More applications of connectivity

Warning

η(this talk)=1

The Meyer-Vietoris inequalities

$$\eta(A) \geq \min(\eta(A \cap B), \eta(A \cup B))$$

Leopold Vietoris, 1891 - 2002

Application: the Meshulam game

For a graph G and an edge e define $G \neg e = G - N(e)$

Theorem (Meshulam):

$$\eta(I(G)) \geq \min(\eta(I(G-e)), \eta(I(G \neg e)) + 1)$$

Proof: let e=ab

- $I(G-e)=I(G)\cup(I(G\neg e)*ab)$
- $(I(G\neg e)*ab)\cap I(G)=I(G\neg e)*\{a,b\}$
- $\eta(I(G\neg e)*\{a,b\})=\eta(I(G\neg e))+1$
- $\eta(A)\geq\min(\eta(A\cap B),\eta(A\cup B))$
- $A=I(G)$, $B= I(G\neg e)*ab$
- $\eta(I(G))\geq\min(\eta(I(G\neg e))+1,\ \eta(I(G-e)))$.

Cooperative colorings

Given graphs $G_1, G_2, \ldots, G_k$ on the same vertex set *V*,

A cooperative coloring is a choice

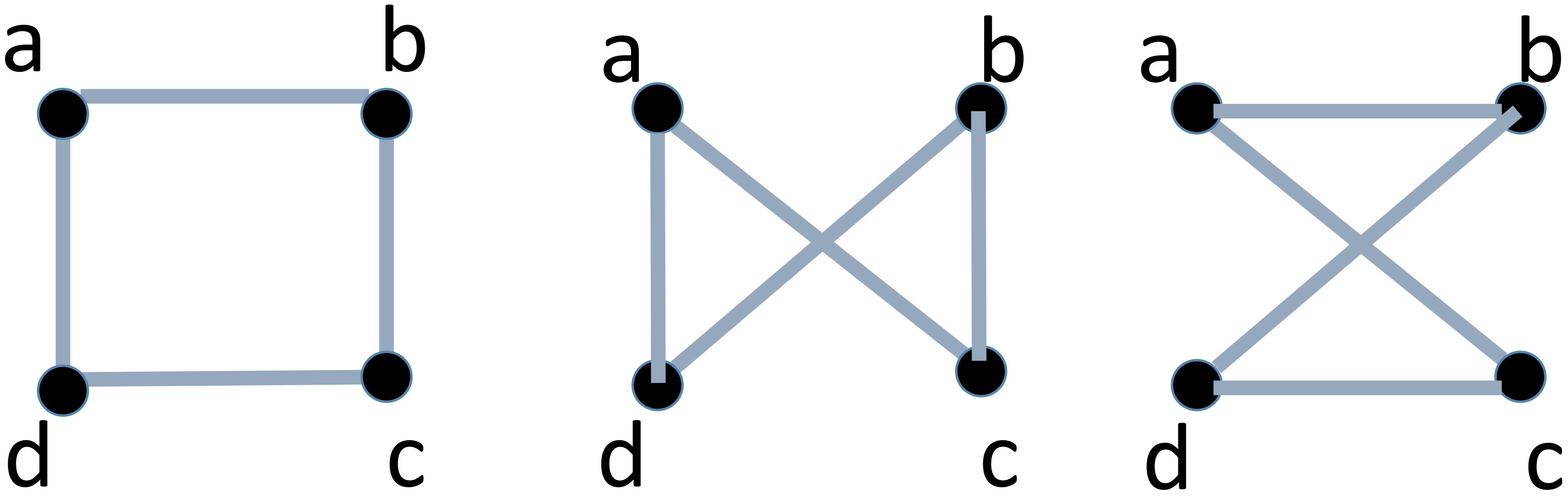
$$A_1 \in I(G_1),\ A_2 \in I(G_2), \ldots, A_k \in I(G_k)$$

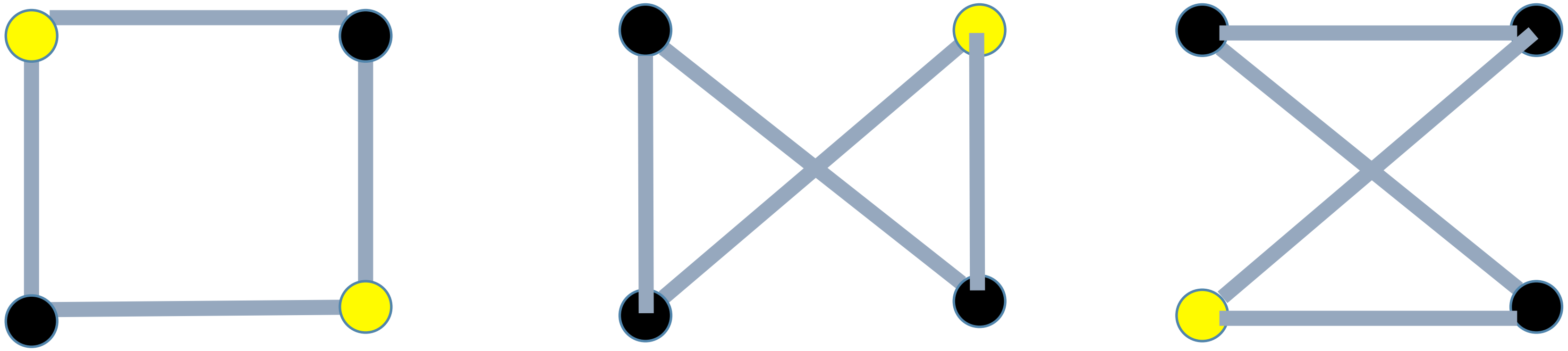
such that

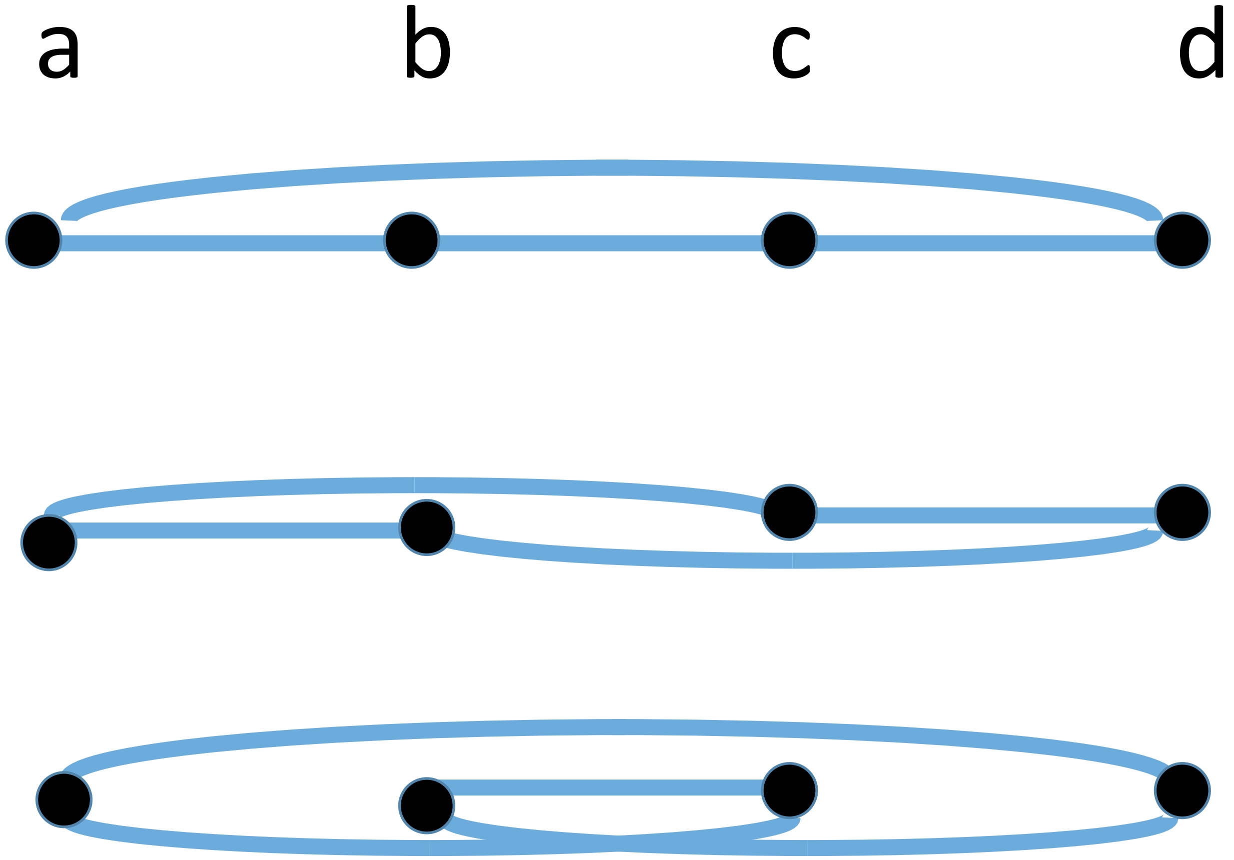
$$\bigcup A_i = V$$

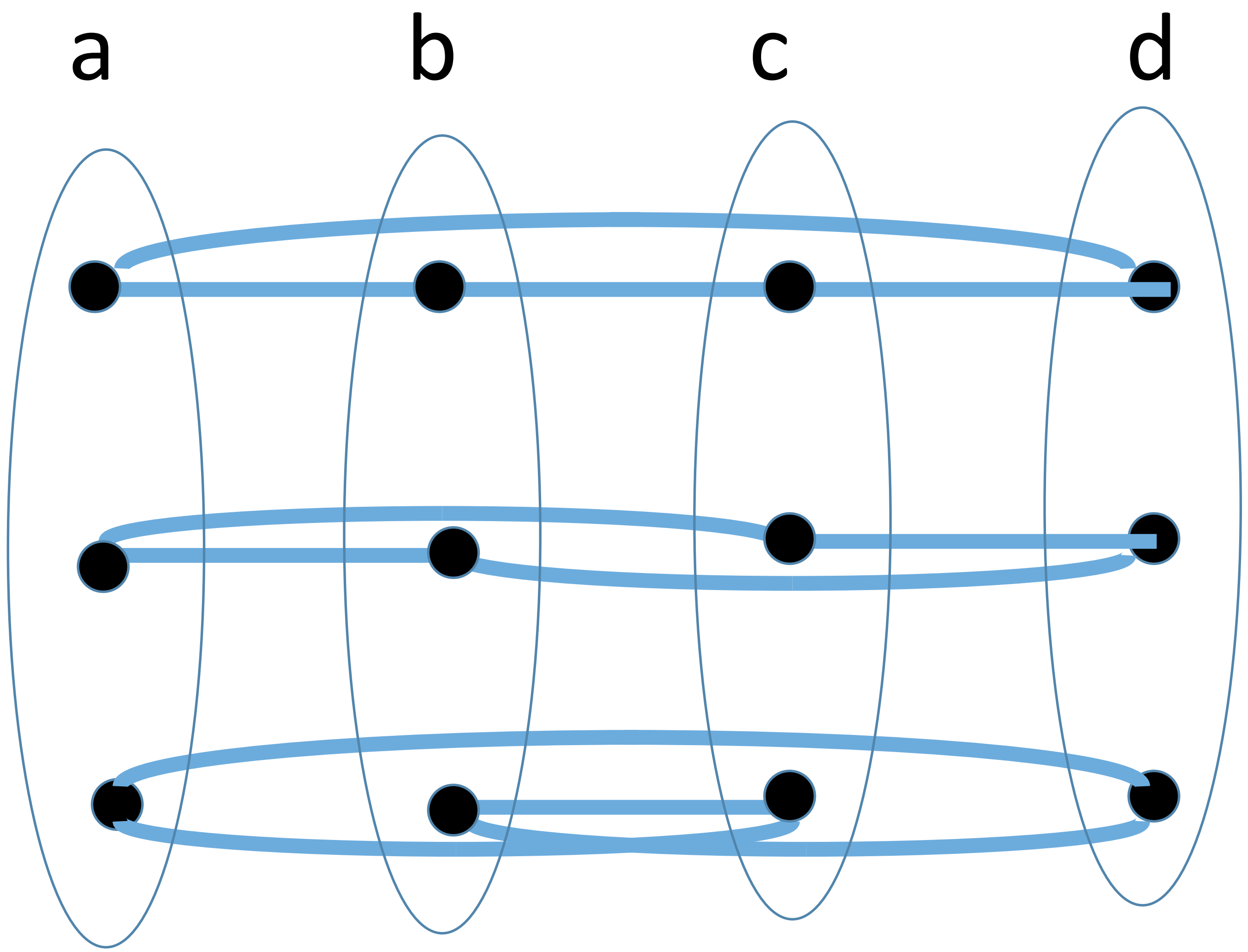
(If all graphs are the same graph *G* then this is ordinary *k*-coloring of G)

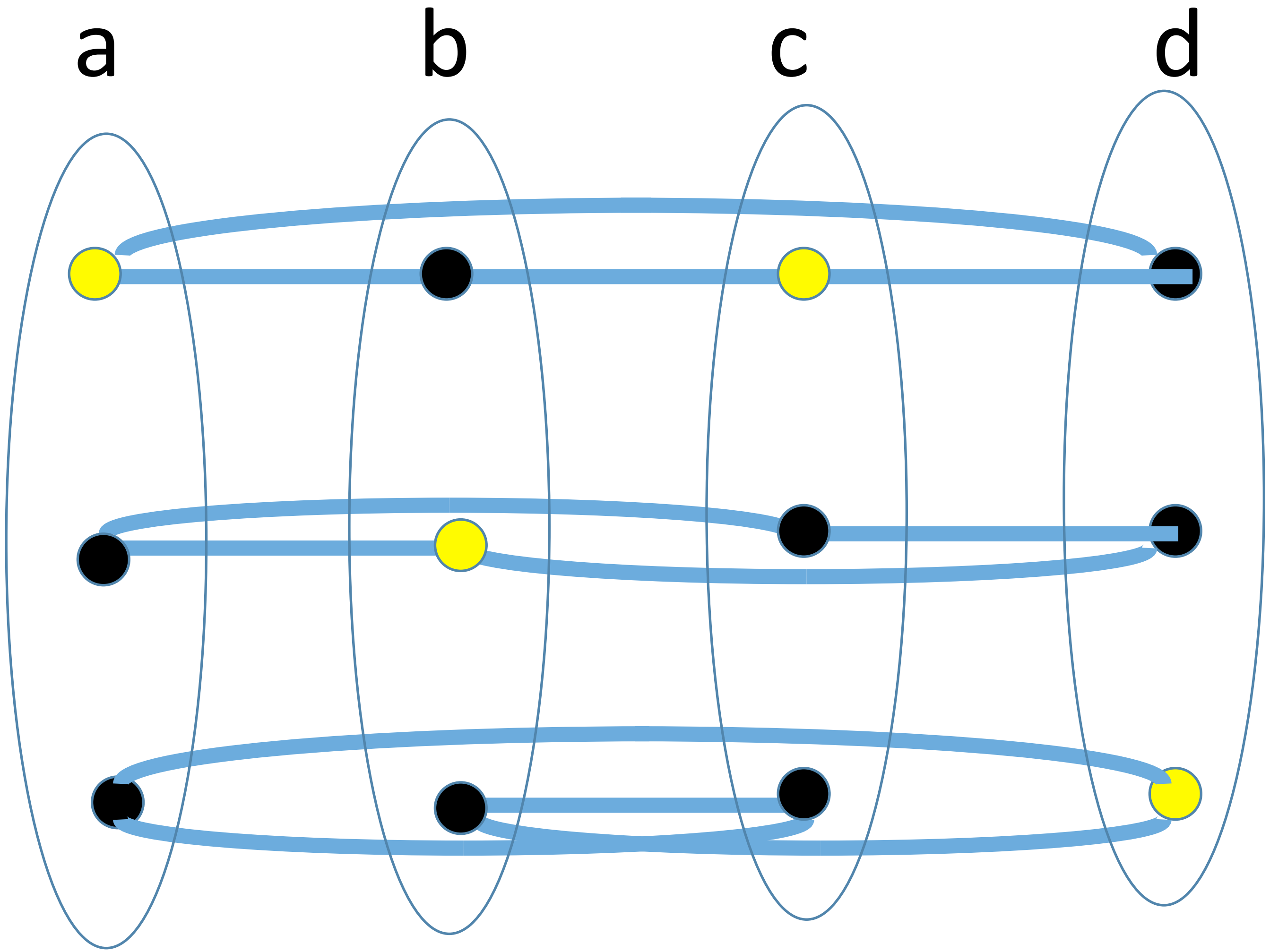
Example:






a
b
c
d

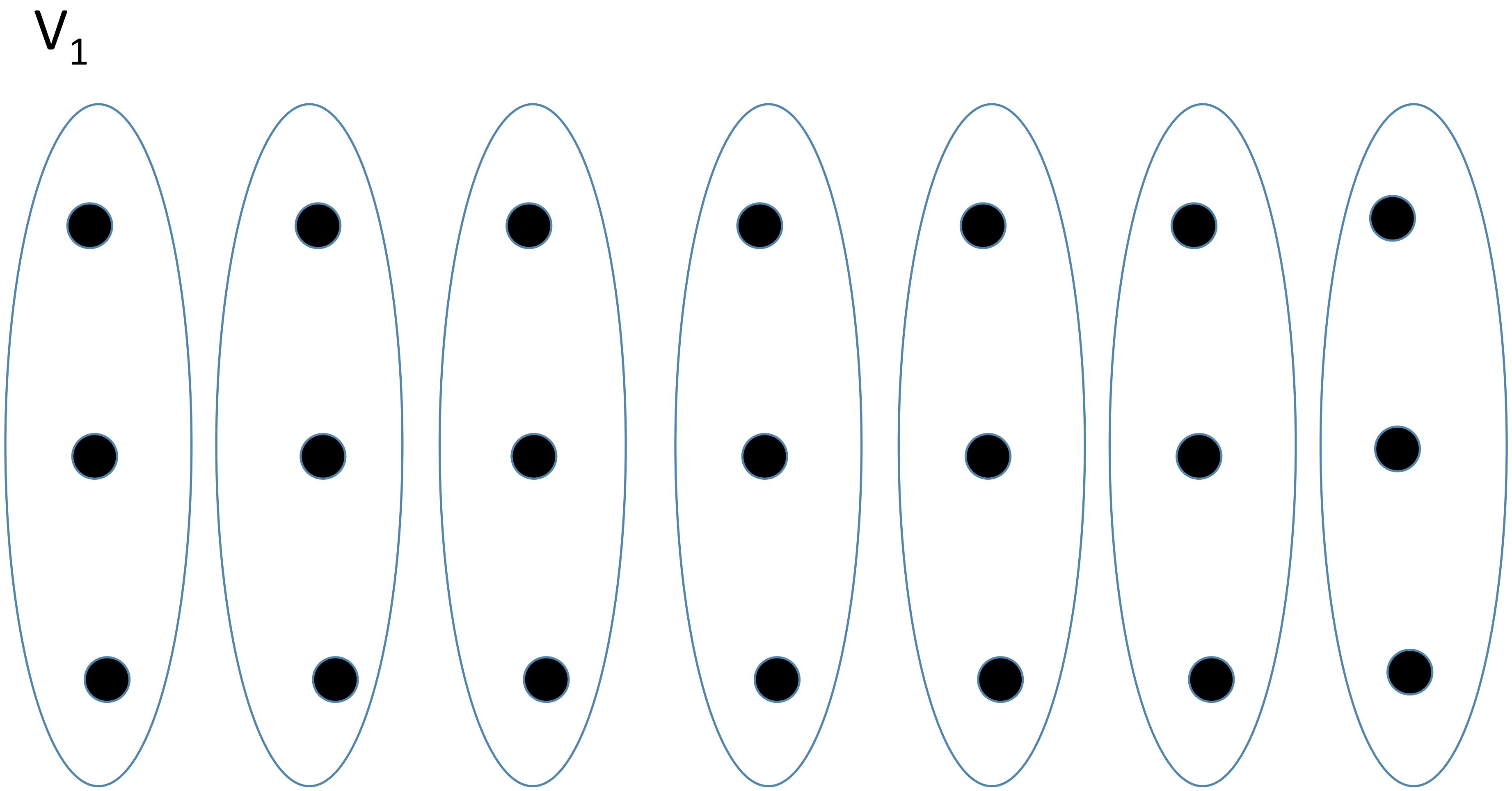

a
b
c
d

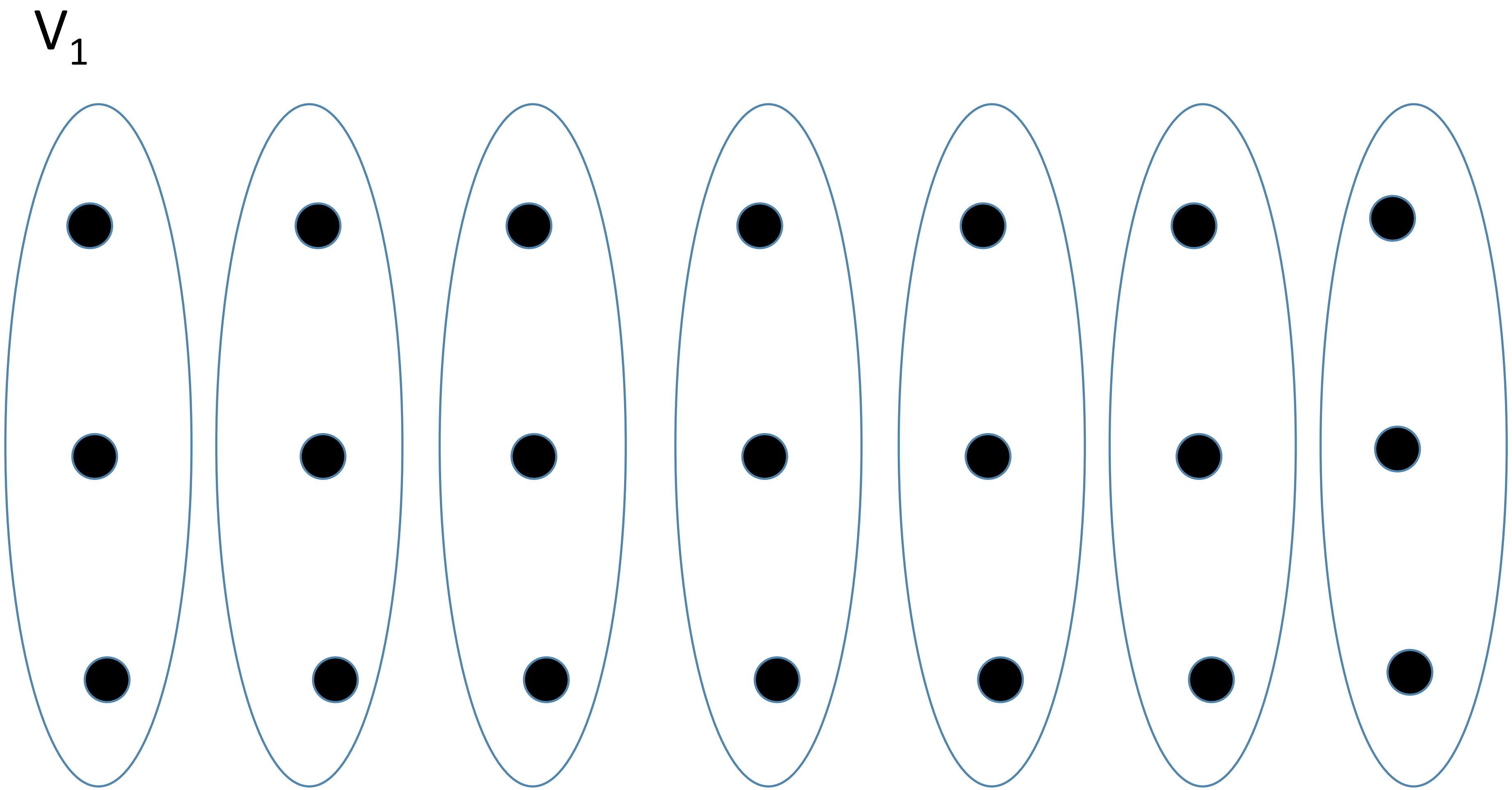

a
b
c
d

Theorem (A+Holzman+Howard+Spruessel)

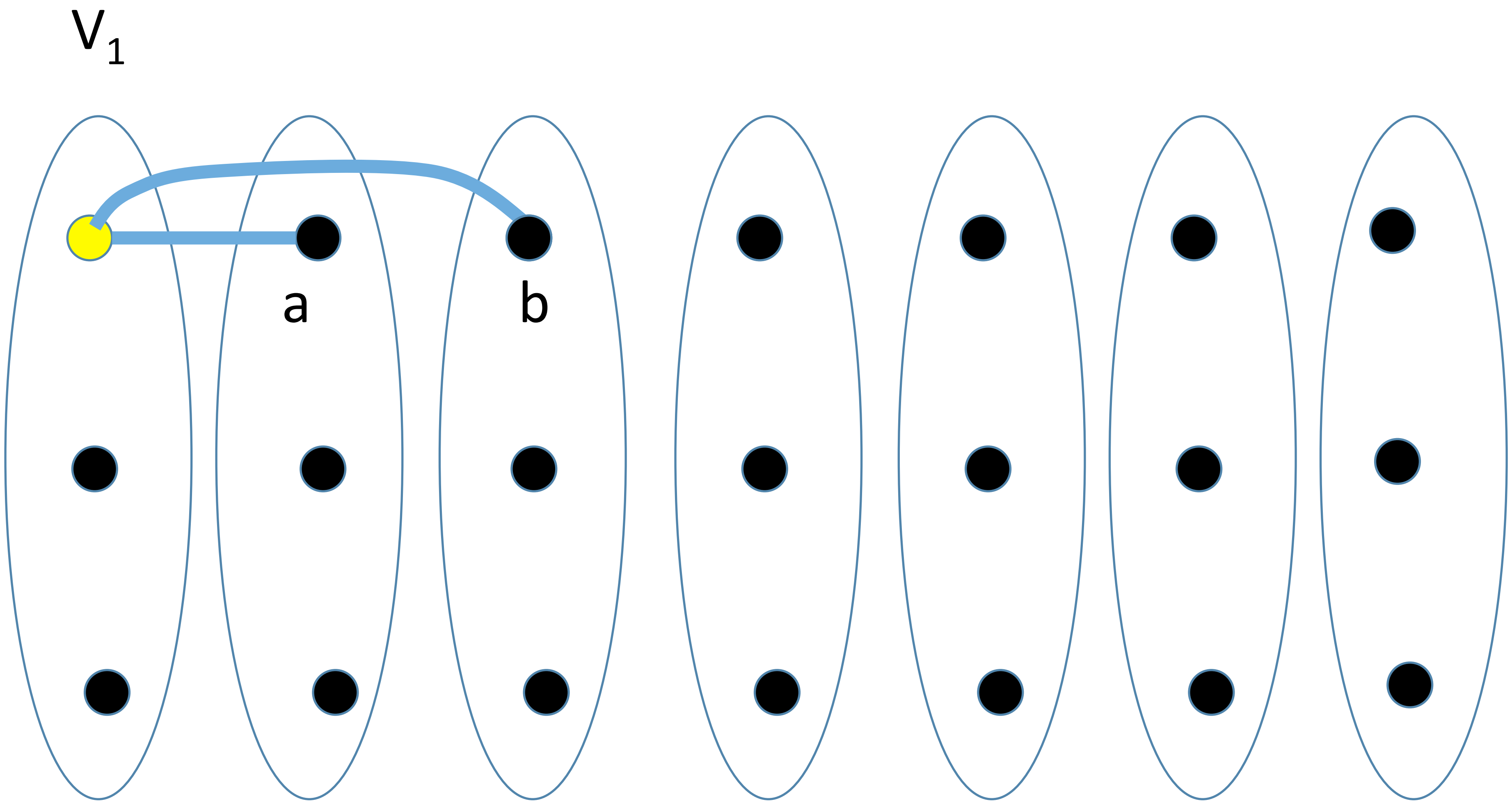
Three cycles on the same vertex set have a cooperative coloring.

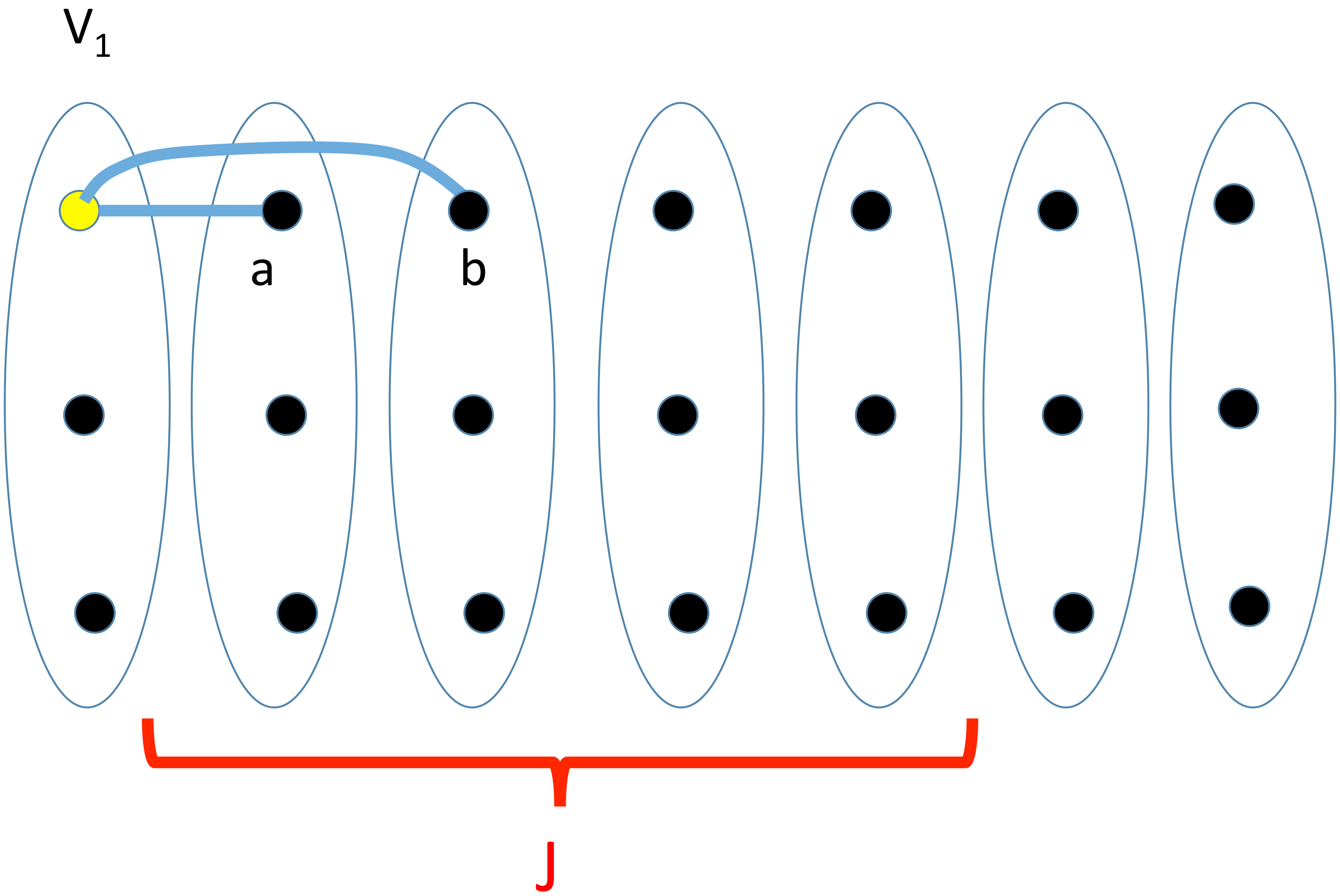
In fact, by three sets of equal sizes (up to 1)


V_1



Choose a vertex from V_1 arbitrarily:


V_1
a
b



V_1
a
b
J

Let $J \subseteq [m] \setminus \{1\}$

We have to show:

$$\eta(I(G[\bigcup_{j \in J} V_j - a - b])) \geq |J|$$

$G[\bigcup_{j\in J} V_j - a - b]$ consists of paths P_k

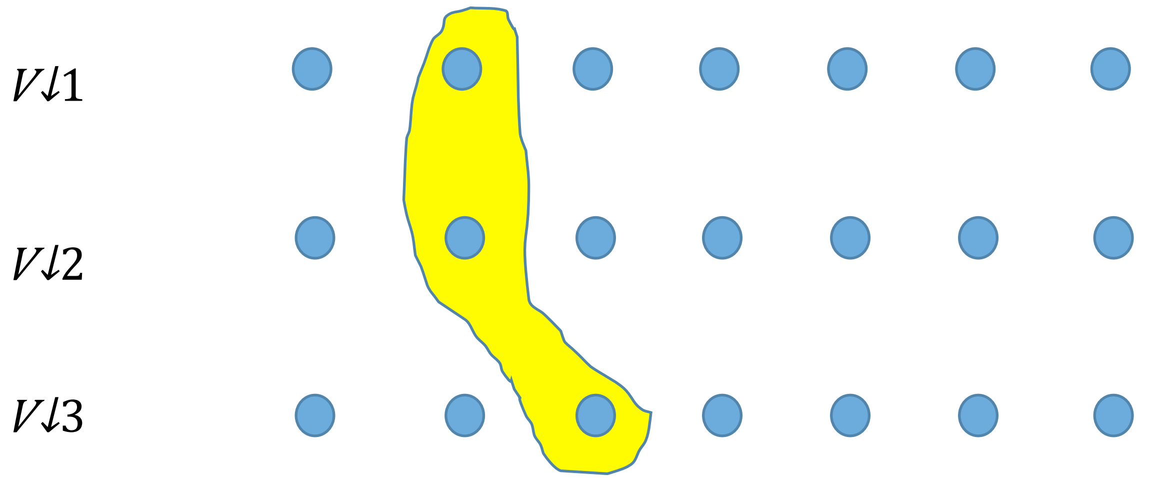
$$\sum P_k \geq 3|J|-2$$

Since $\eta(P_k) \geq \left\lceil \frac{|P_k|}{3} \right\rceil$ and η is additive

$$\eta(I(G[\bigcup_{j\in J} V_j - a - b])) \geq \sum \left\lceil \frac{|P_k|}{3} \right\rceil \geq \left\lceil \sum \frac{|P_k|}{3} \right\rceil \geq |J|$$

For the last inequality remember that $\sum P_k \geq 3|J| - 2$

3-partite hypergraphs



Ryser's conjecture, r=3

Theorem: In a 3-partite hypergraph $\tau \leq 2\nu$.

τ is the covering number, namely the minimal size of a set of vertices that meets all edges,

ν is the matching number – the largest size of a matching

Idea:

If all of V_1 can be matched, then $\tau=\nu=|V_1|$. (Reason: V_1 is a cover)

Reminder: If for every $A\subseteq V_1$ we have $\eta(I(L(K[A])))\geq|A|$ then V_1 is matchable. ($K[A]$ is the set of pairs Konnected to A.)

So, If V_1 is not matchable, some subset of V_1 is connected to "few" pairs of vertices in $V_2\times V_3$.

Definition: For a subset A of V_1 let

$$def(A)=|A|-\eta(I(L(K[A])))$$

Let

$d = \max\{def(A) | A \subseteq V_1\}$

Theorem: $\nu \geq |V_1| - d$.
(follows by a deficiency argument)

Let A be the set for which $def(A) = d$. Then $\eta(I(L(K[A]))) = |A|$ -d.

Observation: If F is a bipartite graph then $\eta(I(L(F)))\geq\nu(F)/2$.

(follows from the fact that $\eta(I(G))\geq\gamma\uparrow i\,(G)$)

Since $\eta(I(L(K[A])))=|A|-d$,
It follows that $\nu(K[A])\leq 2(|A|-d)$.

By König's theorem $\tau(K[A])=\nu(K[A])\leq 2(|A|-d)$.

Let C be a cover of $K[A]$, with $|C|\leq 2(|A|-d)$.

Let $T = C \cup (V\downarrow 1 \setminus A)$

Clearly, C is a cover of the hypergraph (C covers all triples not covered by $V\downarrow 1 \setminus A$).

And

$$|T| = |C| + |V\downarrow 1| - |A| \leq 2(|A| - d) + |V\downarrow 1| - |A| =$$
$$|V\downarrow 1| + |A| - 2d \leq 2(|V\downarrow 1| - d) \leq 2\nu.$$

A hard proof for an easy theorem

Theorem (Gallai): In a family F of intervals on the real line $\tau = \nu$.

Proof: Let $\tau\tau(F) = k$. We shall show that $\nu \geq k$.

We may assume that the intervals are in (0,1).

Every set of $k-1$ points in (0,1) corresponds to a point in Δ_k, the set of points $(x_1, x_2, \ldots, x_k)$ satisfying $\sum x_i = 1$.

The sets A_i satisfy the conditions of KKM, hence there exists a point $x \in \bigcap A_i$.

The k intervals witnessing the fact that $x \in A_i$ are disjoint, proving $\nu \geq k$.

d-intervals

A d-interval is a union of d intervals.

Theorem (Tardos – Kaiser):

In a family of d-intervals $\tau \leq d^{2} \nu$.

The KKMS(hapley) theorem

If the $k-$dimensional simplex is covered by sets A_I , $I\subseteq[k+1]$
so that $conv\, v_j\ j\in J \subseteq \bigcup\{A_I \mid I\subseteq J\}$ for all $J\subseteq[k+1]$ then there exists a family T of subsets of $[k+1]$ such that

(a) $\bigcap A_J\ J\in T \neq \emptyset$, and

(b) T has a fractional matching.