Two fixed points theorems

Some toplogical tools for the way

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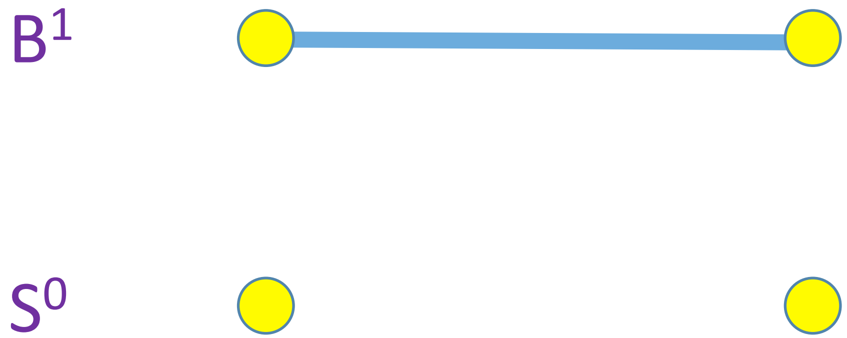
Notation

- B^n is the unit ball in R^n
- S^n is the boundary of B^{n+1}, namely the unit sphere in R^{n+1}



Convention: All functions are continuous

Brouwer's fixed point theorem (1912)

A function f from a closed convex set to itself has a fixed point.

(namely x such that $f(x)=x$)

Borsuk’s theorem (1932)

A function f from S^n to R^n has a point x in S^n such that $f(-x)=f(x)$

Examples:

n=1

On the equator there are two antipodal points with the same temperature.

n=2

On earth there are two antipodal points with the same temperature, and the same barometric pressure.

Re-formulating Brouwer

There is no $f: B^n \longrightarrow S^{n-1}$ such that $f(x)=x$ for every x in S^{n-1}.

Re-formulating Borsuk

There is no $f: S^n \longrightarrow S^{n-1}$ such that $f(-x)=-f(x)$ for every x in S^n.

(such a function is called antipodal)

Reformulation of Borsuk:
There is no f: $S^n \longrightarrow S^{n-1}$ such that $f(-x)=-f(x)$ for every x in S^n.

Proof from Borsuk:

Suppose that such a function exists. Then restricting it to S^{n-1} yields a function f: $S^{n-1} \rightarrow S^{n-1}$ (in particular to R^{n-1}) with no antipodal points going to the same point.

Reformulation of Borsuk:
There is no $f: S^n \longrightarrow S^{n-1}$ such that $f(-x)=-f(x)$ for every x in S^n.

Proof of Borsuk from this version:

Suppose that Borsuk fails, namely there exists a function $f: S^n \rightarrow R^n$ with no x in S^n satisfying $f(-x)=f(x)$

Let $g(x)=f(x)-f(-x)/||f(x)-f(-x)||$

Then $g: S^n \rightarrow S^{n-1}$ is antipodal.

Borsuk is stronger than Brouwer

- Brouwer: There is no f: $B^n \rightarrow S^{n-1}$ such that f(x)=x for every x in S^{n-1}
- Borsuk: There is no g: $S^n \rightarrow S^{n-1}$ such that g(-x)=-g(x) for every x in S^{n-1}
- Assume that there is f as above. Compose it with a projection from B^n to S^n and obtain a g as above.

Brouwer's theorem – an equivalent version: KKM (Knaster – Kuratowski - Mazurkewicz)

If A_i, $i=1\ldots n$ are closed sets (or, together, open sets) in $conv(v_1, v_2, \ldots, v_n)$, and if for every subset J of [n] it is true that

$$\bigcup_{j\in J} A_j \subseteq conv(v_j \; j\in J)$$

(in particular $conv(v_1, v_2, \ldots, v_n) = A_1 \cup A_2 \cup \ldots A_n$)

Then $\bigcap A_j \neq \emptyset$

A discrete version of KKM – Sperner's lemma

Sperner's lemma

If the vertices of a triangulation of $conv(v_1, v_2, \ldots, v_n)$ are colored 1, 2, …,n and if for every subset J of [n] it is true that the vertices in $conv(v_j \; j \in J\}$ are colored only colors from J, then there is a multicolored simplex.