Applications of connectivity

The neighborhood complex

The neighborhood complex N(G) of a graph G is the set of punctured neighborhoods (and their subsets).

Lovasz' theorem

$$\chi(G) > \eta(N(G))$$

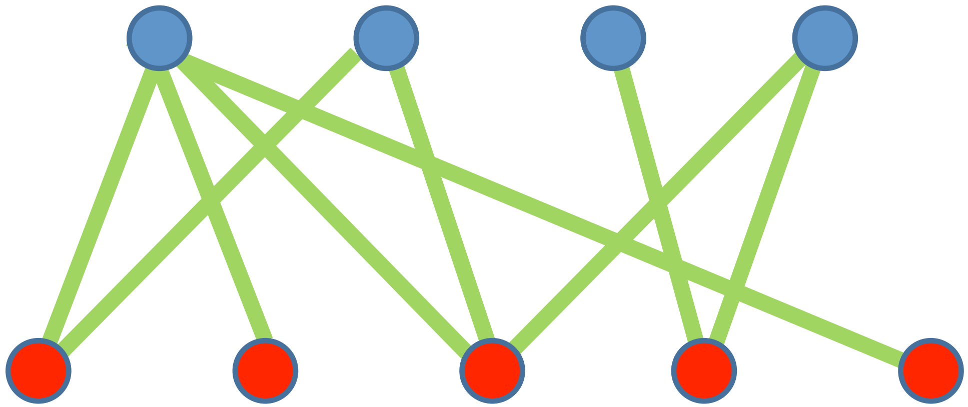
Independent systems of representatives

The Hall setting – SDRs

Given sets $V_1, V_2, \ldots, V_m$, a system of distinct representatives (SDR) is an injective choice function, namely a choice of distinct elements

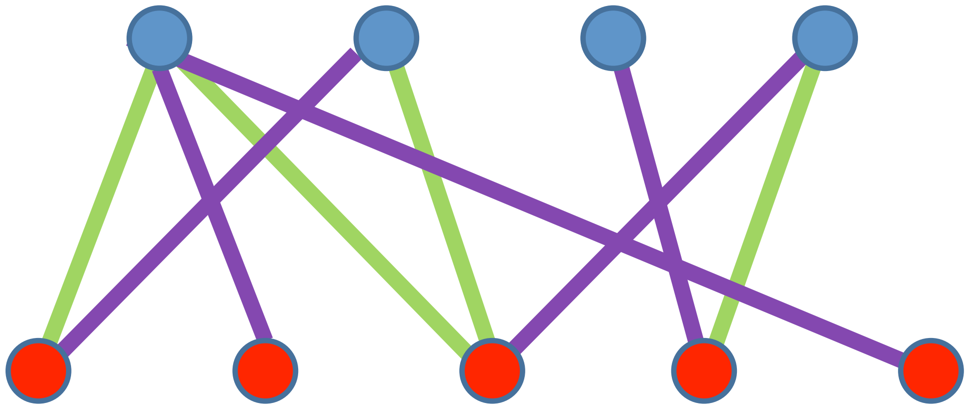
$$v_1 \in V_1, v_2 \in V_2, \ldots, v_m \in V_m$$

The bipartite graph formulation

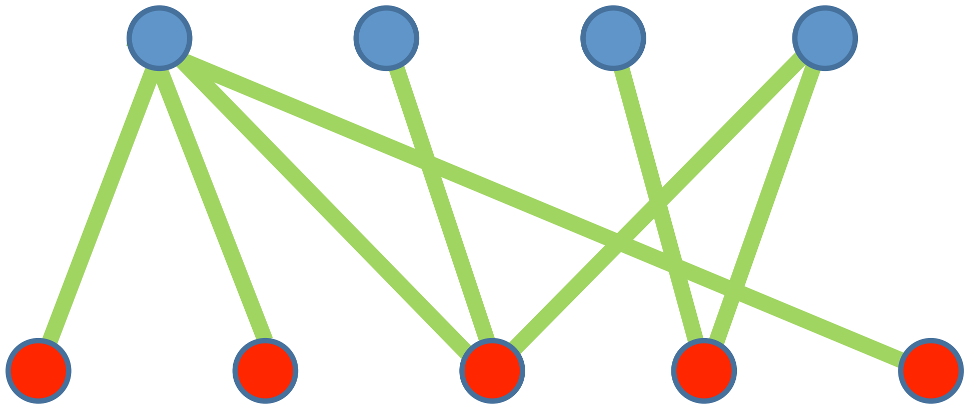


The top points are "men", or "sets", the bottom points are "women" or "elements"

SDR=marriage



No SDR



The 3 rightmost men are connected ("know") to only 2 women

Hall's marriage theorem

Frobenius (1912), König (1915), Hall (1935)

If $|\bigcup_{i\in I} V_i| \geq |I|$ for every $I \subseteq [m]$ then there exists an SDR.

Adding another structure – ISRs

An additional restriction:
the range of the choice function belongs to some given hypergraph.

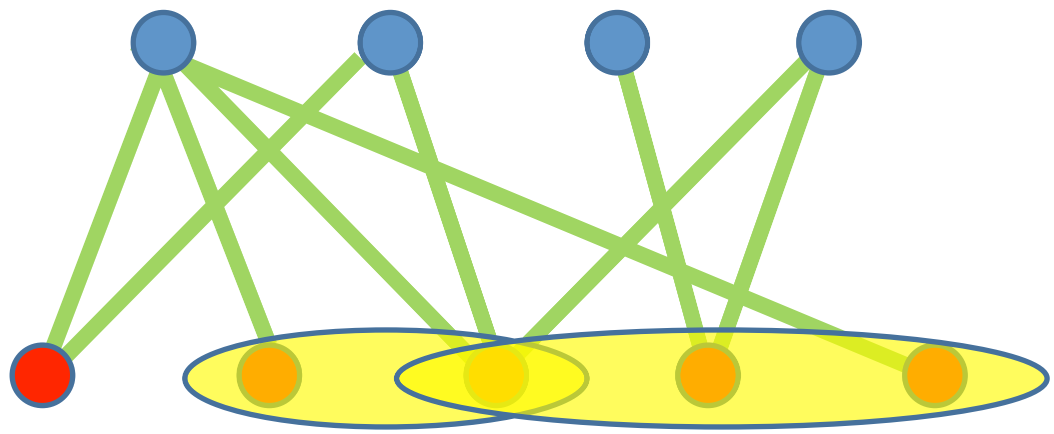
$\mathcal{C}$-ISRs

$\mathcal{C}$ - a complex on $\bigcup_{i \leq m} V_i$

a $\mathcal{C}$– system of representatives (*$\mathcal{C}$-SR*)

is a (not necessarily injective) choice function for the sets V_i whose range belongs to C.

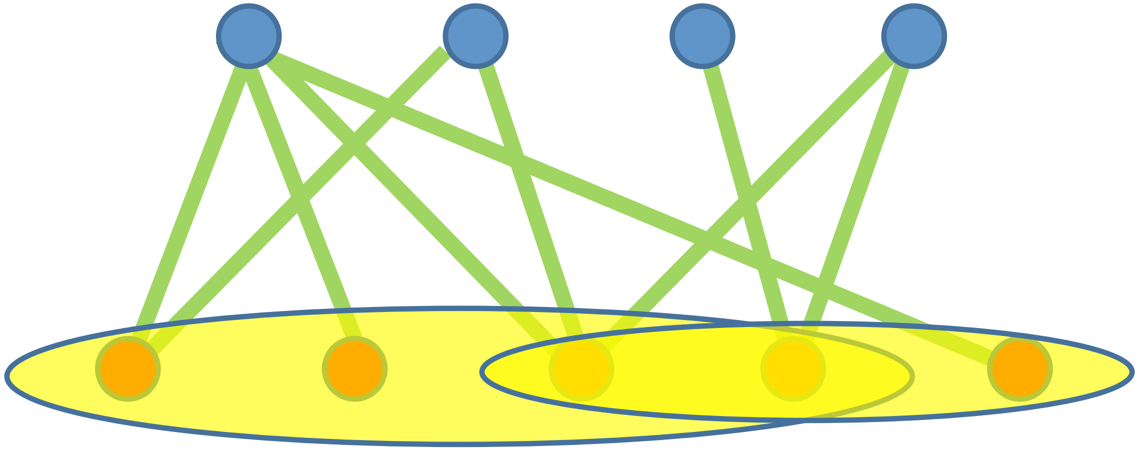
$C\text{–}SRs$ – The bipartite graph formulation



Is there an injective C-SR?

No, just like in the previous example

$C\!-\!SRs$ – The bipartite graph formulation



Here there is an injective C-SR

Re-formulation:

A C-ISR is a choice

$$v_1 \in V_1, v_2 \in V_2, \ldots, v_m \in V_m$$

of not necessarily distinct elements, such that

$$\{v_1, v_2, \ldots, v_m\} \in C$$

Injectivity can also be formulated

Just look at the complex induced on the edges of the bipartite graph namely:

Look at the set of matchings whose women endpoints belong to C.

Note that in this formulation the sets $V\downarrow i$ are disjoint.

We shall always make this assumption.

So – we shall assume that the sets $V\downarrow i$ are disjoint

A classical example – linear independence

Rado's theorem (1942)

If M *is a matroid and*

$rank(\bigcup_{j \in J} V_j) \geq |J|$

for all $J \subseteq [m]$

then there exists an M*-SR.*

Whitney’s theorem

$\eta(M) = rank(M)$ or ∞

Topological Hall

Theorem (A+Haxell, 2000)

If $|\eta(C[\bigcup_{i\in I} V_i])| \geq |I|$ for every $I \subseteq [m]$

then there exists a C-SR.

Graph independence

$C = I(G)$, the complex of independent sets in some graph G.

(a set is in *C if no two vertices in it are connected)*

A special case - ISRs for hypergraphs

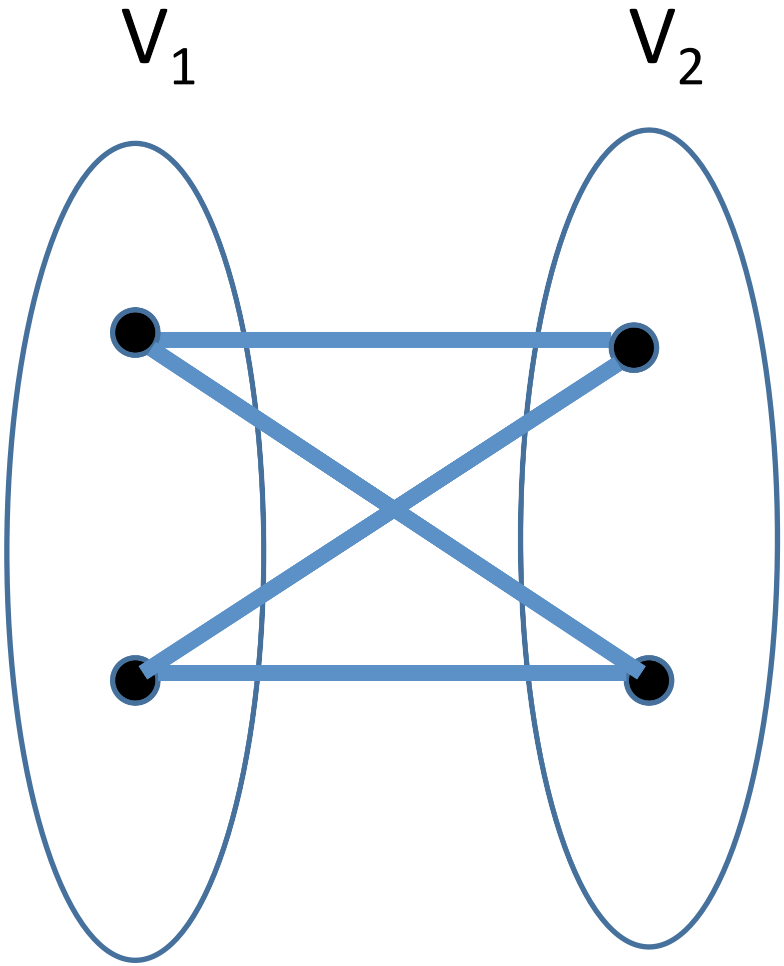
Given hypergraphs $H_1, H_2, \ldots, H_m$

an ISR is a choice of disjoint edges,

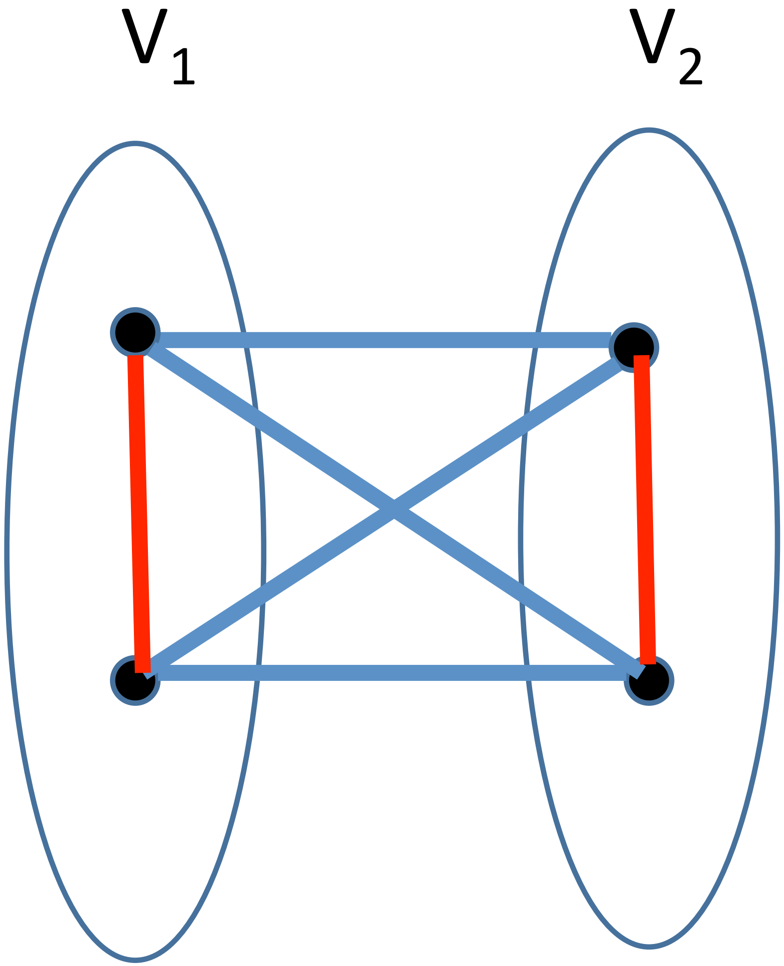
$$e_1 \in H_1, e_2 \in H_2, \ldots, e_m \in H_m$$

So, it is an ISR in the line graph

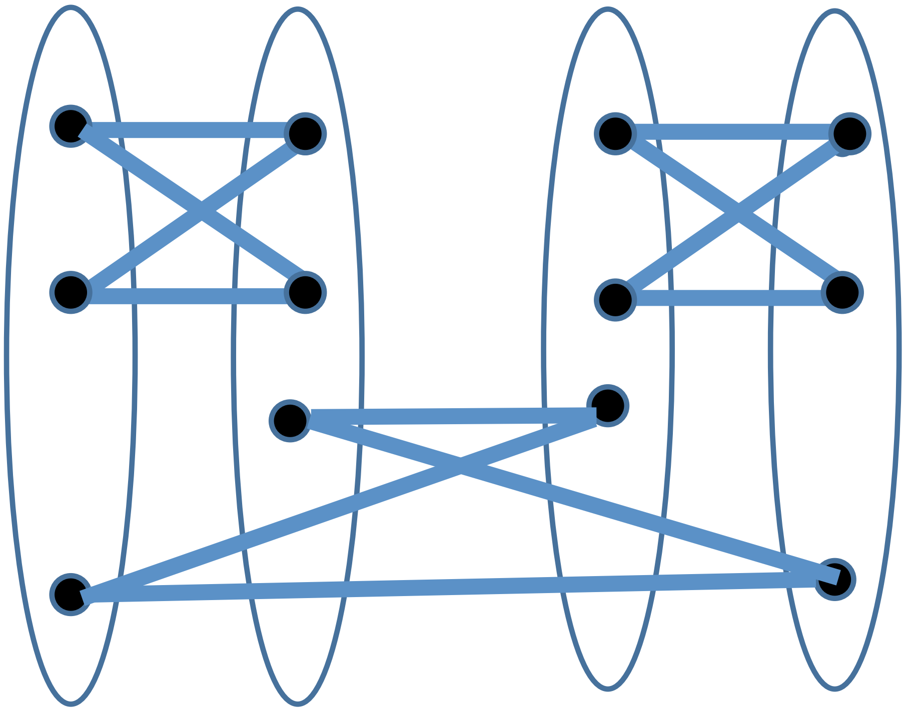
Example - no ISR:



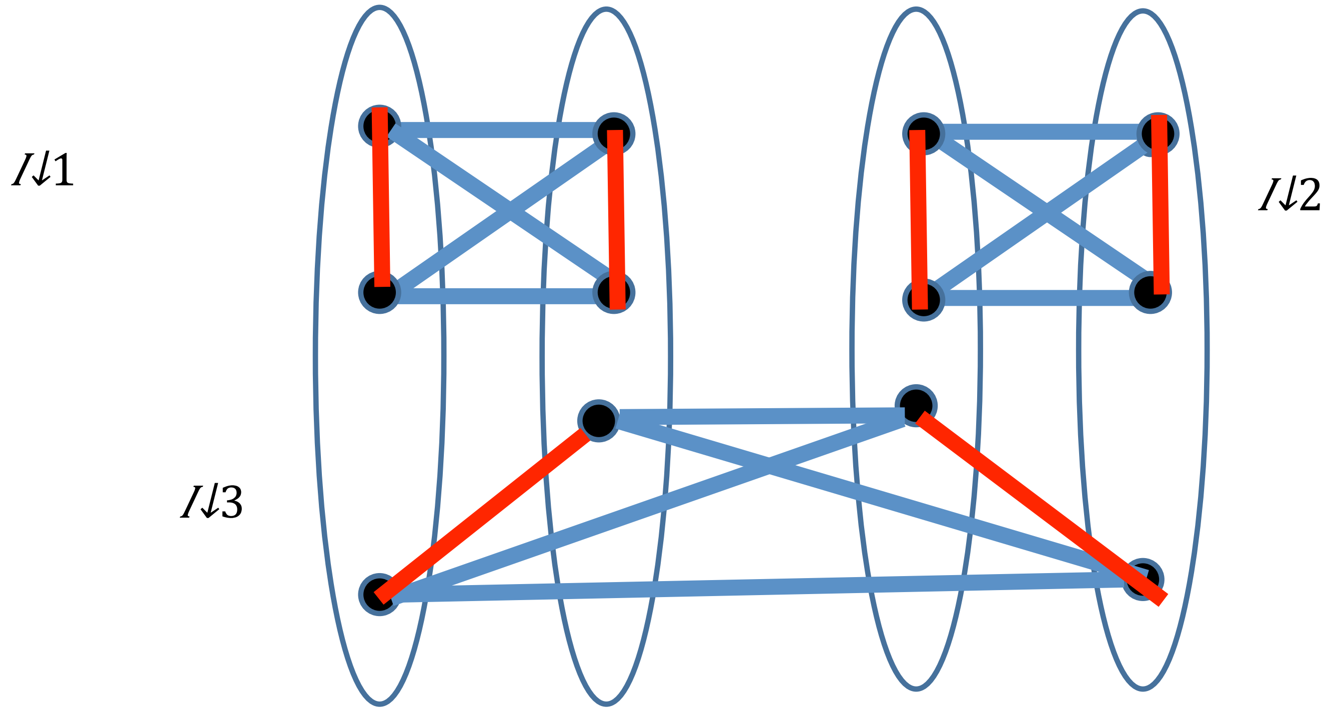
And indeed $m = 2, \quad \eta(I(G)) = 1$



m=4, connectivity 3 (connectivity is additive)



$I(G)=I\downarrow 1 * I\downarrow 2 * I\downarrow 3$



$\eta(I(G)$
$)=1+$
$1+1=$
$3\downarrow$

Translating connectivity to combinatorics - domination

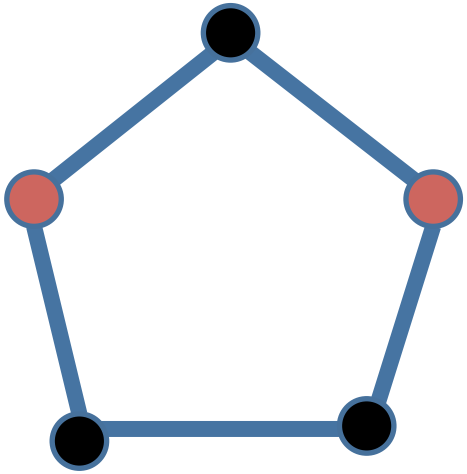
For a set S of vertices, $N(S)$ is the set of neighbors of S, and $N[S]=N(S) \cup S$.

S is dominating if $N[S]=V$, and it is totally dominating if $N(S)=V$.

Domination parameters

- $\gamma(G)$ is the minimal size of a dominating set in *G.*
- $\gamma^t(G)$ is the minimal size of a totally dominating set.
- $\gamma^i(G)$ is the maximum, over all independent sets *I* in *G*, of the minimal size of a set needed to dominate *I.*

$$\gamma(C_5) = 2$$



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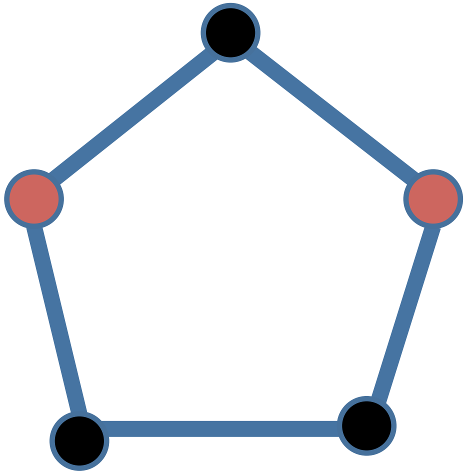
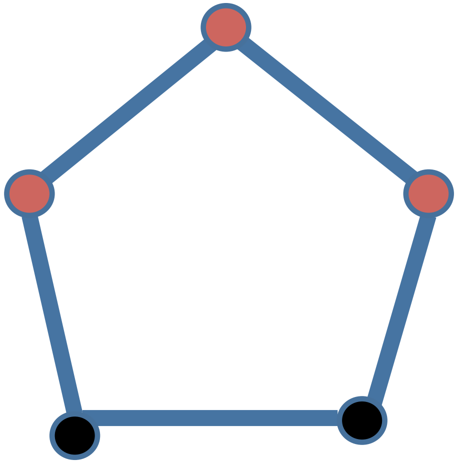


$$\gamma^t(C_5) = 3$$



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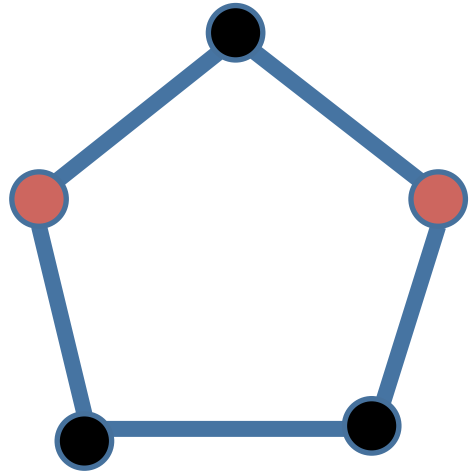

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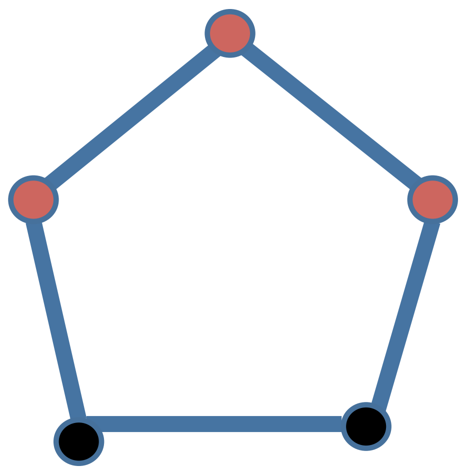
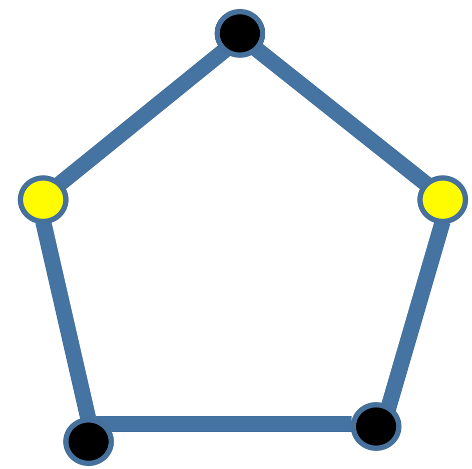
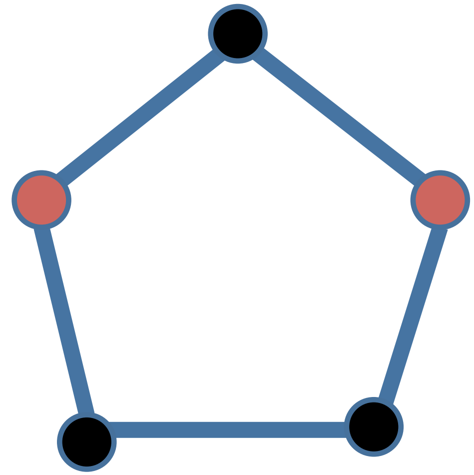
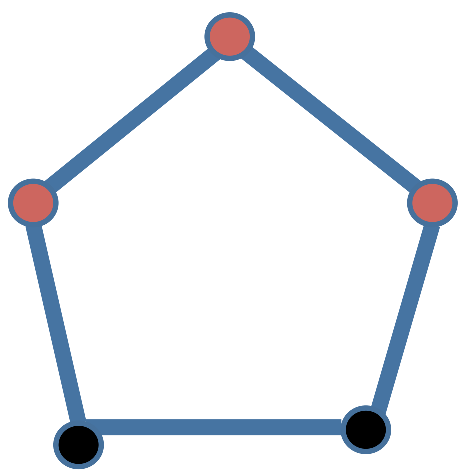




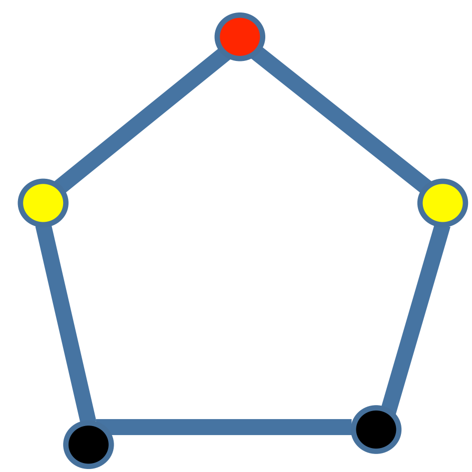
$$\gamma(C_5) = 2$$



$$\gamma^t(C_5) = 3$$



$$\gamma^i(C_5) = 1$$



Theorem (A+Haxell) $\eta(I(G)) \geq \gamma^i(G)$

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Corollary (conjectured by A 1984)

If $H_1, H_2, \ldots, H_m$ are *r*-uniform hypergraphs, and every *k* of them contain *>(k-1)r* disjoint edges, then they have an ISR.

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If $H_1, H_2, \ldots, H_m$ are *r*-uniform hypergraphs, and every *k* of them contain *>(k-1)r* disjoint edges, then they have an ISR.

(the *(k-1)r+1* disjoint edges are independent in the line graph, and every edge in the world dominates (meets) at most *r* of them, so $\gamma^i \geq k$)

Corollary (A 2000) (first open case of a conjecture of Ryser)

In a 3-partite hypergraph, if there are no more than *n* disjoint edges, then there are 2n vertices that meet all edges.

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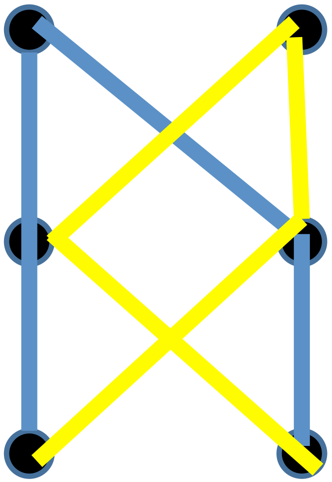
Example, n=1:



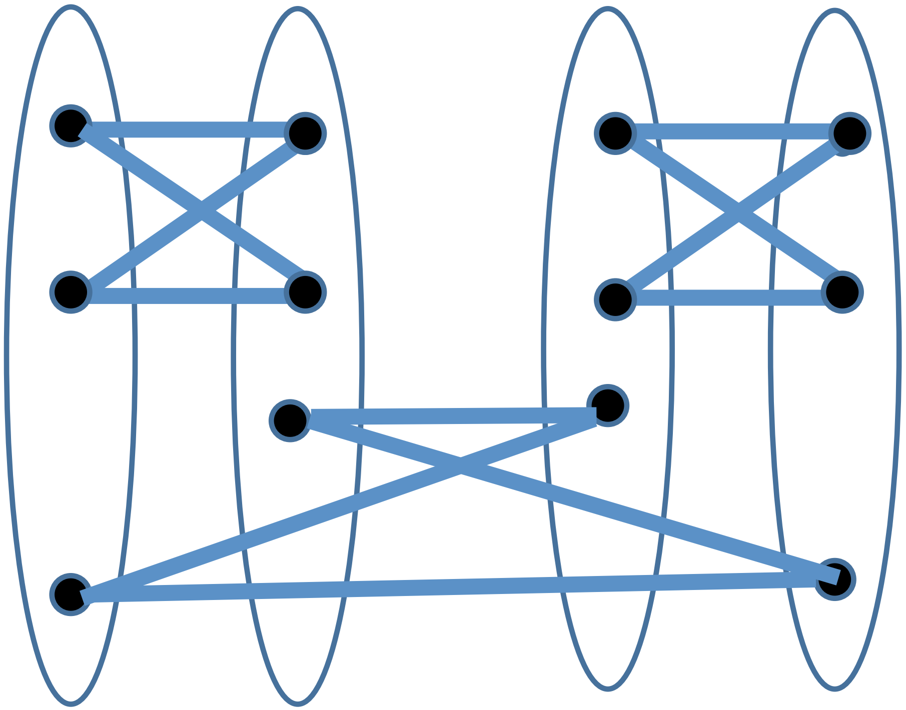
The total domination bound

Theorem (Meshulam) $\eta(I(G)) \geq \frac{\gamma^t(G)}{2}$

Corollary: $\eta(I(G)) \geq \frac{|V|}{2\Delta}$

Corollary (Haxell):
If $|V_i| \geq 2\Delta$ for all *i then there exists an ISR.*

No ISR, d=2, sets of size 2d-1=3
(generalizable to all d that are powers of 2)



The strong coloring conjecture

If $|V_i| = 2\Delta$ for all *i* then there exists a partition of *V* into 2Δ *ISRs.*

Theorem (Haxell)
If $|V_i| = 3\Delta - 1$ for all *i* then there exists a partition of *V* into $3\Delta - 1$ *ISRs.*

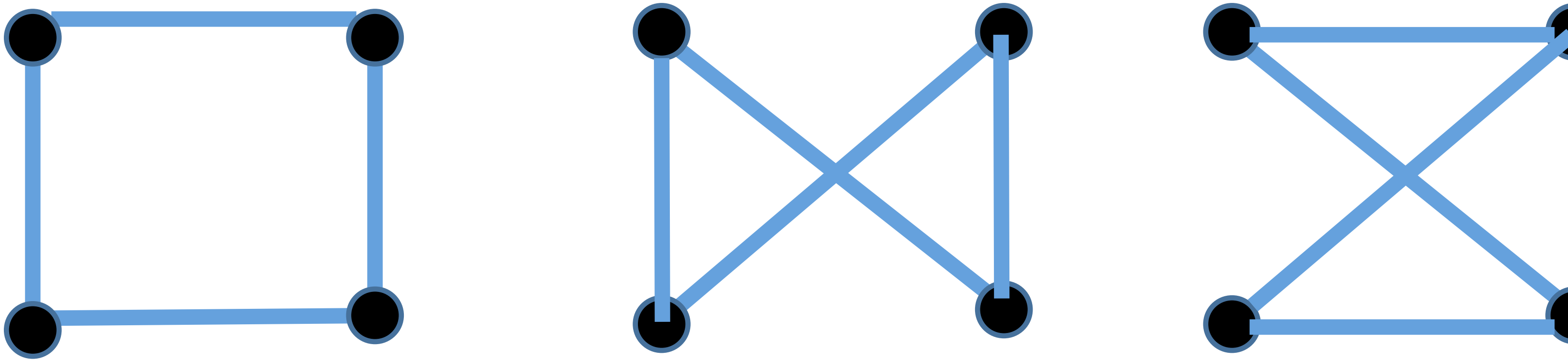
Theorem (Berger+Shkirat) SCC is true for $\Delta = 2$.

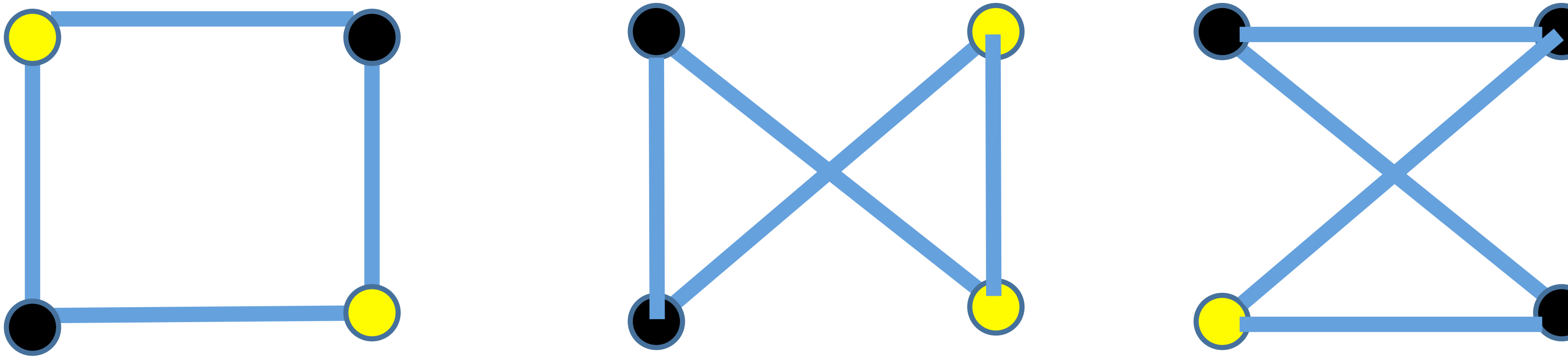
Theorem (A+Berger+Ziv)
If $|V_i| = 2\Delta$ for all *i* then there exists a fractional partition of *V* into 2Δ *ISRs.*

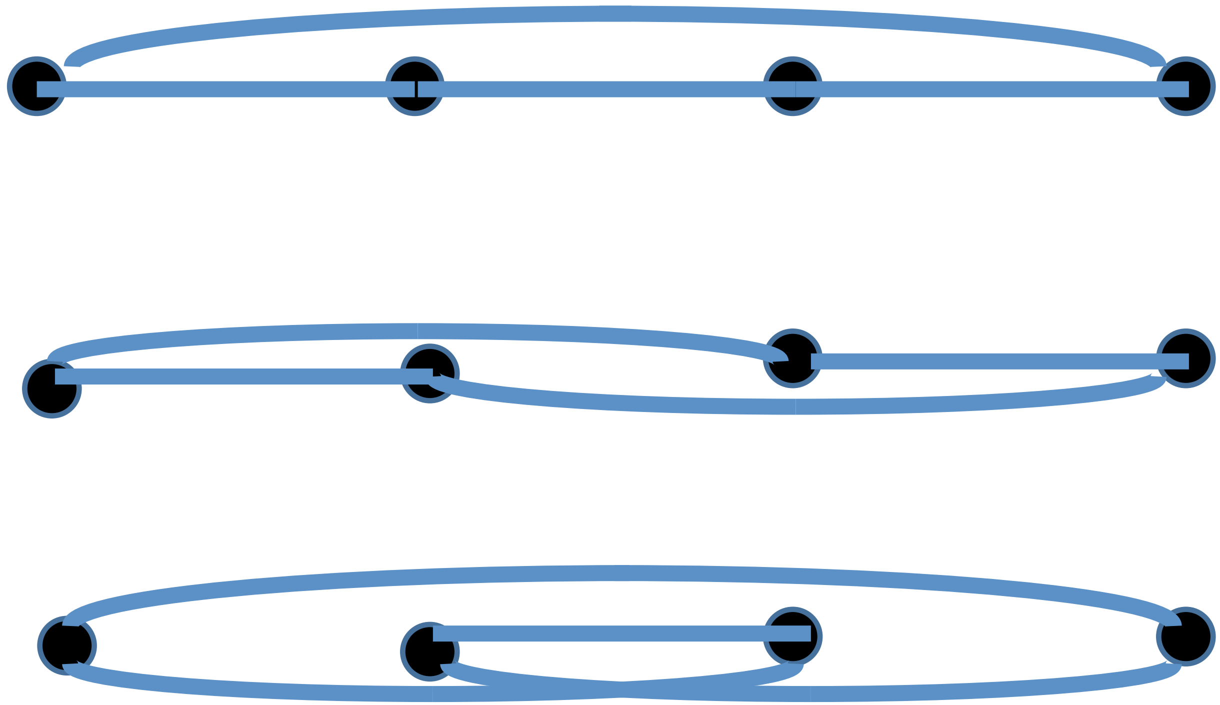
Cooperative colorings

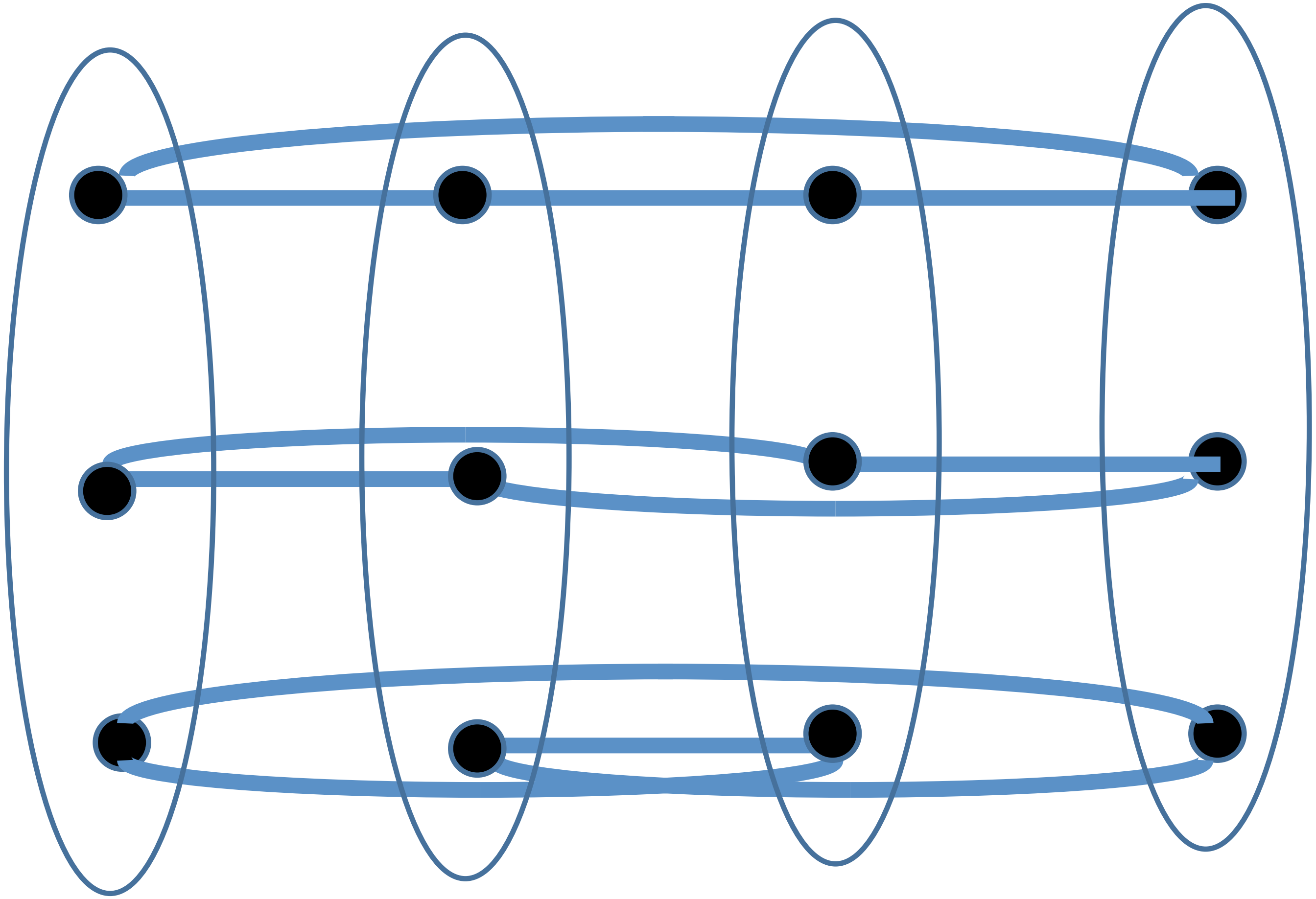
Given graphs $G_1, G_2, \ldots, G_k$ on the same vertex set *V*, a cooperative coloring is a choice $A_1 \in I(G_1), A_2 \in I(G_2), \ldots, A_k \in I(G_k)$ such that $\bigcup A_i = V$.

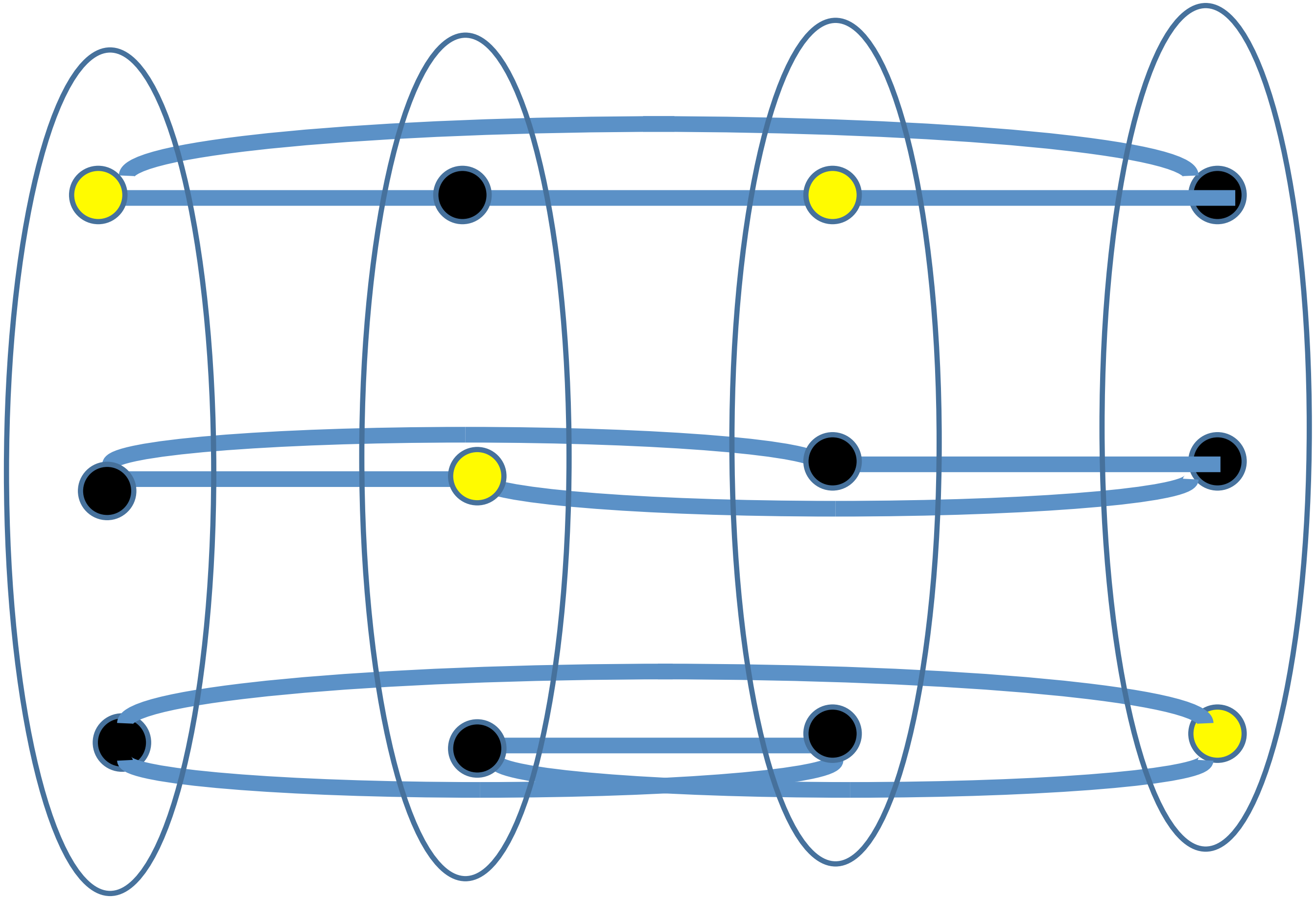
(If all graphs are the same graph *G* then this is ordinary *k*-coloring)





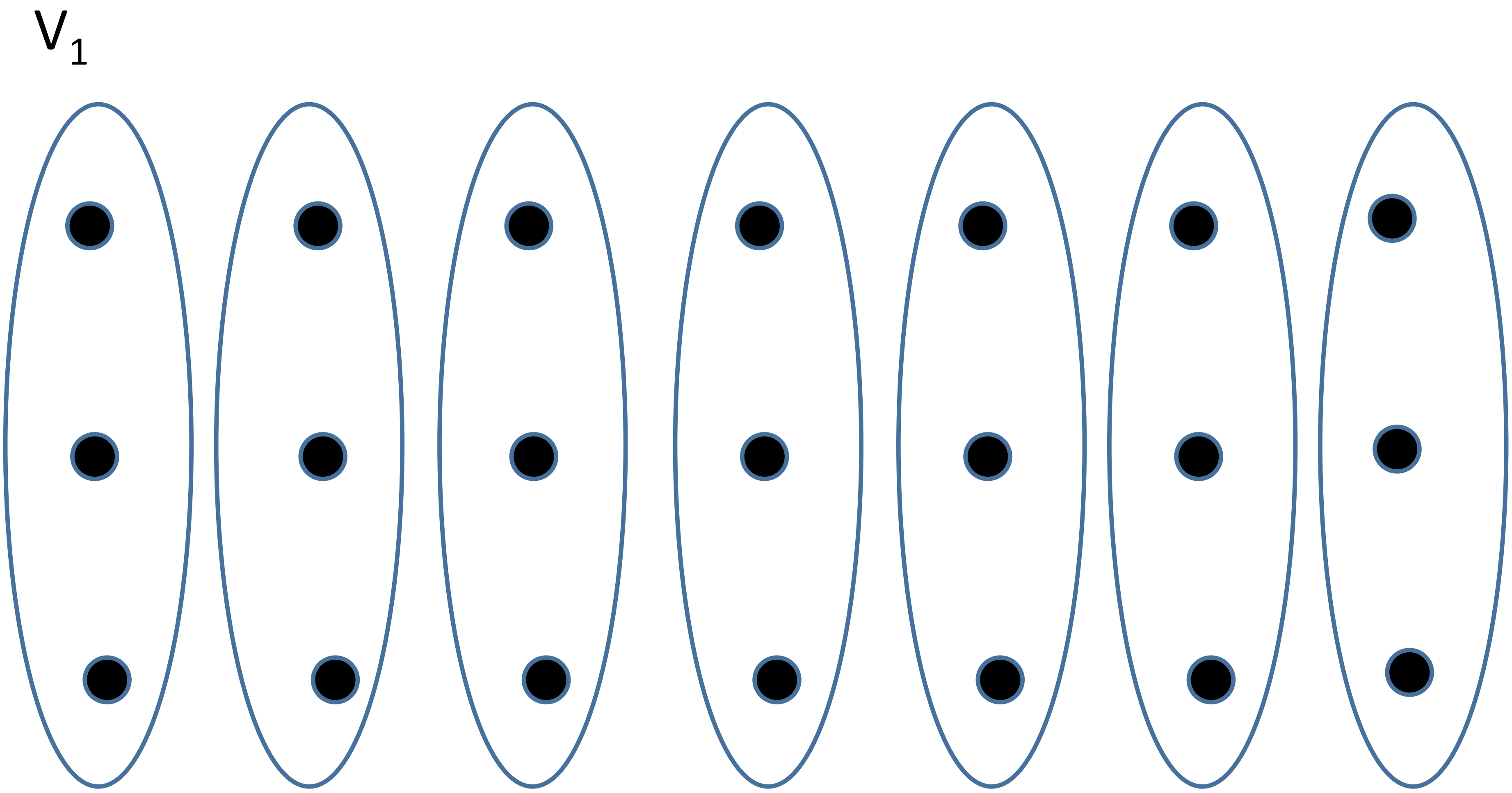


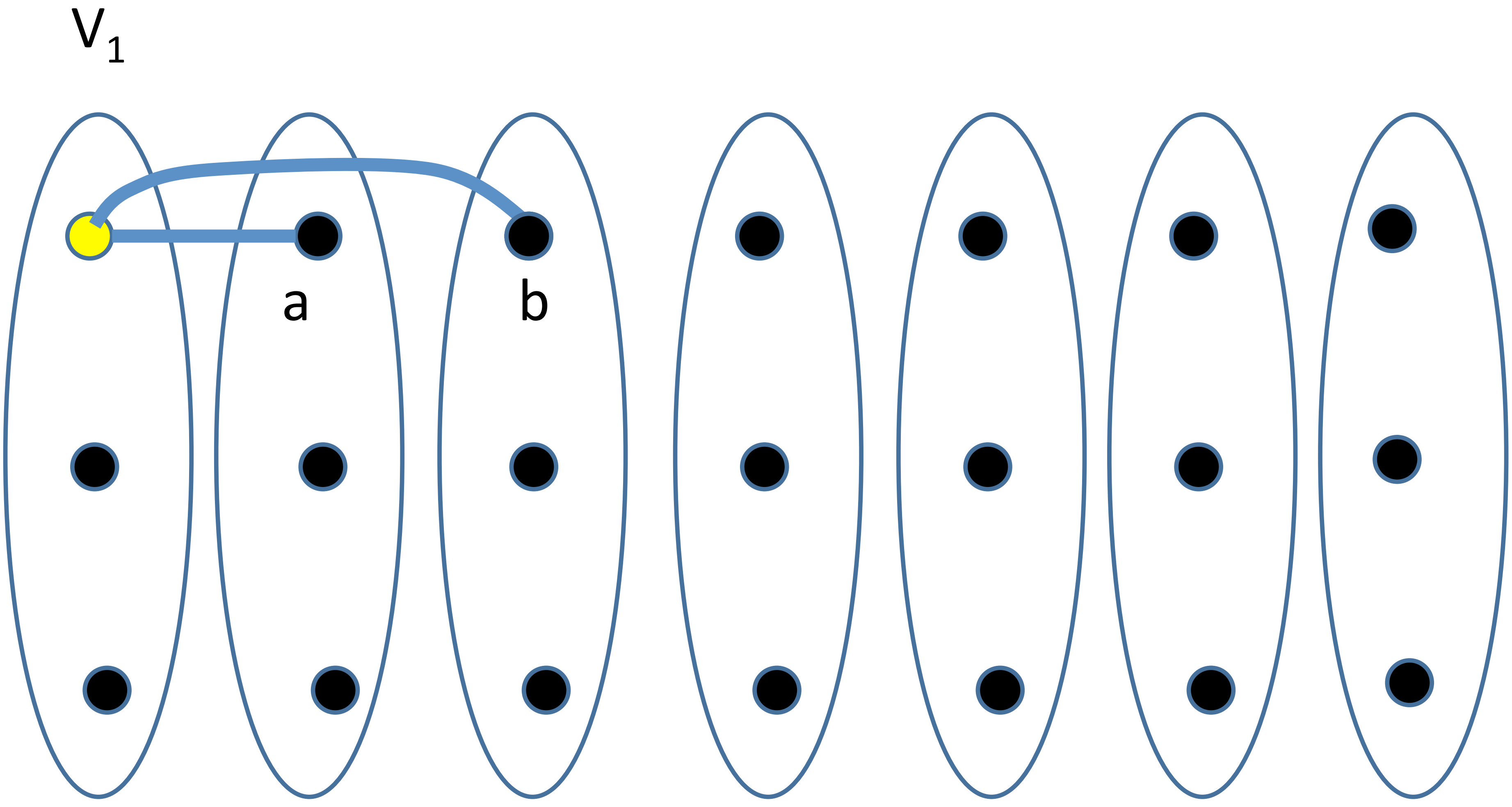


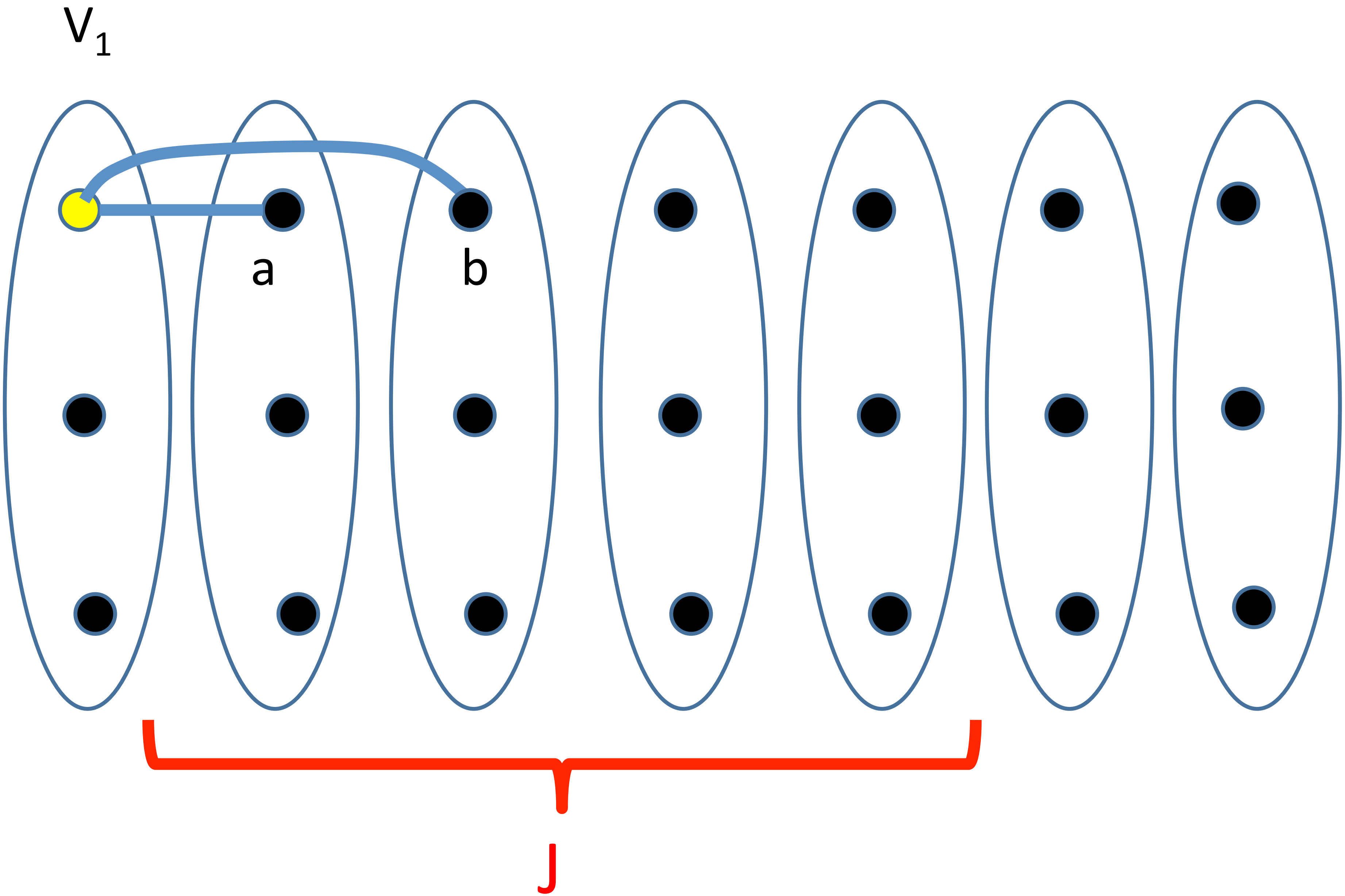


Theorem (A+Holzman+Howard+Spruessel)

Three cycles on the same vertex set have a cooperative coloring.


V_1


V1
a
b



V_1
a
b
J

Let $J \subseteq [m] \setminus \{1\}$

$G[\bigcup_{j\in J} V_j - a - b]$ consists of paths P_k, where

$$\sum P_k = 3|J| - 2$$

Since $\eta(P_k) \geq \left\lceil \frac{|P_k|}{3} \right\rceil$ and η is additive

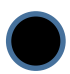
$$\eta(I(G[\bigcup_{j\in J} V_j - a - b])) \geq \sum \left\lceil \frac{|P_k|}{3} \right\rceil \geq |J|$$

So, there exists an ISR for $V_2, V_2, \ldots, V_m$ in G-a-b , proving the theorem.

Connectivity

The connectivity of a complex C, denoted by $\eta(C)$ is the smallest dimension of a hole in C.

Examples:

C=  $\eta(C) = \infty$

C =  $\eta(C) = 1$

The Hsu-Du-Wang conjecture

The Hsu-Du-Wang conjecture

In a $C_{3k}+k$ triangles $\alpha \geq k$

Equivalent: if $|V_i|=3$ ($i=1,2,\ldots k$)
and C_{3k} is a cycle on $\bigcup V_i$ then there exists an ISR.

Easy by topological Hall.
Also easy by Shcrijver.

Thank you

Theorem (Mehsulam)

$$\eta(I(C_n)) = [\frac{n}{3}]$$

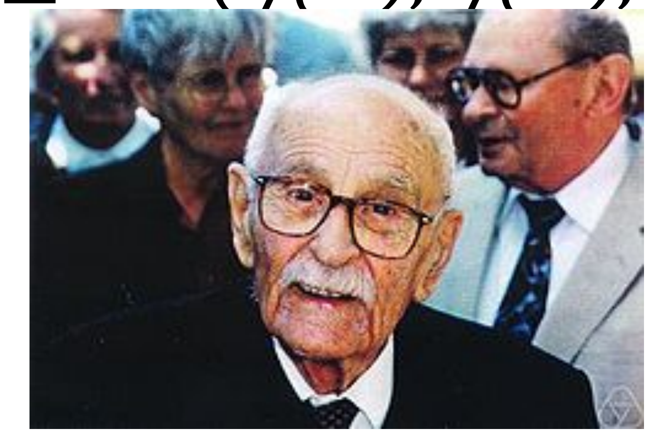
$$\eta(I(P_{3k})) = k$$

$$\eta(I(P_{3k+2})) = k+1$$

$$\eta(I(P_{3k+1})) = \infty$$

The Meyer-Vietoris inequalities

$$\eta(A \cap B) \geq \min(\eta(A), \eta(B), \eta(A \cup B) - 1)$$



Leopold Vietoris, 1891 - 2002

Application: the Meshulam game

- For a graph G and an edge e define $G\neg e = G - N(e)$

- Theorem (Meshulam): $\eta(I(G)) \geq \min(\eta(I(G - e)), \eta(I(G\neg e)) + 1)$

Proof:

- $I(G-e)=(I(G\neg e)*ab)\cup I(G)$
- $(I(G\neg e)*ab)\cap I(G)=I(G\neg e)*\{a,b\}$
- $\eta(I(G\neg e)*\{a,b\})=\eta(I(G\neg e))+1$
- $\eta(I(G\neg e)*ab)=\infty$
- $\eta(A)\geq\min(\eta(B),\eta(A\cap B),\eta(A\cup B))$
- $\eta(I(G))\geq\min(\infty,I(G),\eta(I(G-e)))$