \eta(B)$$

(Well, almost. But sometimes $\eta(A * B) \geq \eta(A) + \eta(B)$)

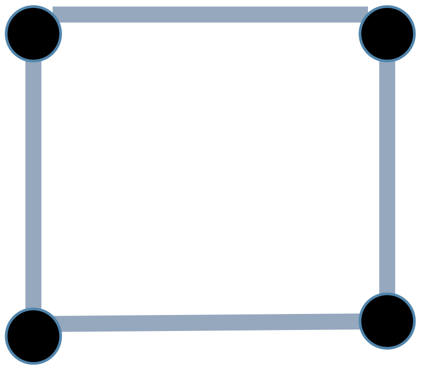
Example

- $\eta(S\upharpoonleft 0)$=1
- $S\upharpoonleft n = S\upharpoonleft 0 * S\upharpoonleft 0 * \ldots * S\upharpoonleft 0$
- And indeed $\eta(S\upharpoonleft n) = 1 + 1 + \ldots + 1$.

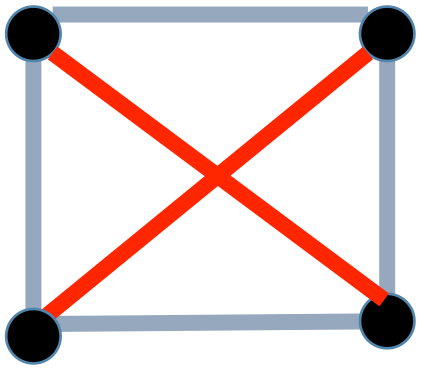
Examples: connectivity of independence complexes

The independence complex of a graph G, denoted by $I(G)$, is the collection of independent sets in G.

C_4

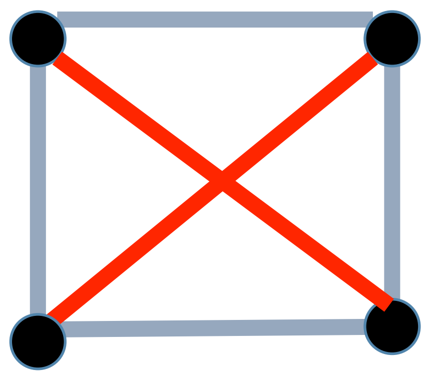


C_4

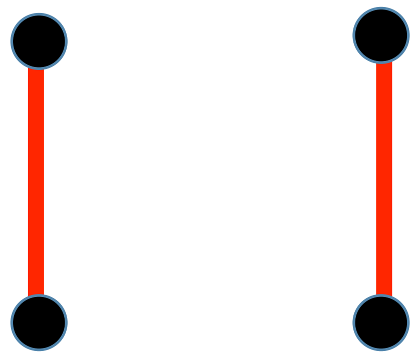


$I(C_4)$

C_4



$I(C_4)$

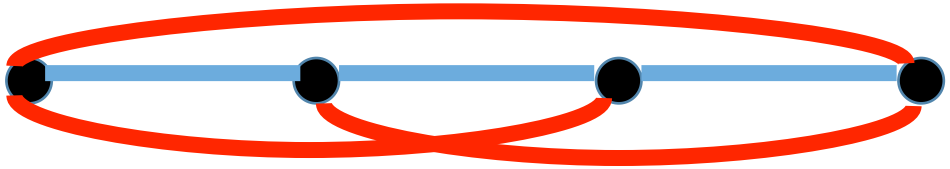


$$\eta(I(C_4)) = 1 \ (disconnected)$$

P_4

$I(P_4)$

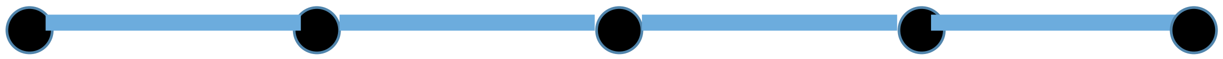
$I(P_4)$



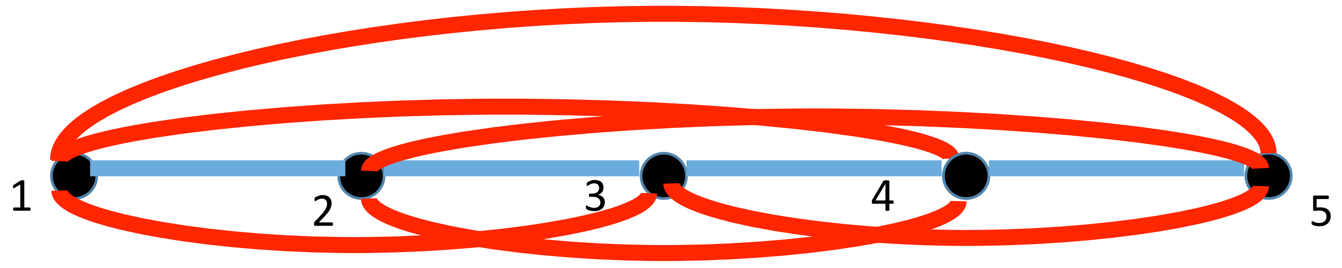
$I(P_4)$

$$\eta(I(P_4)) = \infty$$

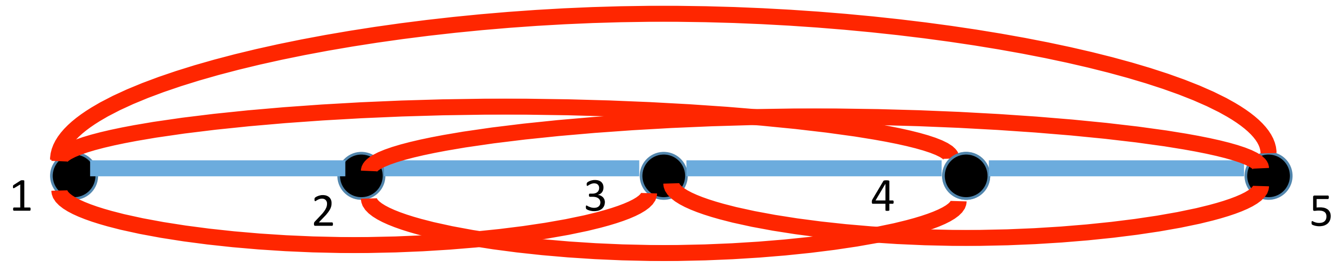
P_5



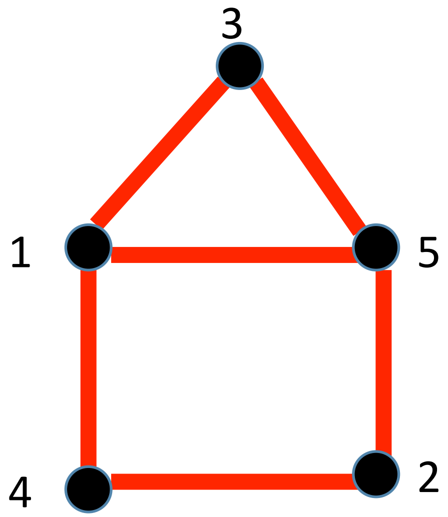
P_5



P_5

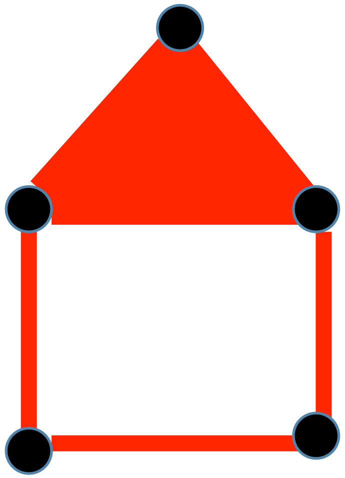


$I(P_5)$



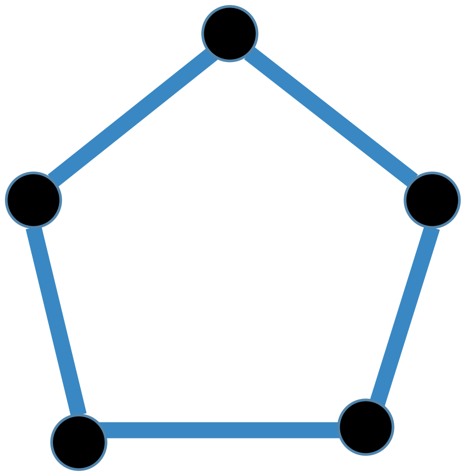
$$\eta(I(P_5)) = 2$$

$I(P_5)$

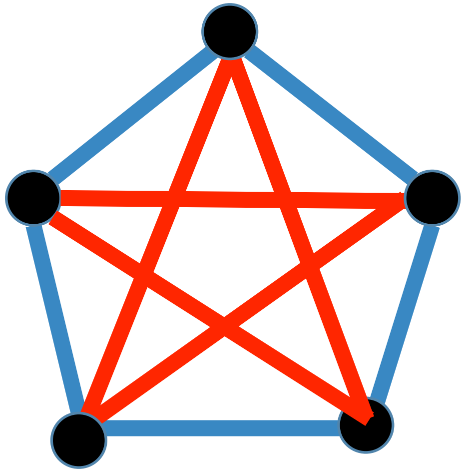


$\eta(I(P_5)) = 2$

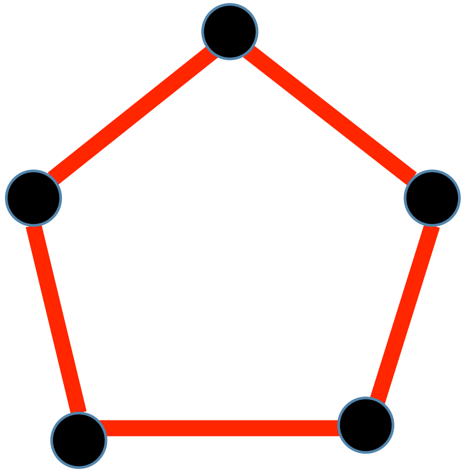
C_5



$I(C_5)$

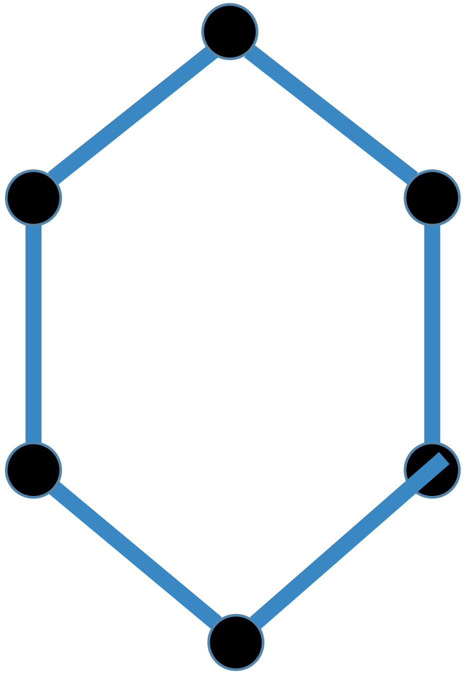


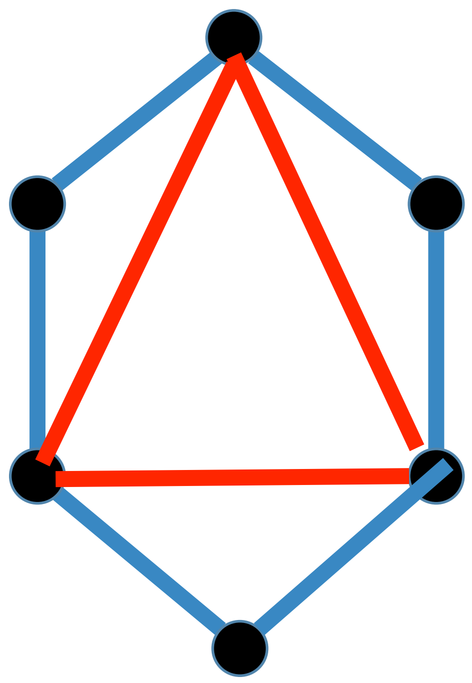
$I(C_5)$

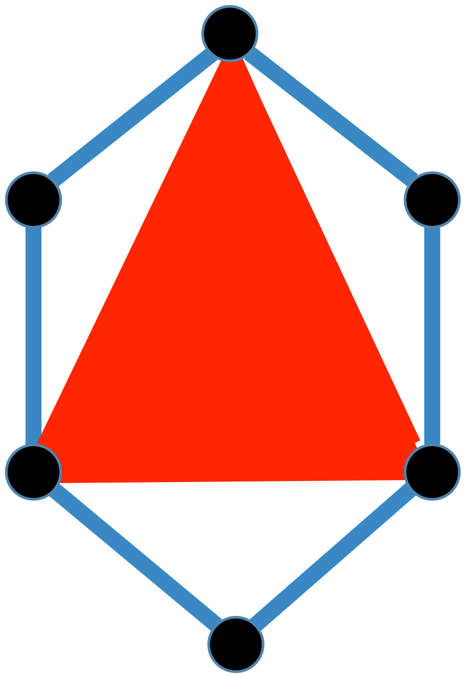


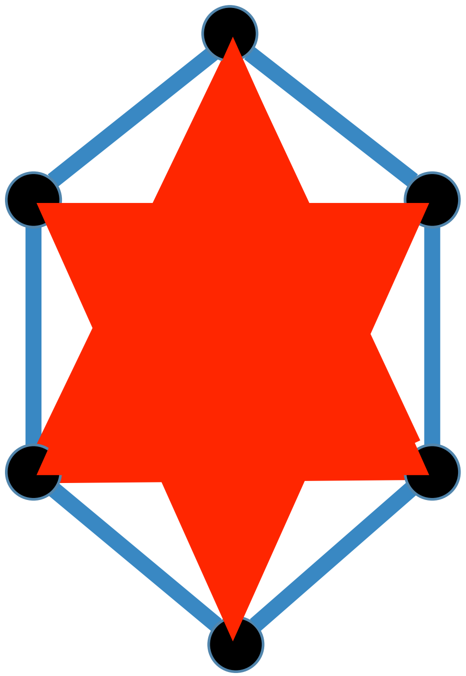
$$\eta(I(C_5)) = 2$$

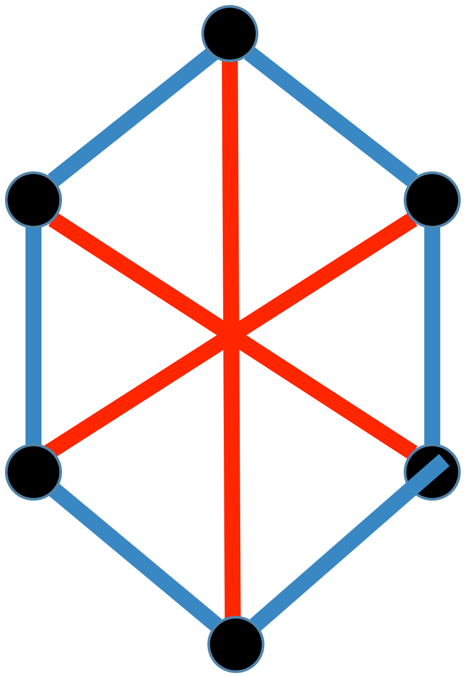
C_6



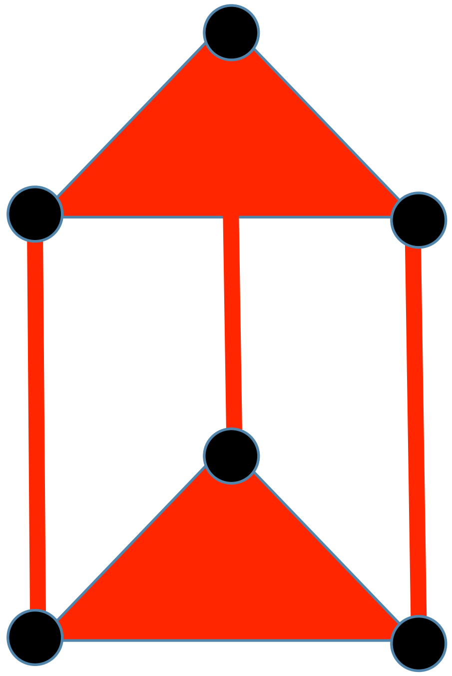








$I(C_6)$



$\eta(I(C_6)) = 2$

The Meyer-Vietoris inequalities

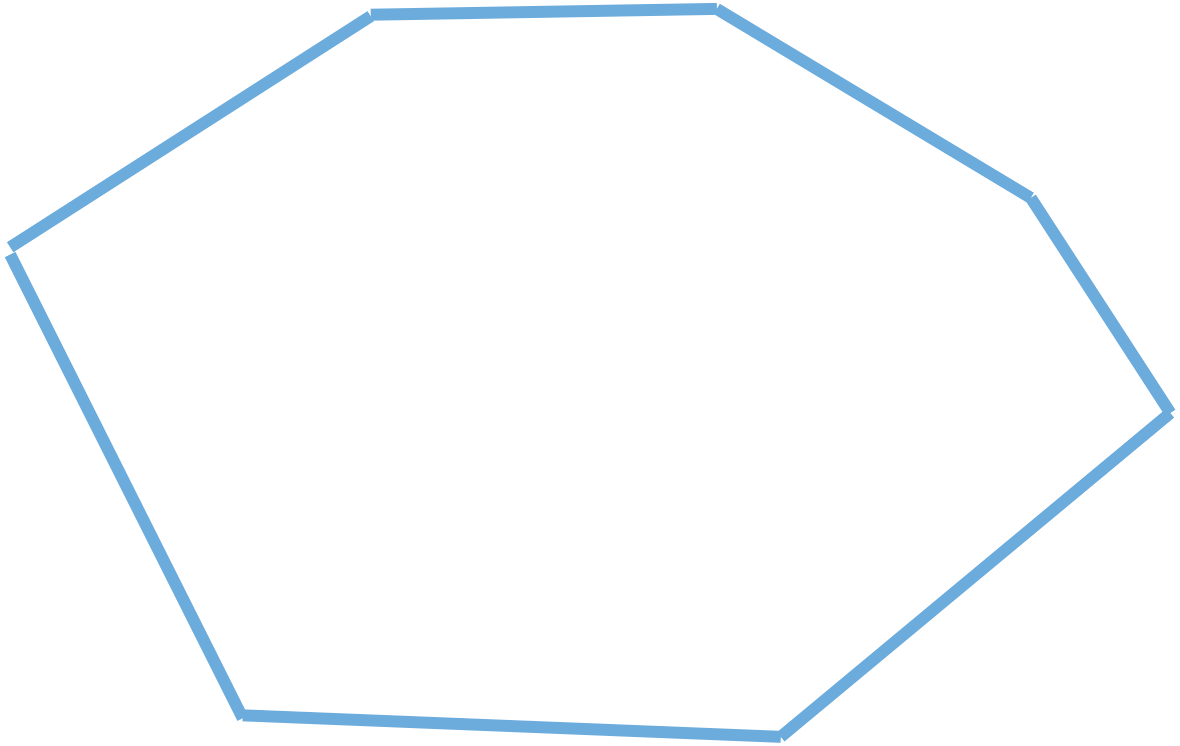
$$\eta(A \cap B) \geq \min(\eta(A), \eta(B), \eta(A \cup B) - 1)$$

Leopold Vietoris, 1891 - 2002

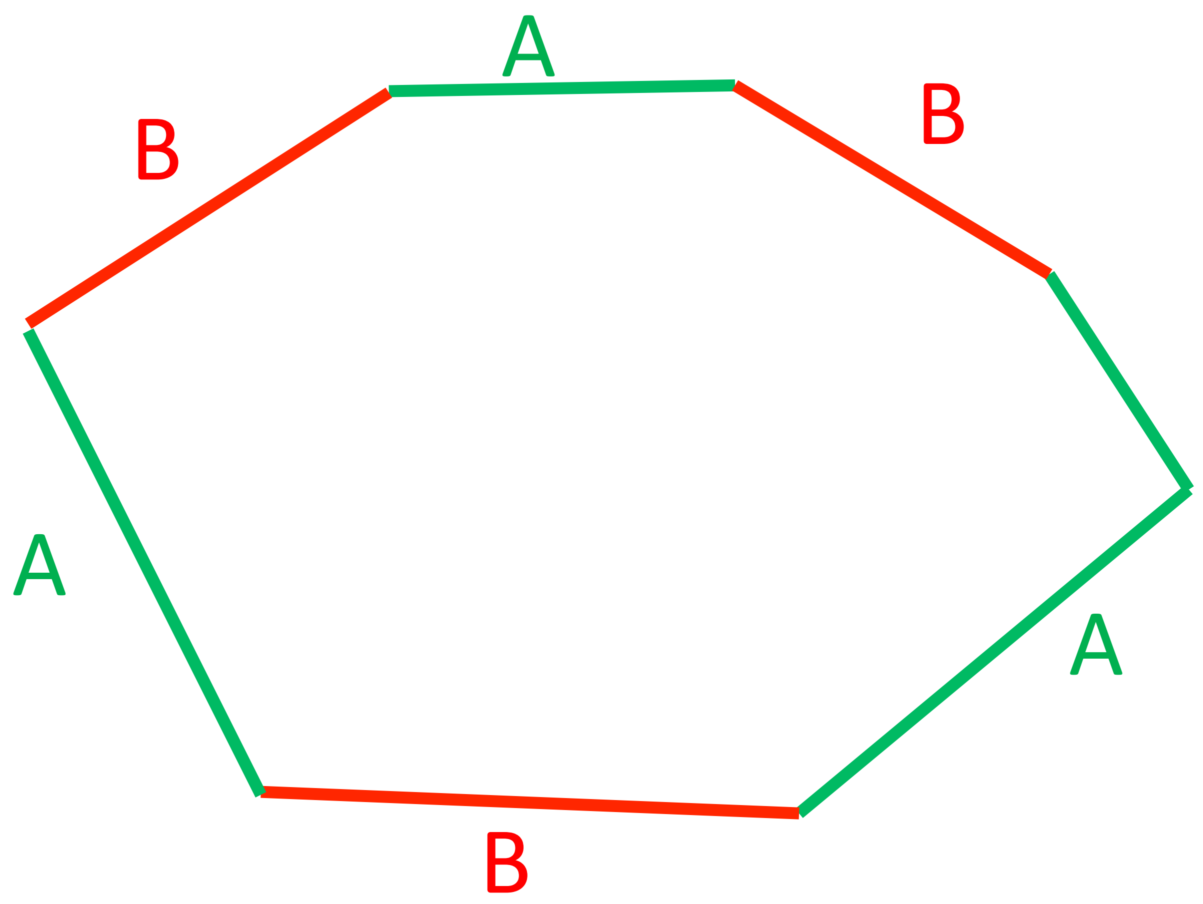
Proof

- We have to show that if $\eta(A),\eta(B),\eta(A\cap \mathrm{B})+1\geq k$ then $\eta(A\cup B)\geq k$.

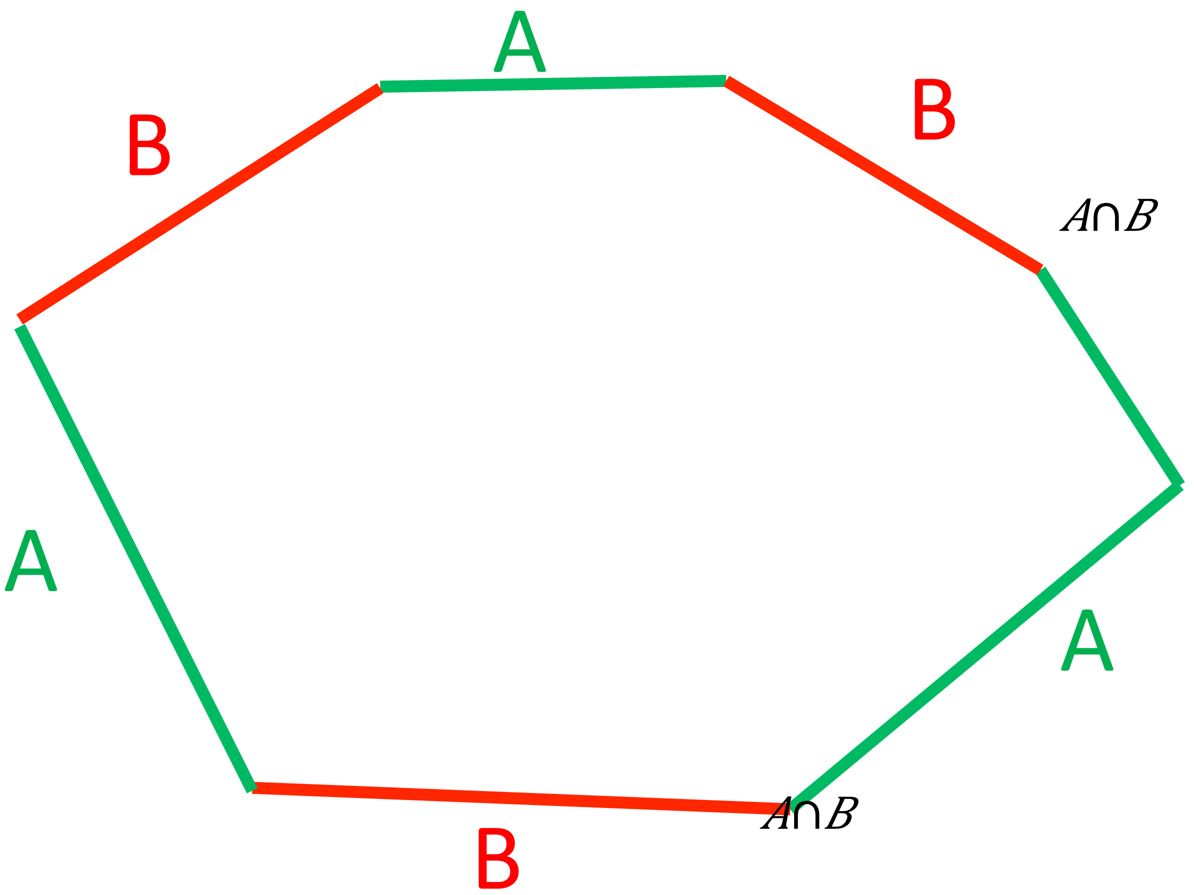
Proof, $k=3$



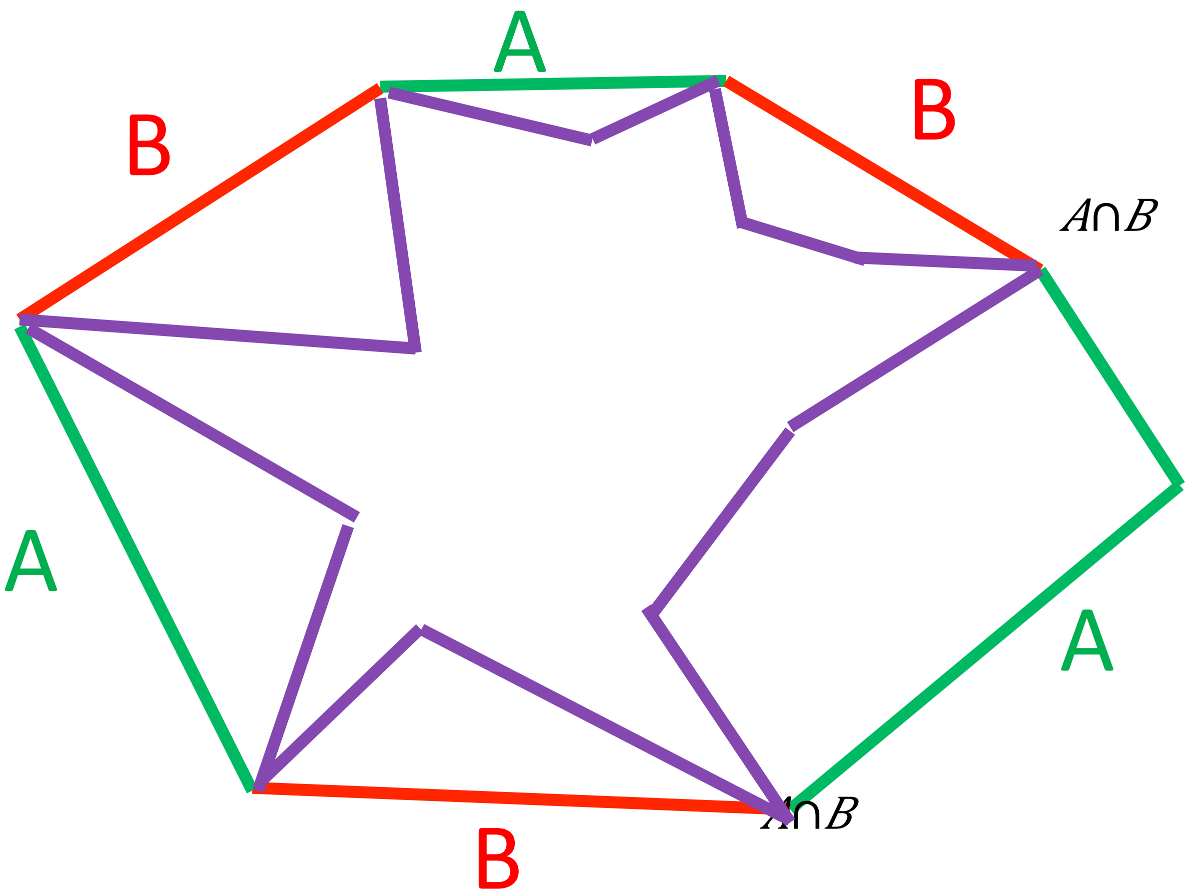
Proof, $k=3$



Proof, $k=3$



Proof, $k=3$ (remember: $\eta(A \cap B) \geq 2$)



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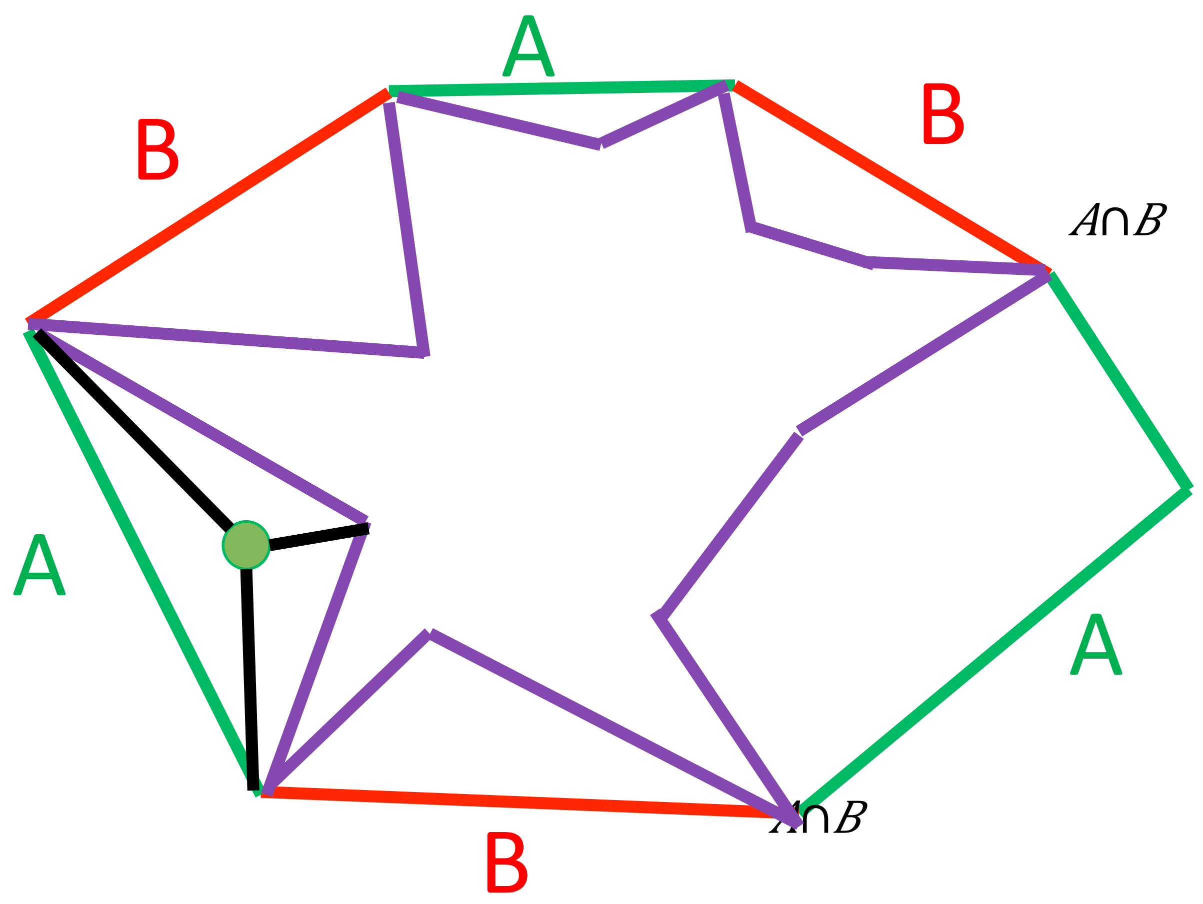

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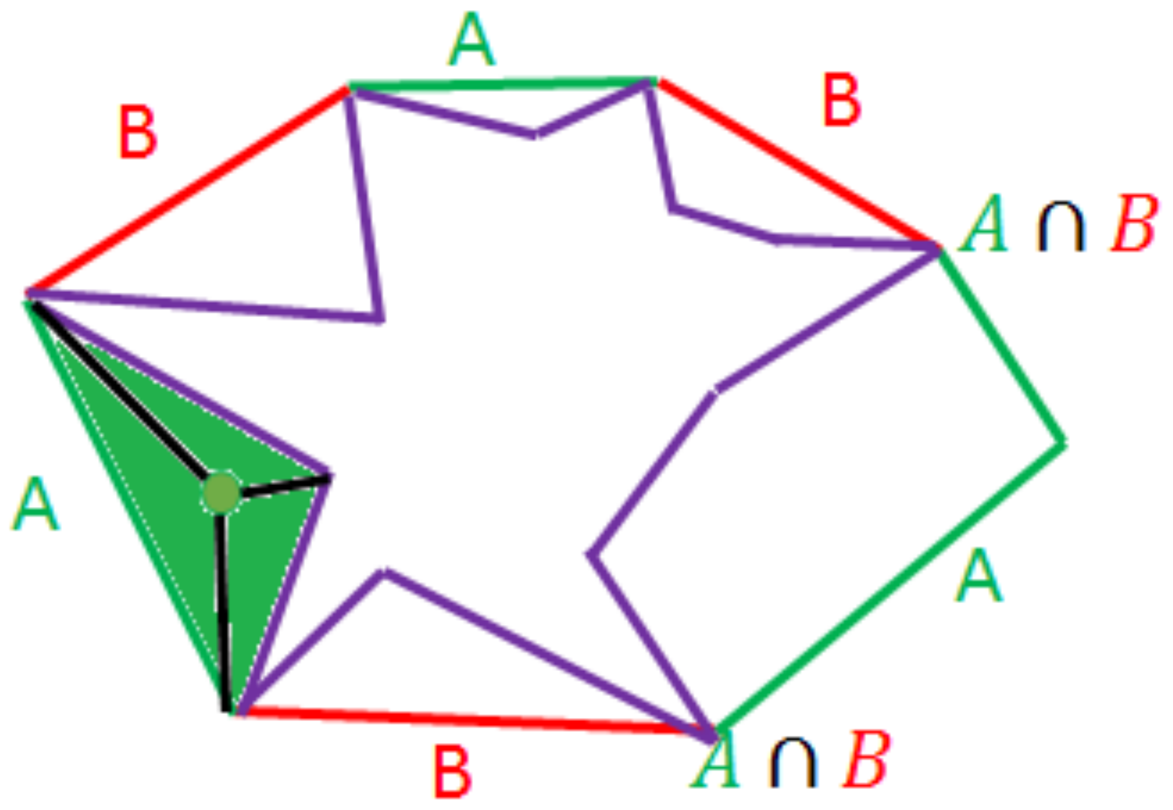

Application: the Meshulam game

- For a graph G and an edge e define $G\neg e = G - N(e)$

- Theorem (Meshulam): $\eta(I(G)) \geq \min(\eta(I(G-e)), \eta(I(G\neg e))+1)$

- Proof: $I(G-e) = (I(G\neg e) * ab) \cup I(G)$
- $(I(G\neg e) * ab) \cap I(G) = I(G) * \{a,b\}$
- $\eta(I(G) * \{a,b\}) = \eta(I(G)) + 1$
- $\eta(I(G\neg e) * ab) = \infty$
- $\eta(A \cup B) \geq \min(\eta(A), \eta(B), \eta(A \cap B) + 1)$ - QED

proof

- Theorem: $\eta(I(G))\geq\min(\eta(I(G-e)),\ \eta(I(G\neg e))+1)$

- $I(G-e)=(I(G\neg e)*ab)\cup I(G)$
- $(I(G\neg e)*ab)\cap I(G)=I(G\neg e)*\{a,b\}$
- $\eta(I(G\neg e)*\{a,b\})=\eta(I(G\neg e))+1$
- $\eta(I(G\neg e)*ab)=\eta(ab)=\infty$
- $\eta(A\cap B)\geq\min(\eta(A),\ \eta(B),\eta(A\cup B)-1)$
- - QED

Another Meyer-Vietoris inequality

- $\eta(A \cap B) \geq \min(\eta(A), \eta(B), \eta(A \cup B) - 1)$

- Exercise: prove this inequality, and use it to prove another “game” inequality: for every vertex v,

$\eta(I(G)) \geq \min(\eta(I(G - v)), \eta(I(G \neg v)) + 1)$

Here $G \neg v = G - N(v)$.